June 16 Math 2306 sec. 53 Summer 2022

Section 5: First Order Equations Models and Applications

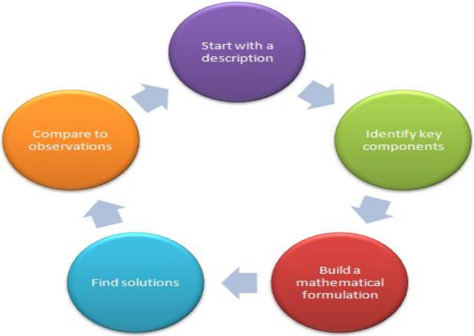


Figure: Mathematical Models give Rise to Differential Equations

Exponential Growth or Decay

If a quantity P changes continuously at a rate proportional to its current level, then it will be governed by a differential equation of the form

$$\frac{dP}{dt} = kP \quad \text{i.e.} \quad \frac{dP}{dt} - kP = 0.$$

Note that this equation is both separable and first order linear. If $k > 0$, P experiences **exponential growth**. If $k < 0$, then P experiences **exponential decay**.

$$P(t) = P(0)\, e^{kt}$$

Series Circuits: RC-circuit

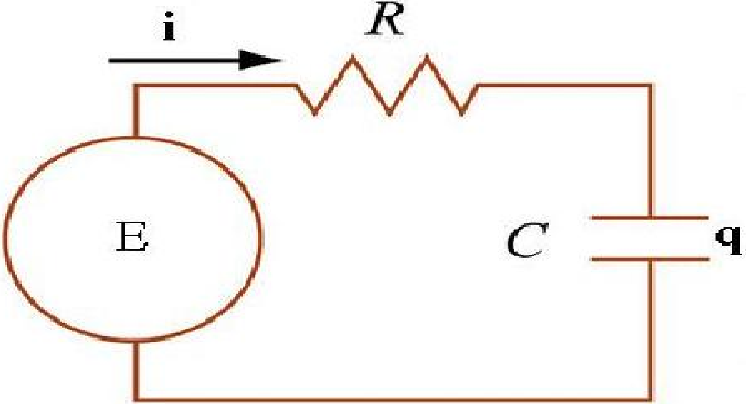


Figure: Series Circuit with Applied Electromotive force E, Resistance R, and Capcitance C. The charge of the capacitor is q and the current $i = \frac{dq}{dt}$.

Series Circuits: LR-circuit

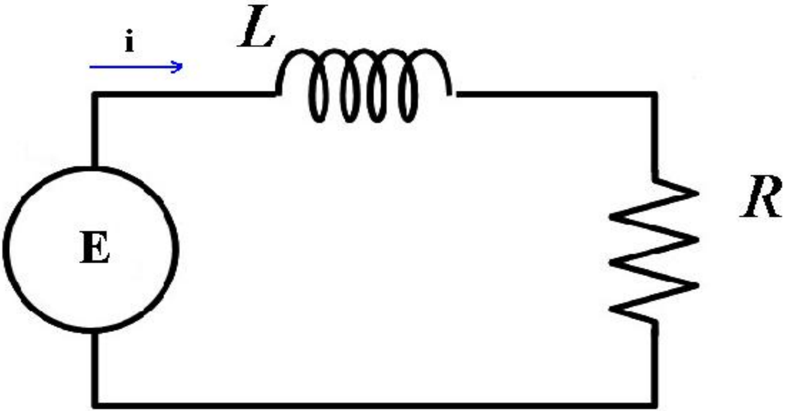


Figure: Series Circuit with Applied Electromotive force E, Inductance L, and Resistance R. The current is i.

Measurable Quantities:

Resistance R in ohms (Ω),
Inductance L in henries (h),
Capacitance C in farads (f),

Implied voltage E in volts (V),
Charge q in coulombs (C),
Current i in amperes (A)

Current is the rate of change of charge with respect to time: $i = \dfrac{dq}{dt}$.

