July 7 Math 3260 sec. 51 Summer 2025

4.6 General Vector Spaces

We defined **Real Vector Spaces**:

1. linear combinations & span,
2. linear dependence & linear independence,
3. subspaces, bases, and dimension

Some examples included vector spaces of matrices, functions, sequences...

$$M_{m\times n},\quad F(D),\quad C^n(I),\quad R^\infty.$$

A **vector** is an element of a vector space which can includes objects other than real *n*-tuples. The term **real** tells us that the scalars are real numbers.

A **real vector space** is a set, V, of objects called vectors together with two operations called **vector addition** and **scalar multiplication** that satisfy the following axioms: For each vector $\vec{x}$, $\vec{y}$, and $\vec{z}$ in V and for any scalars, c and d

1. the sum $\vec{x}+\vec{y}$ is in V, and
2. the scalar multiple $c\vec{x}$ is in V.
3. $\vec{x}+\vec{y}=\vec{y}+\vec{x}$,
4. $(\vec{x}+\vec{y})+\vec{z}=\vec{x}+(\vec{y}+\vec{z})$,
5. There is an additive identity vector in V called the zero vector denoted $\vec{0}_V$, such that $\vec{x}+\vec{0}_V=\vec{x}$ for every $\vec{x}$ in V,
6. For each vector $\vec{x}$ in V, there is an additive inverse vector denoted $-\vec{x}$ such that $-\vec{x}+\vec{x}=\vec{0}_V$.
7. $c(\vec{x}+\vec{y})=c\vec{x}+c\vec{y}$,
8. $(c+d)\vec{x}=c\vec{x}+d\vec{x}$,
9. $c(d\vec{x})=(cd)\vec{x}=d(c\vec{x})$, and
10. $1\vec{x}=\vec{x}$.

A Major Theorem on Vector Spaces

Theorem

Suppose that V is a vector space. Then

1. There is only one additive identity vector in V (i.e., the zero vector of V is unique).
2. Each vector in V has only one additive inverse (i.e., the additive inverse of any vector in V is unique).
3. If $\vec{x}$ is any vector in V, then $0\vec{x} = \vec{0}_V$.
4. If c is any scalar, then $c\vec{0}_V = \vec{0}_V$.

Linear Combination

Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a set of one or more ($k \geq 1$) vectors in a vector space V. A **linear combination** of these vectors is any vector of the form

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_k\vec{v}_k,$$

where $x_1, , x_2, \ldots, x_k$ are scalars. The coefficients, $x_1, x_2, \ldots, x_k$, are often called the **weights**.

Span

Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a set of one or more ($k \geq 1$) vectors in a vector space V. The set of all possible linear combinations of the vectors in S is called the **span** of S. It is denoted by $\mathrm{Span}(S)$ or by $\mathrm{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$.

Linear Independence/Dependence

Let V be a vector space. The collection of vectors $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ in V is said to be **linearly independent** if the homogeneous equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_k\vec{v}_k = \vec{0}_V \tag{1}$$

has only the trivial solution, $x_1 = x_2 = \cdots = x_k = 0$. A set of vectors that is not linearly independent is called **linearly dependent**.

If a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ in V is linearly dependent, an equation of the form

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_k\vec{v}_k = \vec{0}_V$$

with at least one coefficient $x_i \neq 0$ is called a **linear dependence relation**.

Subspace

Let V be a real vector space. A **subspace** of V is a nonempty set, S, of vectors in V such that

- for every $\vec{x}$ and $\vec{y}$ in S, $\vec{x}+\vec{y}$ is in S, and
- for each $\vec{x}$ in S and scalar c, $c\vec{x}$ is in S.

Basis

Let S be a subspace of a vector space V, and let $\mathcal{B}=\{\vec{v}_1,\ldots,\vec{v}_k\}$ be a subset of vectors in S. $\mathcal{B}$ is a **basis** of S provided

- $\mathrm{Span}(\mathcal{B})=S$
- $\mathcal{B}$ is linearly independent.

