July 9 Math 3260 sec. 51 Summer 2025

Chapter 5 Linear Transformations

We want to consider a special class of functions called **linear transformations**. (We haven't defined what this means yet.)

Recall that for a function

$$f : D \to C \quad \text{(read "}f\text{" maps } D \text{ into } C\text{)}$$

- D is the **domain** and C is the **codomain**,
- an **image** is an output, e.g., $y = f(x)$, or a set of outputs, e.g., $f(S) = \{f(x)\,|\,x \in S\}$,
- the **range** is the set of all outputs—i.e., the image of D under f,
- f is called **onto** if $f(D) = C$—i.e., the range equals the codomain,
- f is **one to one** if $f(x) = f(y) \iff x = y$,
- and f is **invertible** if f is onto and one to one (in which case there's an inverse function f^{-1}).

Remark on "Onto"

If a function is not onto, it's always possible to define a new function that is onto.

Case in point: Recall the function $P : R^2 \to R^2$ be defined by $P(\vec{x}) = B\vec{x}$ where $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

$\langle x_1, x_2 \rangle \mapsto \langle 0, x_2 \rangle$

We saw that the range of P is the set $\mathrm{Span}\{\langle 0, 1\rangle\}$, so P is not onto. But we could define the related function

$$\hat{P} : R^2 \to \mathrm{Span}\{\langle 0, 1\rangle\}, \quad P(\vec{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}.$$

This is a *version* of the same function that is onto.

5.2 Linear Transformations for R^n to R^m

Linear Transformation

A **linear transformation** from R^n to R^m is a function $T : R^n \to R^m$ such that for each pair of vectors $\vec{x}$ and $\vec{y}$ in R^n and for any scalar c

1. $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$, and
2. $T(c\vec{x}) = cT(\vec{x})$.

A function having vector spaces as a domain and codomain are called **transformations.**

The two properties in this definition are what we mean by **linear** or **linearity**. Functions that don't have these properties are called nonlinear.

Example

Let $T : R^2 \to R^3$ be defined by $T(\langle x_1, x_2 \rangle) = \langle x_1 x_2, 0, x_1 + x_2 \rangle$.

Find the images of $\langle 0, 0 \rangle$, $\langle 1, 0 \rangle$, $\langle 0, 1 \rangle$, $\langle 1, 1 \rangle$ and $\langle 2, 2 \rangle$.

1. $T(\langle 0, 0 \rangle) =$ $\langle 0(0), 0, 0+0 \rangle = \langle 0, 0, 0 \rangle$
2. $T(\langle 1, 0 \rangle) =$ $\langle 1(0), 0, 1+0 \rangle = \langle 0, 0, 1 \rangle$
3. $T(\langle 0, 1 \rangle) =$ $\langle 0(1), 0, 0+1 \rangle = \langle 0, 0, 1 \rangle$
4. $T(\langle 1, 1 \rangle) =$ $\langle 1(1), 0, 1+1 \rangle = \langle 1, 0, 2 \rangle$
5. $T(\langle 2, 2 \rangle) =$ $\langle 2(2), 0, 2+2 \rangle = \langle 4, 0, 4 \rangle$

$T : R^2 \to R^3 \quad T(\langle x_1, x_2\rangle) = \langle x_1 x_2, 0, x_1 + x_2\rangle$

1. Is $T(\langle 1,0\rangle + \langle 0,1\rangle) = T(\langle 1,0\rangle) + T(\langle 0,1\rangle)$? NO

$T(\langle 1,0\rangle + \langle 0,1\rangle) = T(\langle 1,1\rangle) = \langle 1,0,2\rangle$

$T(\langle 1,0\rangle) + T(\langle 0,1\rangle) = \langle 0,0,1\rangle + \langle 0,0,1\rangle = \langle 0,0,2\rangle$

2. Is $T(2\langle 1,1\rangle) = 2T(\langle 1,1\rangle)$? NO

$T(2\langle 1,1\rangle) = T(\langle 2,2\rangle) = \langle 4,0,4\rangle$

$2T(\langle 1,1\rangle) = 2\langle 1,0,2\rangle = \langle 2,0,4\rangle$

T is not a linear transformation.

Example

Let $T : R^3 \to R^2$ be defined by $T(\langle x_1, x_2, x_3 \rangle) = \langle x_1 - x_2, x_2 - x_3 \rangle$.
Show that T is a linear transformation.

We need to show that $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$ and $T(k\vec{x}) = kT(\vec{x})$ for all $\vec{x}, \vec{y} \in R^3$ and $k \in R$,

Let $\vec{x} = \langle a, b, c \rangle$ and $\vec{y} = \langle d, e, f \rangle$ and $k \in R$ with $a, b, c, d, e, f \in R$.