Component	Potential Drop
Inductor	$L\frac{di}{dt}$
Resistor	Ri i.e. $R\frac{dq}{dt}$
Capacitor	$\frac{1}{C}q$

Kirchhoff's Law

The sum of the voltages around a closed circuit is zero.

In other words, the sum of potential drops across the passive components is equal to the applied electromotive force.

For RC circuit:

PD[1] across resistor $+$ PD across capacitor $=$ Applied Voltage

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

1st order linear eqn. for q.

[1] PD = potential drop

LR Circuit equaion

For LR circuit:

PD across inductor $+$ PD across resistor $=$ Applied Voltage

$$L\frac{di}{dt} + Ri = E(t)$$

1^{st} order linear ODE for i

First Order Circuit Equations

To find the charge on the capacitor in an RC series circuit, solve the IVP

$$\boxed{R\frac{dq}{dt} + \frac{1}{C}q = E(t), \quad \text{subject to} \quad q(0) = q_0}$$

To find the current in an LR series circuit, solve the IVP

$$\boxed{L\frac{di}{dt} + Ri = E(t), \quad \text{subject to} \quad i(0) = i_0}$$

Example

A 200 volt battery is applied to an RC series circuit with resistance 1000Ω and capacitance 5×10^{-6} f. Find the charge $q(t)$ on the capacitor if $i(0) = 0.4A$. Determine the charge as $t \to \infty$.

The ODE is $R\frac{dq}{dt} + \frac{1}{C}q = E$

Here, $R = 1000$, $C = 5\cdot 10^{-6}$, $E(t) = 200$

$i(0) = q'(0) = 0.4$

$$1000\frac{dq}{dt} + \frac{1}{5\cdot 10^{-6}}q = 200$$

Note $\frac{1}{5\cdot 10^{-6}} \div 1000 = \frac{10^6}{5(10^3)} = \frac{10^3}{5} = 200$.

In standard form

$$\frac{dq}{dt} + 200q = \frac{1}{5}, \quad q'(0) = \frac{2}{5}$$

$P(t) = 200$

$$\mu = e^{200t}$$

$$\frac{d}{dt}\left(e^{200t} q\right) = \frac{1}{5} e^{200t}$$

$$e^{200t} q = \int \frac{1}{5} e^{200t}\, dt$$

$$= \frac{1}{5(200)} e^{200t} + k$$

$$q(t) = \frac{\frac{1}{1000} e^{200t} + k}{e^{200t}} = \frac{1}{1000} + k e^{-200t}$$

Apply $q'(0) = \frac{2}{5}$.

$$q'(t) = 0 + k\left(-200e^{-200t}\right)$$

$$q'(0) = -200ke^{0} = \frac{2}{5}$$

$$\Rightarrow k = \frac{2}{5(-200)} = \frac{-1}{500}$$

The charge on the capacitor

$$\boxed{q(t) = \frac{1}{1000} - \frac{1}{500}e^{-200t}}$$

Looking at the long term charge

$$\lim_{t\to\infty} q(t) = \lim_{t\to\infty} \left(\frac{1}{1000} - \frac{1}{500} e^{-200t} \right)$$

$$= \frac{1}{1000}$$

The steady state charge is 0.001 C

A Classic Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

A Classic Mixing Problem

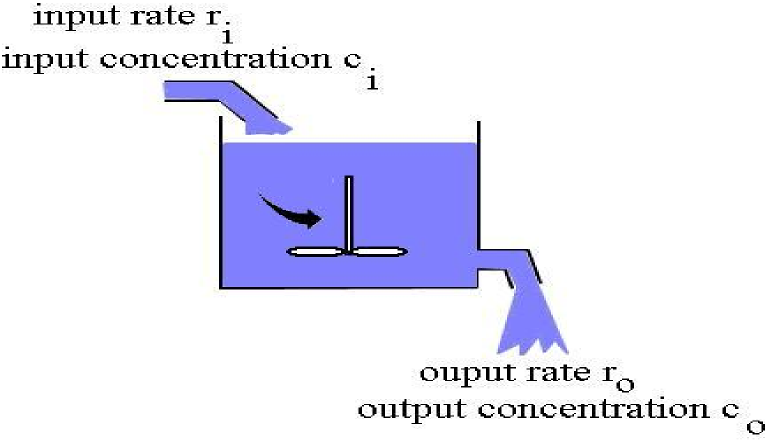


Figure: Spatially uniform composite fluids (e.g. salt & water, gas & ethanol) being mixed. Concentrations of substance change in time.

