June 14 Math 2306 sec. 53 Summer 2022

Section 4: First Order Equations: Linear

A first order, linear ODE in **standard form** is an equation

$$\frac{dy}{dx} + P(x)y = f(x)$$

and look for solutions on an interval where P and f are continuous. We know that solutions will look like[1]

$$y = y_c + y_p$$

where y_c is called the complementary solution and y_p is called a particular solution.

[1] If $f(x) = 0$, the equation is called homogeneous. For such equations, the y_p is also zero.

General Solution of First Order Linear ODE

We have a process for solving a first order linear ODE.

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and isolate y.

Recall

Last time, we solved the IVP $\dfrac{dy}{dx} - \dfrac{1}{x}y = 2x$ with $y(1) = 5$. We found the integrating factor

$$\mu(x) = e^{\int -\frac{1}{x}\,dx} = x^{-1}.$$

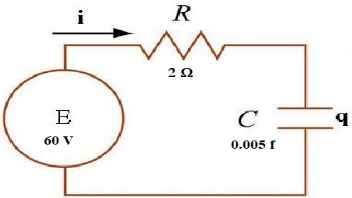
Verify that

$$\frac{d}{dx}(\mu y) = \mu\left(\frac{dy}{dx} - \frac{1}{x}y\right)$$

$$\frac{d}{dx}(x^{-1}y) = x^{-1}\frac{dy}{dx} + (-x^{-2})y = x^{-1}\frac{dy}{dx} - x^{-2}y$$

$$x^{-1}\left(\frac{dy}{dx} - \frac{1}{x}y\right) = x^{-1}\frac{dy}{dx} - x^{-1}\left(\frac{1}{x}\right)y = x^{-1}\frac{dy}{dx} - x^{-2}y$$

Steady and Transient States



$i = \frac{dq}{dt}$

Figure: The charge $q(t)$ on the capacitor in the given curcuit satisfies a first order linear equation.

$$2\frac{dq}{dt} + 200q = 60, \quad q(0) = 0.$$

Standard form $\quad \frac{dq}{dt} + 100q = 30$

$P(t) = 100 \quad \Rightarrow \quad \mu = e^{\int P(t)dt} = e^{\int 100 dt} = e^{100t}$

$$e^{100t}\left(\frac{dq}{dt}+100q\right)=e^{100t}(30)$$

$$\frac{d}{dt}\left(e^{100t}q\right)=30e^{100t}$$

$$\int\frac{d}{dt}\left(e^{100t}q\right)dt=\int 30e^{100t}\,dt$$

$$e^{100t}q=\frac{30}{100}e^{100t}+k$$

$$q=\frac{\frac{3}{10}e^{100t}+k}{e^{100t}}$$

$$* \int e^{at}\,dt=\frac{1}{a}e^{at}+C$$

$$q=\frac{3}{10}+ke^{-100t}$$

general solution

Apply $q(0)=0$

$$q(0) = 0 = \frac{3}{10} + k e^{-100(0)}$$

$$\frac{3}{10} + k = 0 \implies k = \frac{-3}{10}$$

$$\boxed{\text{The solution to the IVP is} \quad q(t) = \frac{3}{10} - \frac{3}{10} e^{-100t}}$$

Steady and Transient States

Note that the solution, the charge, consists of a complementary and a particular solution, $q = q_p + q_c$.

$$q(t) = \frac{3}{10} - \frac{3}{10}e^{-100t}$$

$$q_c(t) = -\frac{3}{10}e^{-100t} \quad \text{and} \quad q_p(t) = \frac{3}{10}$$

Evaluate the limit

$$\lim_{t\to\infty} q_c(t) = \lim_{t\to\infty} \frac{-3}{10} e^{-100t} = 0$$

Steady and Transient States

The complementary solution contains the information given by the initial condition, and for some physical systems like this the complementary solution decays.

Definition: Such a decaying complementary solution is called a **transient state**.

Note that due to this decay, after a while $q(t) \approx q_p(t)$.