$T(\vec{x}) = T(\langle a, b, c \rangle) = \langle a-b, b-c \rangle$

$T(\vec{y}) = T(\langle d, e, f \rangle) = \langle d-e, e-f \rangle$

$\vec{x} + \vec{y} = \langle a+d, b+e, c+f \rangle$

$T(\vec{x}+\vec{y}) = T(\langle a+d, b+e, c+f \rangle) = \langle (a+d) - (b+e), (b+e) - (c+f) \rangle$

$T: R^3 \to R^2 \quad T(\langle x_1, x_2, x_3\rangle) = \langle x_1 - x_2, x_2 - x_3\rangle$

$T(\vec{x}+\vec{y}) = \langle a+d-b-e, b+e-c-f\rangle$

$= \langle a-b+d-e, b-c+e-f\rangle$

$= \langle a-b, b-c\rangle + \langle d-e, e-f\rangle$

$= T(\vec{x}) + T(\vec{y})$

$T(k\vec{x}) = T(k\langle a, b, c\rangle) = T(\langle ka, kb, kc\rangle)$

$= \langle ka-kb, kb-kc\rangle$

$= \langle k(a-b), k(b-c)\rangle$

$= k\langle a-b, b-c\rangle = kT(\vec{x})$

T is a linear transformation.

Recall Properties of Matrix-Vector Product

If A is an $m \times n$ matrix, $\vec{x}$ and $\vec{y}$ are vectors in R^n, and c is a scalar, then

- $A\vec{x}$ is a vector in R^m.
- $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$, and
- $A(c\vec{x}) = cA\vec{x}$.

Lemma

If A is an $m \times n$ matrix and $T : R^n \to R^m$ is defined by $T(\vec{x}) = A\vec{x}$, then T is a linear transformation.

Not only does the matrix-vector product define a linear transformation. Turns out, **every** linear transformation from R^n to R^m is a matrix-vector product!

Theorem

Suppose that $T : R^n \to R^m$ is a linear transformation. Then there is a unique $m \times n$ matrix A, such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in R^n$.

Furthermore, the matrix[a] A is the matrix whose column vectors are

$$\mathrm{Col}_j(A) = T\left(\vec{e}_j\right)$$

where $\mathcal{E} = \left\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\right\}$ is the standard basis for R^n.

[a] We'll call A the **standard matrix** for the transformation T.

The **columns** of A are the images of the standard unit vectors.

A is *unique* if we are considering inputs and outputs relative to the standard basis $\mathcal{E}$.

Example

Find the standard matrix for the linear transformation $T : R^2 \to R^3$ given by

$$T(\langle x_1, x_2 \rangle) = \langle x_1 + 3x_2, 2x_1 + 4x_2, -2x_2 \rangle.$$

Let's call the matrix A. The columns of A will be $T(\vec{e}_1)$ and $T(\vec{e}_2)$.

$T(\vec{e}_1) = T(\langle 1, 0 \rangle) = \langle 1, 2, 0 \rangle$

$T(\vec{e}_2) = T(\langle 0, 1 \rangle) = \langle 3, 4, -2 \rangle$

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix}$$

$$T(\langle x_1, x_2\rangle) = \langle x_1 + 3x_2, 2x_1 + 4x_2, -2x_2\rangle.$$

Check: $A\vec{x} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix} \langle x_1, x_2 \rangle$

$= \langle x_1 + 3x_2, 2x_1 + 4x_2, -2x_2 \rangle$ ✓

Theorem

If $T : R^n \to R^m$ is a linear transformation, then

$$T(\vec{0}_n) = \vec{0}_m.$$

Remark: This can be used as a test to rule out that something is a linear transformation. That is, if for some $T : R^n \to R^m$, $T(\vec{0}_n) \neq \vec{0}_m$, then T can't be a linear transformation.

Caveat: This doesn't say that $T(\vec{0}_n) = \vec{0}_m$ by itself guarantees linearity.