If a subspace S of a vector space V has a basis with k vectors in it ($k<\infty$), then every basis of S has k vectors in it. We'll define dimension based on this.

Dimension

Let S be a subspace of a vector space V. We define the **dimension** of S as follows:

- If $S = \{\vec{0}_V\}$, then we define $\dim(S) = 0$.
- If S has a basis consisting of k vectors, where $k < \infty$, then we define $\dim(S) = k$.
- If S is not spanned by any finite set of vectors, then we say that S is infinite dimensional.

Examples of finite dimensional vector spaces include R^n, $M_{m\times n}$, $\mathbb{P}_n$, etc.
Examples of infinite dimensional vector spaces include R^∞, $F(D)$, $C^n(I)$, etc.

Section 4.7.4.2 Function Spaces $C^n(I)$

If D is a subset R, then $F(D)$ is the vector space of real valued functions with domain D.

$$F(D) = \{f \mid f : D \to R\}$$

For $f, g \in F(D)$ and scalar c, we defined the two operations

$$\text{vector addition} \quad (f+g)(x) = f(x) + g(x), \quad \text{for each } x \in D$$

$$\text{scalar multiplication} \quad (cf)(x) = cf(x), \quad \text{for each } x \in D$$

The operations of vector addition and scalar multiplication are the ones that are familiar from basic calculus.

The vector space $F(D)$ is an example of an infinite dimensional vector space.

We want to consider the case that the domain D is an interval (which we'll call I for *interval*).

Some Basic Results from Calculus

Let $I = (a, b)$ be some interval in R, and suppose f and g are in $F(I)$ and c is a scalar.

- If f and g are continuous on I, then $f + g$ and cf are continuous on I.
- If f and g are differentiable on I, then $f + g$ and cf are differentiable on I, and

$$(f + g)' = f' + g', \quad \text{and} \quad (cf)' = cf'$$

- If f is differentiable on I, then f is continuous on I.

$C^n(I)$

The notation $C^0(I)$ denotes the subset of $F(I)$ of functions that are continuous on I. Is $C^0(I)$ a subspace of $F(I)$?

$C^0(I)$ is not empty, $z(x) = 0$ is in $C^0(I)$

It's closed since $f, g \in C^0(I) \Rightarrow f+g$ and cf are in $C^0(I)$. C^0 is a subspace of $F(I)$.

The notation $C^1(I)$ denotes the subset of $C^0(I)$ of functions that are continuously differentiable on I. Is $C^1(I)$ a subspace of $C^0(I)$? Is $C^1(I)$ a subspace of $F(I)$?

Yes $C^1(I)$ is closed and nonempty.

$C^n(I)$

- The set $C^n(I)$ is the subspace of $F(I)$ of functions that are at least n-times continuously differentiable on I.

- The set $C^\infty(I)$ is the subspace of $F(I)$ of functions with continuous derivatives of all orders on the interval I.

- The set $C^\infty(R)$ is the subspace of $F(R)$ of functions with continuous derivatives of all orders on all of R.

Examples of $C^\infty(R)$ functions include many familiar favorites:

$$e^x, \quad \sin(x), \quad \cos(\pi x), \quad \tan^{-1}(x), \quad \text{etc.}$$

$f(x) = e^x$, $p(x) = x$

Last time, we establised that the set $\{1, x, x^2\}$ is linearly independent in $C^\infty(R)$. In fact, the set $\{1, x, x^2, \ldots, x^n\}$ is linearly independent in $C^\infty(R)$ for each positive integer n.