Building an Equation

The rate of change of the amount of salt

$$\frac{dA}{dt} = \left(\begin{array}{c} \textit{input rate} \\ \textit{of salt} \end{array}\right) - \left(\begin{array}{c} \textit{output rate} \\ \textit{of salt} \end{array}\right)$$

The input rate of salt is

$$\text{fluid rate in} \cdot \text{concentration of inflow} \quad = \quad r_i(c_i).$$

The output rate of salt is

$$\text{fluid rate out} \cdot \text{concentration of outflow} \quad = \quad r_o(c_o).$$

Building an Equation

The concentration of the outflowing fluid is

$$c_o = \frac{\text{total salt}}{\text{total volume}} = \frac{A(t)}{V(t)} = \frac{A(t)}{V(0) + (r_i - r_o)t}.$$

$$\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V}.$$

This equation is first order linear.

In standard form $\quad \frac{dA}{dt} + \frac{r_o}{V} A = r_i c_i$

If $A(0)$ is the initial salt, we have an IVP.

Solve the Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

$A \sim \text{lb}$

$t \sim \text{min}$

$\frac{dA}{dt} \sim \frac{\text{lb}}{\text{min}}$

$$\frac{dA}{dt} + \frac{r_o}{V} A = r_i c_i$$

$$r_i = 5 \frac{\text{gal}}{\text{min}}, \quad c_i = 2 \frac{\text{lb}}{\text{gal}}, \quad r_o = 5 \frac{\text{gal}}{\text{min}}$$

$$c_o = \frac{A}{V} = \frac{A}{V(0) + (r_i - r_o)t} = \frac{A}{500 + (5-5)t} = \frac{A}{500} \frac{\text{lb}}{\text{gal}}$$

$$\frac{dA}{dt} + \frac{5}{500} A = 10, \quad A(0) = 0$$

1st order linear ODE

$$P(t) = \frac{1}{100} \Rightarrow \mu = e^{\int \frac{1}{100} dt} = e^{\frac{1}{100}t}$$

$$\frac{d}{dt}\left(e^{\frac{1}{100}t} A\right) = 10\, e^{\frac{1}{100}t}$$

$$e^{\frac{1}{100}t} A = \int 10\, e^{\frac{1}{100}t} dt = 10(100)\, e^{\frac{1}{100}t} + k$$

$$A = \frac{1000\, e^{\frac{1}{100}t} + k}{e^{\frac{1}{100}t}} = 1000 + k e^{-\frac{1}{100}t}$$

Using $A(0) = 0$

$$A(0) = 1000 + k e^{0} = 0 \Rightarrow k = -1000$$

$$\boxed{\text{The amount of salt is} \quad A = 1000 - 1000\,e^{-\frac{1}{100}t} \ \text{lb}}$$

Calling the concentration of salt in the tank $C(t)$.

$$C(t) = \frac{A(t)}{V} = \frac{1000 - 1000e^{-\frac{1}{100}t}}{500}$$

$$C(5) = 2 - 2e^{-\frac{1}{100}(5)} \approx 0.0975 \ \frac{\text{lb}}{\text{gal}}$$

Note As $t \to \infty$ the concentration

$$C(t) \to 2 \ \frac{\text{lb}}{\text{gal}}$$

$r_i \neq r_o$

Suppose that instead, the mixture is pumped out at 10 gal/min. Determine the differential equation satisfied by $A(t)$ under this new condition.