Fundamental Subspaces: Range and Kernel

Let $T : R^n \to R^m$ be a linear transformation, and let A be its standard matrix. The **range** of T is defined by

$$\text{range}(T) = \{T(\vec{x}) \mid \vec{x} \in R^n\}.$$

and the **kernel** of T, denoted $\ker(T)$ is defined by

$$\ker(T) = \left\{\vec{x} \in R^n \mid T(\vec{x}) = \vec{0}_m\right\}.$$

Moreover, $\text{range}(T)$ is a subspace of R^m, $\ker(T)$ is a subspace of R^n, and

$$\text{range}(T) = \mathcal{CS}(A), \quad \text{and} \quad \ker(T) = \mathcal{N}(A).$$

It follows from the FTLA that

$$\dim\big(\text{range}(T)\big) + \dim\big(\ker(T)\big) = n.$$

Example

Identify the range and kernel of $T : R^3 \to R^2$ given by

$$T(\langle x_1, x_2, x_3 \rangle) = \langle 2x_1 + x_2 - 2x_3, 2x_1 + 3x_2 + 6x_3 \rangle.$$

Let's find the standard matrix A. We need $T(\vec{e}_1)$, $T(\vec{e}_2)$, $T(\vec{e}_3)$.

$T(\vec{e}_1) = T(\langle 1,0,0 \rangle) = \langle 2, 2 \rangle$

$T(\vec{e}_2) = T(\langle 0,1,0 \rangle) = \langle 1, 3 \rangle$

$T(\vec{e}_3) = T(\langle 0,0, \quad \rangle) = \langle -2, 6 \rangle$

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 2 & 3 & 6 \end{bmatrix}$$

$T(\langle x_1, x_2, x_3 \rangle) = \langle 2x_1 + x_2 - 2x_3, 2x_1 + 3x_2 + 6x_3 \rangle.$

range (T) = $CS(A)$ and ker(T) = $\mathcal{N}(A)$

let's find rref (A)

$A \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \end{bmatrix}$

* This set is a basis

range (T) = Span $\{\langle 2,2 \rangle, \langle 1,3 \rangle\}$

$A\vec{x} = \vec{0}_2 \Rightarrow$ $x_1 = 3x_3$, $x_2 = -4x_3$, x_3 is free

$\vec{x} = x_3 \langle 3, -4, 1 \rangle$

ker(T) = Span $\{\langle 3, -4, 1 \rangle\}$.

$T(\langle x_1, x_2, x_3\rangle) = \langle 2x_1 + x_2 - 2x_3, 2x_1 + 3x_2 + 6x_3\rangle$.

1. Is T onto? Is $T(\vec{x}) = \vec{y}$ consistent for all $\vec{y} \in \mathbb{R}^2$?

   $[A \mid \vec{y}] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & -3 & z_1 \\ 0 & 1 & 4 & z_2 \end{array}\right]$ always consistent

   T is onto.

2. Is T one to one?

   $T(\vec{0}_3) = T(\langle 3, -4, 1\rangle) = \vec{0}_2$

   T is not one to one.

3. Is T invertible?

   No, T is not both onto and one to one.

Range & Kernel Dimensions

Onto & One to One Indicators

Theorem

Let $T : R^n \to R^m$ be a linear transformation. Then

1. T is onto if and only if $\dim(\text{range}(T)) = m$, and
2. T is one-to-one if and only if $\dim(\ker(T)) = 0$.

The second statement can be rephrased as saying that T is one-to-one if and only if

$$T(\vec{x}) = \vec{0}_m$$

has only the trivial solution, $\vec{x} = \vec{0}_n$.

Onto, One-to-One & Standard Matrix

Suppose $T : R^n \to R^m$ is a linear transformation with standard matrix A.

Since $\text{range}(T) = \mathcal{CS}(A)$, T is onto if and only if A has a pivot in every row.

Since $\ker(T) = \mathcal{N}(A)$, T is one-to-one if and only if all columns of A are pivot columns.

Note that since $\dim\big(\text{range}(T)\big) + \dim\big(\ker(T)\big) = n$, the only way for T to be both onto and one-to-one is for $m = n$. That is, $T : R^n \to R^n$ and A is a square matrix.

Invertible Linear Transformations

Inverse of a Linear Transformation

Let $T : R^n \to R^n$ be a linear transformation with standard matrix A. Then T is invertible if and only if A is an invertible matrix. In this case,

$$T^{-1}(\vec{x}) = A^{-1}\vec{x}$$

for each $\vec{x}$ in R^n.

The standard matrix for T^{-1} is the inverse of the standard matrix for T.

5.3 Visualizing Linear Transformations

We want to consider certain linear mappings from R^2 to R^2 that correspond to geometric transformations.

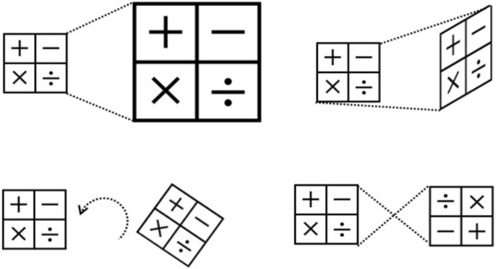


Figure: Scaling, shearing, rotations, reflections

Scaling Transformation

Let $r > 0$ and define $T : R^2 \longrightarrow R^2$ by $T(\vec{x}) = r\vec{x}$.
T is a **dilation** if $r > 1$ and a **contraction** if $0 < r < 1$.