Section 4.7.4.3 Function Spaces of Polynomials

Finite Dimensional Subspace of $C^\infty(R)$

Consider the set $\mathcal{B}_n = \{1, x, x^2, \dots, x^n\}$ in $C^\infty(R)$. Let

$$\mathbb{P}_n = \mathrm{Span}(\mathcal{B}_n).$$

$\mathbb{P}_n$ is the set of polynomials in x of degree at most n with real coefficients. For

$$p(x) = p_0 + p_1 x + \dots + p_n x^n \quad \text{and} \quad q(x) = q_0 + q_1 x + \dots + q_n x^n$$

in $\mathbb{P}_n$ and scalar c,

vector addition $\quad (p+q)(x) = (p_0+q_0)+(p_1+q_1)x+\cdots+(p_n+q_n)x^n, \quad$ and

scalar multiplication $\quad (cp)(x) = cp_0 + cp_1 x + \dots + cp_n x^n.$

Example

Let $\mathcal{P}_{2,1}$ be the subset of $\mathbb{P}_2$ of polynomials $p(x) = p_0 + p_1 x + p_2 x^2$ that satisfy $p(1) = 0$. Which of the following are elements of $\mathcal{P}_{2,1}$?

✓ 1. $f(x) = 1 - 2x + x^2$ $\quad f(1) = 1 - 2(1) + (1^2) = 0$

✗ 2. $g(x) = 1 - 3x + 4x^2$ $\quad g(1) = 1 - 3(1) + 4(1^2) = 2 \neq 0$

✓ 3. $p(x) = 1 + 3x - 4x^2$ $\quad p(1) = 1 + 3(1) - 4(1^2) = 0$

✓ 4. $z(x) = 0$ $\quad z(1) = 0$

Example Continued...

Show that $\mathcal{P}_{2,1} = \{p \in \mathbb{P}_2 \mid p(1) = 0\}$ is a subspace of $\mathbb{P}_2$ by finding a spanning set (show that $\mathcal{P}_{2,1}$ can be defined as a span).

Let $p(x)$ be in $\mathcal{P}_{2,1}$. $p(x) = p_0 + p_1 x + p_2 x^2$.

$$p(1) = p_0 + p_1(1) + p_2(1^2) = p_0 + p_1 + p_2 = 0$$

So $p_0 = -p_1 - p_2$

$$p(x) = -p_1 - p_2 + p_1 x + p_2 x^2$$

$$= -p_1 + p_1 x - p_2 + p_2 x^2$$

$p(x) = p_1(-1+x) + p_2(-1+x^2)$

so $\mathcal{P}_{2,1} = \text{Span}\{x-1, x^2-1\}$.

As a span $\mathcal{P}_{2,1}$ is a subspace of $\mathbb{P}_2$.

4.8 Working with Coordinate Vectors

Coordinate Vectors

Suppose that V is a vector space and suppose that S is a finite dimensional subspace of V. Suppose that $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_k\}$ is an ordered basis for S. Then the (unique) vector

$$[\vec{x}]_{\mathcal{B}} = \langle c_1, c_2, \ldots, c_k \rangle \in R^k$$

such that

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$$

is called the **coordinate vector** of $\vec{x}$ with respect to the ordered basis $\mathcal{B}$.

Note that no matter what sort of objects the vectors in S are, the coordinate vectors are vectors in R^k.

Two Lemmas

Lemma

Suppose that S is a subspace of a vector space V and $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_k\}$ is an ordered basis of S. If $\vec{x}$ and $\vec{y}$ are any two vectors in S and c is any scalar then

1. $[\vec{x} + \vec{y}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}}$ and
2. $[c\vec{x}]_{\mathcal{B}} = c\,[\vec{x}]_{\mathcal{B}}$.

Lemma

Suppose S is a subspace of a vector space V and $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_k\}$ is an ordered basis of S. Then $\vec{0}_V$ is the only vector in S that has coordinate vector $\vec{0}_k$. In other words, the following statement holds for all vectors $\vec{x} \in S$:

$$[\vec{x}]_{\mathcal{B}} = \vec{0}_k \text{ if and only if } \vec{x} = \vec{0}_V.$$

Theorem

Suppose S is a subspace of a vector space V and $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_k\}$ is an ordered basis of S. Let $T = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m\}$ be any set of vectors in S, and let $C_T = \left\{[\vec{x}_1]_{\mathcal{B}}, [\vec{x}_2]_{\mathcal{B}}, \ldots, [\vec{x}_m]_{\mathcal{B}}\right\}$ be the set of vectors in R^k consisting of the coordinate vectors of the elements of T with respect to the basis $\mathcal{B}$. Then T is linearly independent in V if and only if C_T is linearly independent in R^k.