$r_i = 5, \quad c_i = 2 \qquad r_o = 10$

The volume $V(t) = V(0) + (r_i - r_o)t = 500 - 5t$

$$c_o = \frac{A}{500-5t}$$

$$\frac{dA}{dt} + \frac{r_o}{V} A = r_i c_i$$

$$\Rightarrow \qquad \frac{dA}{dt} + \frac{10}{500-5t} A = 10$$

A Nonlinear Modeling Problem

A population $P(t)$ of tilapia changes at a rate jointly proportional to the current population and the difference between the constant carrying capacity[2] M of the environment and the current population. Determine the differential equation satsified by P.

$$\frac{dP}{dt} \propto P \; (\underbrace{\text{difference between } P \text{ and } M}_{M-P})$$

$$\frac{dP}{dt} = kP(M-P)$$

for some constant k.

[2]The carrying capacity is the maximum number of individuals that the environment can support due to limitation of space and resources.

Logistic Differential Equation

The equation

$$\frac{dP}{dt} = kP(M-P), \quad k, M > 0$$

is called a **logistic growth equation.**
Solve this equation and show that for any $P(0) \neq 0$, $P \to M$ as $t \to \infty$.

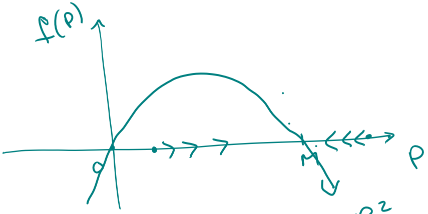
The ODE is separable : $\frac{dP}{dt} = \underset{g(t)}{k} \underbrace{(P(M-P))}_{h(P)}$

The ODE is a Bernoulli equation

$$\frac{dP}{dt} - kMP = -kP^2 \qquad n=2$$

$$\frac{dP}{dt} = kP(M-P) = f(P)$$

plot $(P, f(P))$



$f(P) = kMP - kP^2$

Let's solve $\frac{dP}{dt} - kMP = -kP^2$, $P(0) = P_0$

Let $u = P^{1-2} = P^{-1}$

If $\frac{dP}{dt} + Q(t)P = f(t)$ then $\frac{du}{dt} + (1-n)Q(t)u = (1-n)f(t)$

Here $Q(t) = -kM$, $(1-n)Q(t) = -1(-kM) = kM$

$f(t) = -k$, $(1-n)f(t) = -1(-k) = k$

u solves

$$\frac{du}{dt} + kMu = k$$

$Q(t) = kM$, $\mu = e^{\int kM\,dt} = e^{kMt}$

$$\frac{d}{dt}\left[e^{kMt}u\right] = k e^{kMt}$$

$$e^{kMt}u = \int k e^{kMt}\,dt$$

$$= \frac{k}{kM}e^{kMt} + C$$

$$u = \frac{1}{M} + C e^{-kMt}$$

$$u = P^{-1} = \frac{1}{P} \Rightarrow P = \frac{1}{u}$$

$$P = \frac{1}{\frac{1}{M} + C e^{-kMt}} \cdot \frac{M}{M}$$

$$P(t) = \frac{M}{1 + CM e^{-kMt}}$$

Apply $P(0) = P_0$

$$P(0) = P_0 = \frac{M}{1 + CM e^{0}}$$

$$(1 + CM)P_0 = M$$

$$P_0 + CMP_0 = M \implies CMP_0 = M - P_0$$

$$\implies CM = \frac{M - P_0}{P_0}$$

$$P(t) = \frac{M}{1 + \left(\frac{M-P_0}{P_0}\right) e^{-kMt}} \cdot \frac{P_0}{P_0}$$

$$\boxed{P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}}}$$

soln to the logistic eqn.

For $P_0 \neq 0$,

$$\lim_{t \to \infty} P(t) = \lim_{t \to \infty} \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}}$$

$$= \frac{MP_0}{P_0 + 0} = \frac{MP_0}{P_0} = M.$$