Find the standard matrix of T.

we need $T(\vec{e}_1)$ and $T(\vec{e}_2)$

$T(\vec{e}_1) = r\vec{e}_1 = \langle r, 0\rangle$, $T(\vec{e}_2) = r\vec{e}_2 = \langle 0, r\rangle$

$$A = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} = r\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = rI_2$$

The Geometry of Dilation/Contraction

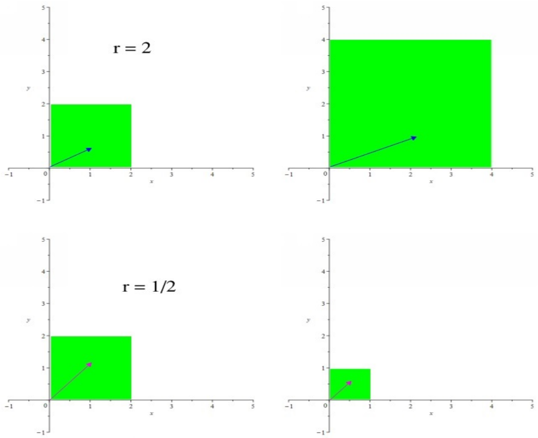


Figure: The 2×2 square in the plane under the dilation $\vec{x} \mapsto 2\vec{x}$ (top) and the contraction $\vec{x} \mapsto \frac{1}{2}\vec{x}$ (bottom). Each includes an example of a single vector and its image.

A Shear Transformation on R^2

Find the standard matrix for the linear transformation from $R^2 \to R^2$ that maps $\vec{e}_2$ to $\vec{e}_1 + \vec{e}_2$ and leaves $\vec{e}_1$ unchanged.

Calling the transformation S, we need $S(\vec{e}_1)$ and $S(\vec{e}_2)$.

$S(\vec{e}_1) = \vec{e}_1 = \langle 1, 0 \rangle$

$S(\vec{e}_2) = \vec{e}_1 + \vec{e}_2 = \langle 1, 0 \rangle + \langle 0, 1 \rangle = \langle 1, 1 \rangle$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

A Shear Transformation on R^2

For the shear transformation with standard matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ find the images of the vectors

$$\vec{x}_1 = \langle 0, -1 \rangle, \quad \vec{x}_2 = \langle 1, -1 \rangle, \quad \text{and} \quad \vec{x}_3 = \langle 1, 1 \rangle$$

$S(\vec{x}_1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \langle 0, -1 \rangle = \langle -1, -1 \rangle$

$S(\vec{x}_2) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \langle 1, -1 \rangle = \langle 0, -1 \rangle$

$S(\vec{x}_3) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \langle 1, 1 \rangle = \langle 2, 1 \rangle$

A Shear Transformation on R^2

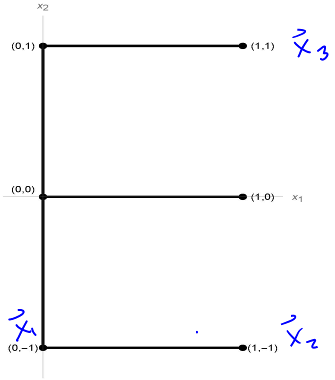


Figure: The letter "E" from line segments connecting select points from $\{(0,-1),(0,0),(0,1),(1,-1),(1,0),(1,1)\}$.

A Shear Transformation on R^2

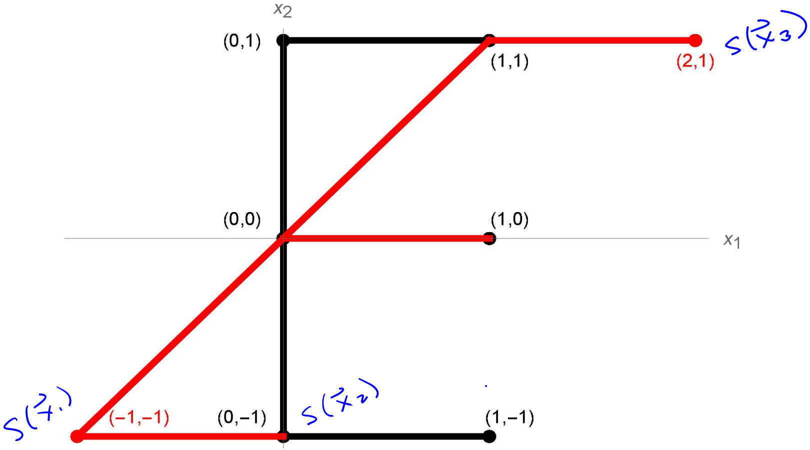


Figure: Letter "E" mapped under the shear transformation $\vec{x} \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{x}$.