The power of this theorem is that it will allow us to translate a problem in some finite dimensional vector space to R^k. Then we can use tools, like row reduction, to bear.

Example

Let $S = \{p, q, r\}$, where $p(x) = 1 + 4x - 2x^2 + 3x^3$,

$$q(x) = 1 + 3x - 3x^2 + x^3, \quad \text{and} \quad r(x) = 2 + 4x - 8x^2 - 2x^3.$$

Determine whether S is linearly dependent or linearly independent in $\mathbb{P}_3$. If linearly dependent, find a linear dependence relation.

We'll use coordinate vectors with basis

$\mathcal{B} = \{1, x, x^2, x^3\}$.

$[p]_{\mathcal{B}} = \langle 1, 4, -2, 3\rangle$ Consider $A\vec{x} = \vec{0}$

$[q]_{\mathcal{B}} = \langle 1, 3, -3, 1\rangle$

$[\vec{r}]_{\mathcal{B}} = \langle 2, 4, -8, -2\rangle$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 4 & 3 & 4 \\ -2 & -3 & -8 \\ 3 & 1 & -2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since $\{ [p]_B, [q]_B, [r]_B \}$ are linearly dependent,

S' is linearly dependent.

Moreover,

$$r(x) = -2p(x) + 4q(x)$$

$$\Rightarrow \quad 2p(x) - 4q(x) + r(x) = z(x)$$

Example

Let $S = \{p, q, r\}$, where $p(x) = 1 + 4x - 2x^2 + 3x^3$,

$$q(x) = 1 + 3x - 3x^2 + x^3, \quad \text{and} \quad r(x) = 2 + 4x - 8x^2 - 2x^3.$$

Check:

$2p(x) - 4q(x) + r(x) =$

$(2-4+2) + (8-12+4)x + (-4+12-8)x^2 + (6-4-2)x^3 = z(x)$

Chapter 5 Linear Transformations

In this chapter, we will consider a special class of functions called **linear transformations.** The inputs and outputs that we'll be interested in will be vectors.

Let's start with some notation and concepts related to functions more generally.

Domain, Codomain, Images, & Range

"f maps D into C"

$$\underbrace{f}_{\text{Rule}} : \underbrace{D}_{\text{set of all inputs}} \longrightarrow \underbrace{C}_{\text{set where outputs live}}$$

- D is the **domain** of the function.
- C is where the outputs are. It's called the **codomain**.
- For x in D, if $f(x) = y$, we call y the **image of x under f**.
- If S is a subset of D, then the **image of S under f** is the collection of all images for each x in S.

$$f(S) = \{y \in C \mid y = f(x) \text{ for at least one } x \in S\} = \{f(x) \mid x \in S\}$$

- $f(D)$ is the **range** of f. This is the set of all actual images. We can write $f(D) = \text{range}(f)$.

Codomain -vs- Range

Codomain = the set that contains the outputs

Range = the set of actual outputs

Example: Consider $f : R \to R$ defined by $f(x) = e^x$. The **codomain** is R because that's how the function is being defined. But recall that

$$e^x > 0, \quad \text{for all real } x.$$

So the **range** is the interval $(0, \infty)$.

Example: Consider $g : (0, \infty) \to R$ defined by $g(x) = \ln(x)$. For this function

$$\text{the } \textbf{codomain} = R = \text{the } \textbf{range}.$$

Onto, One-to-One, & Invertibility

$$f : D \to C$$

f maps D into C

Onto

If $f(D) = C$, that is, if the range is equal to the codomain, we say that f is **onto.** In this case, we say

"f maps D **onto** C."