Definition: Such a corresponding particular solution is called a **steady state**.

Bernoulli Equations

Suppose $P(x)$ and $f(x)$ are continuous on some interval (a, b) and n is a real number different from 0 or 1 (not necessarily an integer). An equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

is called a **Bernoulli** equation.

Observation: This equation has the flavor of a linear ODE, but since $n \neq 0, 1$ it is necessarily nonlinear. So our previous approach involving an integrating factor does not apply directly. Fortunately, we can use a change of variables to obtain a related linear equation.

Solving the Bernoulli Equation

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

Let the new dependent variable

$$u = y^{1-n}$$

Find the ODE for u. Find $\frac{du}{dx}$

$$\frac{du}{dx} = (1-n)\, y^{1-n-1} \frac{dy}{dx} = (1-n)\, y^{-n} \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y^n}{1-n} \frac{du}{dx}$$

Plug this into the ODE

$$\frac{y^n}{1-n}\frac{du}{dx} + P(x)\,y = f(x)\,y^n$$

Divide by $\frac{y^n}{1-n}$ (i.e. multiply by $(1-n)\,y^{-n}$)

$$\frac{du}{dx} + (1-n)P(x)\underbrace{y^{-n}y}_{y^{1-n}\text{, that's } u} = (1-n)f(x)\underbrace{y^{-n}y^n}_{1}$$

$$\frac{du}{dx} + (1-n)\,P(x)\,u = (1-n)\,f(x)$$

This is a 1st order linear eqn.

Form is $\frac{du}{dx} + P_1(x)u = f_1(x)$

where $P_1(x) = (1-n)P(x)$ and

$f_1(x) = (1-n)f(x)$

Since $u = y^{1-n}$, $y = u^{\frac{1}{1-n}}$

Example

Solve the initial value problem $y' - y = -e^{2x}y^3$, subject to $y(0) = 1$.

This is a Bernoulli equation w/

$n = 3$.

$$u = y^{1-n} = y^{1-3} = y^{-2}$$

$$\frac{du}{dx} = -2y^{-3}\frac{dy}{dx} \implies \frac{dy}{dx} = \frac{-y^3}{2}\frac{du}{dx}.$$

$$-\frac{y^3}{2}\frac{du}{dx} - y = -e^{2x}y^3$$ Divide by $\frac{-y^3}{2}$

$$\frac{du}{dx} - \left(\frac{-2}{y^3}\right)y = -e^{2x}y^3\left(\frac{-2}{y^3}\right)$$ mult by $\frac{-2}{y^3}$

note $\frac{y}{y^3} = y^{-2}$

The ODE is

$$\frac{du}{dx} + 2u = 2e^{2x}$$

$$\frac{dy}{dx} - y = -e^{2x}y^3 \qquad P(x) = -1, \quad f(x) = -e^{2x}$$

$1-n = 1-3 = -2$

$$(1-n)P(x) = -2(-1) = 2$$

$$(1-n)f(x) = -2(-e^{2x}) = 2e^{2x}$$

Our ODE for u is

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)f(x)$$

$$\frac{du}{dx} + 2u = 2e^{2x}$$

$$P_1(x) = 2 \implies \mu = e^{\int P_1(x)\,dx} = e^{\int 2\,dx} = e^{2x}$$

$$\frac{d}{dx}\left(e^{2x}u\right) = 2e^{2x}e^{2x} = 2e^{4x}$$

Integrate

$$e^{2x}u = \int 2e^{4x}\,dx$$

$$= \frac{2}{4}e^{4x} + C$$

$$\implies u = \frac{\frac{1}{2}e^{4x} + C}{e^{2x}} = \frac{1}{2}e^{2x} + Ce^{-2x}$$

$$u = y^{-2} \implies y = u^{-1/2} = \frac{1}{\sqrt{u}}$$

The solutions to the ODE are

$$y = \frac{1}{\sqrt{\frac{1}{2}e^{2x} + Ce^{-2x}}}.$$ Apply $y(0)=1$

$$y(0) = 1 = \frac{1}{\sqrt{\frac{1}{2}e^{0} + Ce^{0}}} = \frac{1}{\sqrt{\frac{1}{2} + C}}$$

$$\Longrightarrow \quad \sqrt{\frac{1}{2} + C} = 1 \quad \Rightarrow \quad \frac{1}{2} + C = 1^2 = 1 \quad \Rightarrow \quad C = \frac{1}{2}$$

$$\boxed{\begin{array}{l}\text{The solution to the IVP is} \\[1em] \qquad y = \dfrac{1}{\sqrt{\frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x}}}\end{array}}$$