Shearing Transformations

The matrix for a shearing transformation looks like one of

$$\underbrace{\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}}_{\text{horizontal shear}} \quad \text{or} \quad \underbrace{\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}}_{\text{vertical shear}}$$

(A shear matrix is what you get from I_2 by doing one row replacement: $kR_i + R_j \to R_j$.)

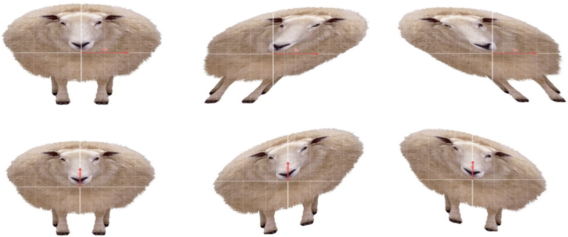
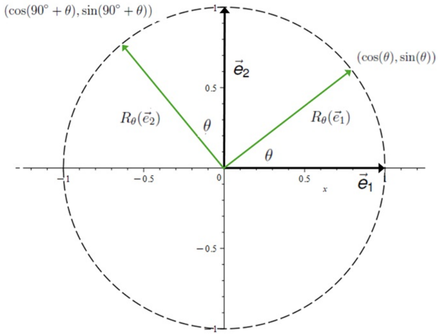


Figure: **Sheared Sheep** (Top) Horizontal Shear (right $k > 0$ and left $k < 0$), (Bottom) Vertical Shear (up $k > 0$ and down $k < 0$).

A Rotation on R^2

Let $R_\theta : R^2 \longrightarrow R^2$ be the rotation transformation that rotates each point in R^2 counter clockwise about the origin through an angle θ. Find the standard matrix for R_θ.



$R_\theta(\vec{e}_1) = \langle \cos\theta, \sin\theta \rangle$

$\cos(90^\circ + \theta) = \cos 90^\circ \cos\theta - \sin 90^\circ \sin\theta$
$= -\sin\theta$

$\sin(90^\circ + \theta) = \sin 90^\circ \cos\theta + \sin\theta \cos 90^\circ$
$= \cos\theta$

$R_\theta(\vec{e}_2) = \langle -\sin\theta, \cos\theta \rangle$

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

A Rotation in R^2

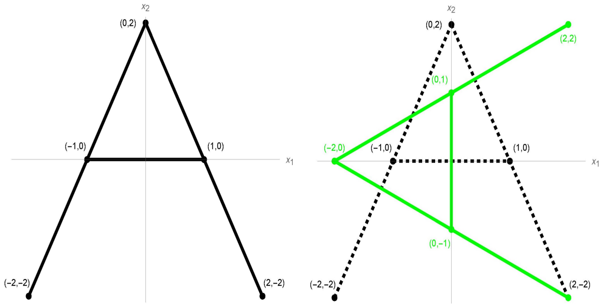
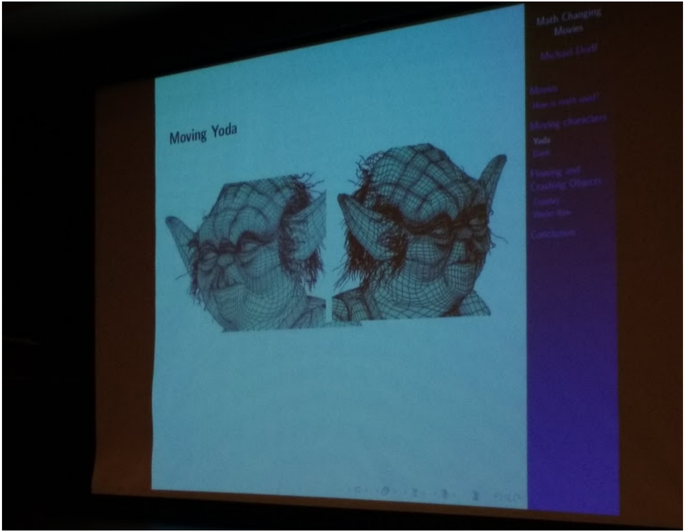


Figure: The letter “A” under a rotation transformation R_{90°.