If f maps D onto C, then for each $y \in C$

$$f(x) = y \quad \text{is consistent.}$$

By consistent, we mean has at least one solution.

Onto, One-to-One, & Invertibility

$$f : D \to C$$

One-to-One

If for each $y \in \text{range}(f)$, the equation

$$f(x) = y$$

has exactly one solution, we say that f is **one-to-one.**

If f is one-to-one, then for each y such that $f(x) = y$ is consistent

$$f(x) = y \quad \text{has a unique solution.}$$

Onto, One-to-One, & Invertibility

$$f : D \to C$$

Invertible

If f maps D onto C and f is one-to-one, then we say that f is **invertible**. If $f : D \to C$ is invertible, then there is a corresponding **inverse function** denoted by f^{-1} such that

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x, \quad \text{for each } x \in D \text{, and}$$

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = x, \quad \text{for each } x \in C.$$

$f^{-1} : C \to D$ is the function defined by

$$f^{-1}(y) = x \text{ where } x \text{ is the unique solution of } f(x) = y.$$

Example

Let $T : R^2 \to R^2$ be defined by $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

1. Find the images of $\vec{x} = \langle 1, 1 \rangle$, $\vec{x} = \langle -2, 1 \rangle$, and $\vec{x} = \langle x_1, x_2 \rangle$.
2. What is $\text{range}(T)$? Does T map R^2 onto R^2?
3. Is T one-to-one?
4. Is T invertible?

$$T(\langle 1, 1 \rangle) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \langle 1, 1 \rangle = \langle 0(1) + (-1)(1),\ 1(1) + 0(1) \rangle$$

$$= \langle -1, 1 \rangle$$

$$T(\langle -2, 1 \rangle) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \langle -2, 1 \rangle = \langle -1, -2 \rangle$$

$$T(\langle x_1, x_2 \rangle) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \langle x_1, x_2 \rangle = \langle -x_2, x_1 \rangle$$

2. For any vector $\langle y_1, y_2 \rangle$ in $\mathbb{R}^2$.

$$T(\langle y_2, -y_1 \rangle) = \langle y_1, y_2 \rangle$$

The range is all of $\mathbb{R}^2$.

3. Is T one to one?

$$T(\langle x_1, x_2 \rangle) = T(\langle y_1, y_2 \rangle)$$

$$\langle -x_2, x_1 \rangle = \langle -y_2, y_1 \rangle$$

$$\Rightarrow -x_2 = -y_2 \Rightarrow x_2 = y_2 \quad \text{and}$$

$$x_1 = y_1$$

$\langle x_1, x_2 \rangle = \langle y_1, y_2 \rangle$

That is, $T(\vec{x}) = T(\vec{y}) \iff \vec{x} = \vec{y}$

T is one to one.

4. Yes, T is onto and one to one, so T is invertible.

Example

Let $P : R^2 \to R^2$ be defined by $P(\vec{x}) = B\vec{x}$ where $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

1. Find the images of $\vec{x} = \langle 1, 1\rangle$, $\vec{x} = \langle -2, 1\rangle$, and $\vec{x} = \langle x_1, x_2\rangle$
2. What is $\mathrm{range}(P)$? Does P map R^2 onto R^2?
3. Is P one-to-one?
4. Is P invertible?

1. $P(\langle 1,1\rangle) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\langle 1,1\rangle = \langle 0,1\rangle$

$P(\langle -2,1\rangle) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\langle -2,1\rangle = \langle 0,1\rangle$

$P(\langle x_1,x_2\rangle) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\langle x_1,x_2\rangle = \langle 0,x_2\rangle$

2. range $(P) = \mathrm{Span}\{\langle 0,1\rangle\}$.

P is not onto.

3. $P(\langle 1,1\rangle) = P(\langle -2,1\rangle) = \langle 0,1\rangle$

but $\langle 1,1\rangle \neq \langle -2,1\rangle$

P is not one to one

4. P is not invertible.