Section 5: First Order Equations Models and Applications

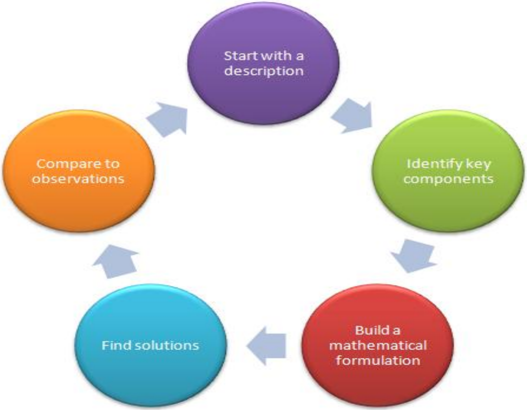


Figure: Mathematical Models give Rise to Differential Equations

Population Dynamics

A population of dwarf rabbits grows at a rate proportional to the current population. In 2011, there were 58 rabbits. In 2012, the population was up to 89 rabbits. Estimate the number of rabbits expected in the population in 2021.

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A population of dwarf rabbits grows at a rate proportional to the current population. In 2011, there were 58 rabbits. In 2012, the population was up to 89 rabbits. Estimate the number of rabbits expected in the population in 2021.

We need variables. The population is changing in time, so let's introduce variables

$$t \sim \text{time} \quad \text{and} \quad P(t) \sim \text{is the population (density) at time } t.$$

We need to express the following mathematically:

The population's rate of change is proportional to the population.

$\frac{dP}{dt}$

$$\frac{dP}{dt} = kP \quad \text{for some constant } k.$$

1st order ODE for P.

Let's take t in years w/ $t=0$ in 2011.

With this choice, we know that

$P(0) = 58$ and $P(1) = 89$

Choosing the first as an initial condition, we have an IVP

$$\frac{dP}{dt} = kP, \quad P(0) = 58$$

Solve by separation of variables

$$\frac{1}{P}\,dP = k\,dt$$

$$\int \frac{1}{P}\,dP = \int k\,dt \Rightarrow \ln|P| = kt + C$$

Assume $P>0$, $\quad P = e^{kt+C} = e^{C} e^{kt}$

Letting $A = e^c$, $P(t) = Ae^{kt}$. Using $P(0) = 58$

$$P(0) = 58 = Ae^0 \implies A = 58$$

The population $P(t) = 58e^{kt}$; we can find k using $P(1) = 89$.

$$P(1) = 89 = 58e^k \implies e^k = \frac{89}{58}$$

$$\implies k = \ln\left(\frac{89}{58}\right)$$

Hence

$$P(t) = 58\, e^{t\ln\left(\frac{89}{58}\right)}.$$

Since $t=0$ in 2011, 2021 is $t=10$.

This model predicts

$$P(10) = 58\, e^{10 \ln\left(\frac{89}{58}\right)} \approx 4200$$

in 2021.

Exponential Growth or Decay

If a quantity P changes continuously at a rate proportional to its current level, then it will be governed by a differential equation of the form

$$\frac{dP}{dt} = kP \quad \text{i.e.} \quad \frac{dP}{dt} - kP = 0.$$

Note that this equation is both separable and first order linear. If $k > 0$, P experiences **exponential growth**. If $k < 0$, then P experiences **exponential decay**.

Decay is usually expressed

$$\frac{dP}{dt} = -kP \quad \text{w/} \quad k > 0$$