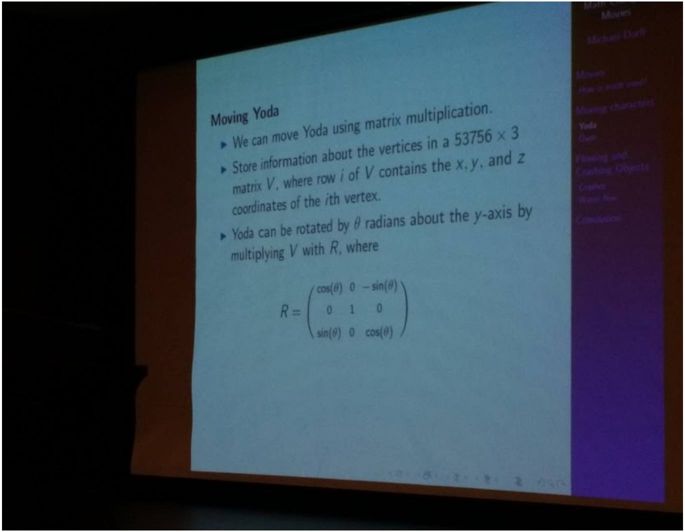
The standard matrix

$$A_{90^\circ} = \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Rotation in Animation



Rotation in Animation



Rotation in Curve Generation

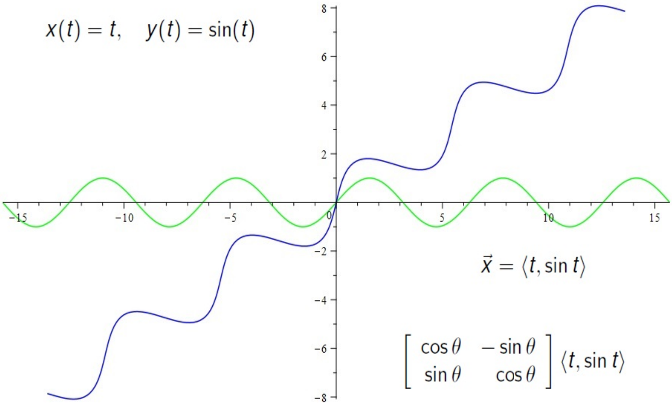


Figure: The curve $y = \sin(x)$ plotted as a vector valued function along with a version rotated through and angle $\theta = \frac{\pi}{6}$.

Example

Show that R_θ is invertible by showing that $R_{-\theta} = R_\theta^{-1}$.

we can show that $A_{-\theta} = A_\theta^{-1}$

$$A_{-\theta} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$A_{-\theta} A_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta(-\sin\theta) + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad A_{-\theta} = A_\theta^{-1}$$

Note $\quad A_{-\theta} = A_\theta^{-1} \Rightarrow A_\theta^{-1} = A_\theta^{T}$

Reflection Through Axis

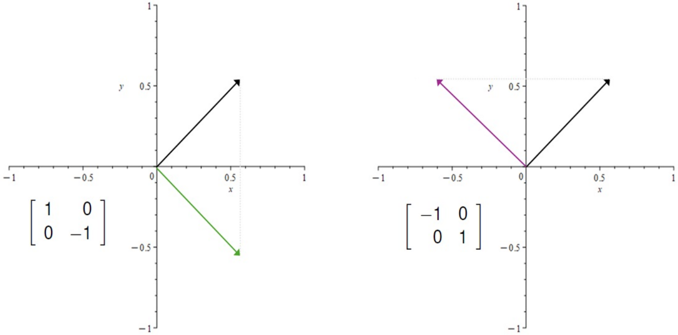


Figure: The matrix to reflect through the x_1-axis (left) or x_2-axis (right).

$$P_{x_1} : R^2 \to R^2 \quad P_{x_1}(\langle x_1, x_2 \rangle) = \langle x_1, -x_2 \rangle$$

$$P_{x_2} : R^2 \to R^2 \quad P_{x_2}(\langle x_1, x_2 \rangle) = \langle -x_1, x_2 \rangle$$

Summary of Geometric Transformations on R^2

- **Scaling**: $\vec{x} \mapsto rI_2\vec{x}$, is a dilation if $r > 0$ and contraction if $0 < r < 1$.
- **Shear**: $\vec{x} \mapsto A\vec{x}$ where $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ or $A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ for constant k.
- **Rotation** (counter clockwise about the originthrough angle θ): $\vec{x} \mapsto A_\theta\vec{x}$ where $A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
- **Reflection** (through an axis):
  $P_{x_1}(\langle x_1, x_2\rangle) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\langle x_1, x_2\rangle = \langle x_1, -x_2\rangle$, or
  $P_{x_2}(\langle x_1, x_2\rangle) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\langle x_1, x_2\rangle = \langle -x_1, x_2\rangle$.

We can combine these with the composition of transformations.

5.4 Linear Transformation of Lines

One way of distinguishing linear transformations from R^n to R^m is that a linear transformation

maps a line to a line or to a point.

We first need to determine how to characterize a line using vectors in R^n. Generally speaking, we will do this with a point (some point on the line) and a direction vector (a vector parallel to the line).

Let's consider the case in R^2. We have some point $P = (p_1, p_2)$ on our line, and the line is parallel to some vector $\vec{d} = \langle d_1, d_2 \rangle$ (with at least one of d_1 or d_2 nonzero).

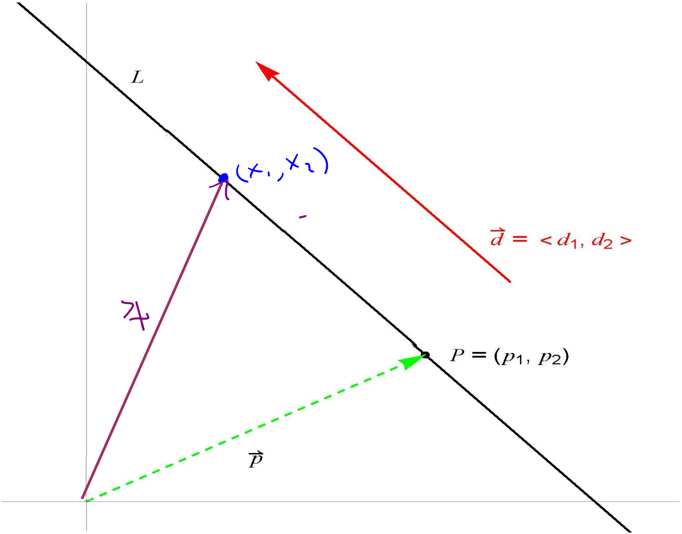


Figure: The point $P = (p_1, p_2)$ is on the line that is parallel to the vector $\vec{d} = \langle d_1, d_2 \rangle$. We want an equation for some arbitrary point $X = (x_1, x_2)$ on the line L.

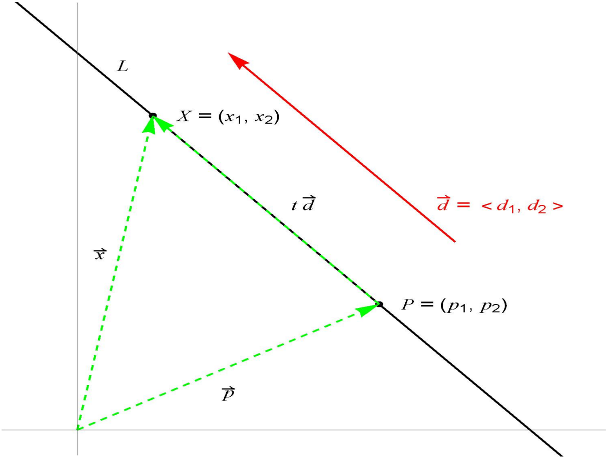


Figure: The vector $\vec{x} = \langle x_1, x_2 \rangle$ will be the vector $\vec{p} = \langle p_1, p_2 \rangle$ plus some scalar multiple of $\vec{d}$. We can get all of the points on L by letting the scalar t vary. That is L is described by the equation $\vec{x} = \vec{p} + t\vec{d}, \quad -\infty < t < \infty$.

A Line in R^2

If $P = (p_1, p_2)$ is any point on the line L that is parallel to the nonzero vector $\vec{d} = \langle d_1, d_2 \rangle$, then the line L is the collection of points corresponding to the vector

$$\vec{x} = \vec{p} + t\vec{d}, \quad -\infty < t < \infty. \tag{1}$$

Equation (1) is called a **vector parametric equation** or a **vector equation** of the line L. The set of component equations

$$\begin{array}{rcl} x_1 & = & p_1 + td_1 \\ x_2 & = & p_2 + td_2 \end{array}, \quad -\infty < t < \infty$$

are called **parametric equations** for the line L.

We can similarly consider a line in R^n containing the point $P = (p_1, p_2, \ldots, p_n)$ and parallel to the nonzero vector $\vec{d} = \langle d_1, d_2, \ldots, d_n \rangle$. The line is described by the vector equation

$$\vec{x} = \vec{p} + t\vec{d}, \quad -\infty < t < \infty.$$

Example

Let $T : R^2 \to R^2$ be defined by $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$. Let L_1 be the line through the point $P = (1, 1)$ and parallel to the vector $\vec{d} = \langle -1, 1 \rangle$, and let L_2 be the line through P and parallel to the vector $\vec{g} = \langle -3, 1 \rangle$.

1. Find the vector parametric equations for L_1 and L_2.
2. Find $T(L_1)$ and $T(L_2)$, the images of L_1 and L_2 under T.

For L_1 $\vec{x} = \vec{p} + t\vec{d}$ $\vec{x} = \langle 1, 1 \rangle + t\langle -1, 1 \rangle$ $t \in \mathbb{R}$

For L_2: $\vec{x} = \vec{p} + t\vec{g}$ $\vec{x} = \langle 1, 1 \rangle + t\langle -3, 1 \rangle$

$T(L_1) = \{ T(\vec{x}) \mid \vec{x} \in L_1 \}$

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

$$T(\vec{x}) = T(\langle 1,1\rangle + t\langle -1,1\rangle)$$

$$= T(\langle 1,1\rangle) + t\,T(\langle -1,1\rangle)$$

$$A\langle 1,1\rangle = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\langle 1,1\rangle = \langle 4,8\rangle$$

$$A\langle -1,1\rangle = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\langle -1,1\rangle = \langle 2,4\rangle$$

$$T(L_1) = \{\langle 4,8\rangle + t\langle 2,4\rangle \mid t \in \mathbb{R}\}$$

For L_2 $\quad \vec{x} = \langle 1,1\rangle + t\langle -3,1\rangle$

$$T(\vec{x}) = T(\langle 1,1\rangle) + t\,T(\langle -3,1\rangle)$$

$$T(\langle -3,1\rangle) = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\langle -3,1\rangle = \langle 0,0\rangle$$

For $\vec{x}$ on L_2, $T(\vec{x}) = \langle 4,8\rangle + t\langle 0,0\rangle$

$= \langle 4,8\rangle$

$T(L_2) = \{\langle 4,8\rangle\}$.

T maps L_1 to a line, but it maps L_2 to a single point. The direction vector for L_2 has image <0,0> (i.e., it is in the kernel of T).

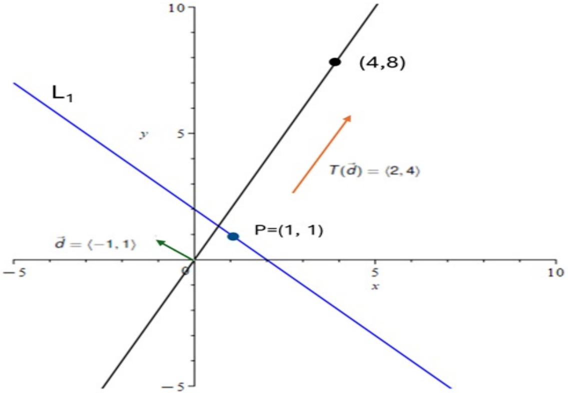


Figure: L_1 and its image $T(L_1)$ under T. The image of the line is a line.

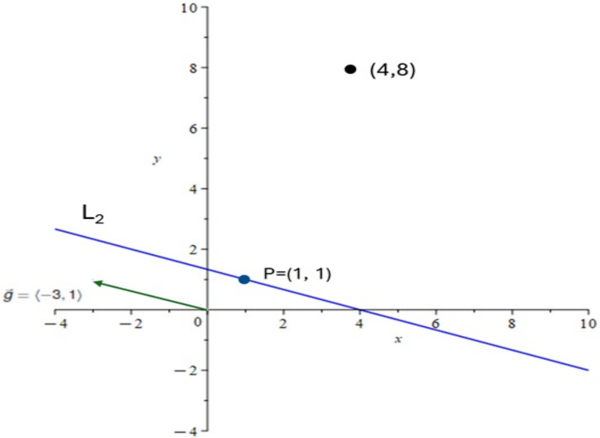


Figure: L_2 and its image $T(L_2)$ under T. The image of this line is a point. Note that we got a single point because the direction vector $\vec{g}$ for L_2 is in the kernel of T.

Theorem

Suppose that $T : R^n \to R^m$ is a linear transformation and suppose that L is a line in R^n. Specifically, suppose that

$$L : \vec{x} = \vec{p} + t\vec{d} \tag{2}$$

where $\vec{d} \neq \vec{0}_n$.
Then $T(L)$ is either a point or a line in R^m. Specifically,

1. If $\vec{d} \notin \ker(T)$, then $T(L)$ is a line in R^m.
2. If $\vec{d} \in \ker(T)$, then $T(L)$ is a point in R^m.

A consequence is that if L_1 and L_2 are parallel lines (i.e., have the same direction vector $\vec{d}$), then either $T(L_1)$ and $T(L_2)$ is a pair of parallel lines ($\vec{d} \notin \ker(T)$) or is a pair of points.