June 1 Math 3260 sec. 51 Summer 2023

Recall

We defined a **linear system** of (algebraic) equations in n variables $x_1, \ldots, x_n$ to be one of the form

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & \vdots & & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m. \end{array}$$

We said that a **solution** is an n-tuple $(s_1, \ldots, s_n)$ that reduces all equations to an identity upon substitution, and a **solution set** is the set of all solutions. Two systems are **equivalent** if they have the same solution set.

Three Flavors of Solution Set

Theorem

A linear system of equations has exactly one of the following:

i No solution, or

ii Exactly one solution, or

iii Infinitely many solutions.

We said that a system is

- **inconsistent** if it has no solutions (case i) and
- **consistent** if it has at least one solution (cases ii and iii).

Linear Systems & Matrices

Given a linear system, we can identify two related matrices,

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & \vdots & & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m. \end{array}$$

The **coefficient matrix** and the **augmented matrix**.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Operations on Systems and Matrices

We had three operations that we can perform on a system of equations (swap, scale, and replace) that maintain the solution set.

Elementary row operations are the analogous operations that we can perform on the rows of a matrix.

Elementary Row Operations

i Interchange row i and row j (**swap**), $R_i \leftrightarrow R_j$.

ii Multiply row i by any nonzero constant k (**scale**), $kR_i \to R_i$.

iii Replace row j with the sum of itself and k times row i (**replace**), $kR_i + R_j \to R_j$.

Row Equivalent Matrices

Definition

Two matrices are called **row equivalent** if one can be obtained from the other by performing a sequence of elementary row operations.

Theorem

If the augmented matrices of two linear systems of equations are row equivalent, then the linear systems of equations are equivalent.

Section 1.2: Row Reduction and Echelon Forms

Definition

A matrix is in **echelon form**, also called *row echelon form (ref)*, if it has the following properties:

i Any row of all zeros are at the bottom.

ii The first nonzero number (called the *leading entry*) in a row is to the right of the first nonzero number in all rows above it.

iii All entries below a leading entry are zeros.[a]

[a] This condition is superfluous but is included for clarity.

an ref

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 7 \end{bmatrix}$$

not an ref

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Reduced Echelon Form

Definition

A matrix is in **reduced echelon form**, also called *reduced row echelon form (rref)* if it is in echelon form and has the additional properties

iv The leading entry of each row is 1 (called a *leading* 1), and
v each leading 1 is the only nonzero entry in its column.

an rref

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

not an rref

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Example (finding ref's and rref's)

Find an echelon form for the following matrix using elementary row operations.

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 6 \\ 0 & 3 & 2 \end{bmatrix}$$

Goal $\begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix}$ or $\begin{bmatrix} x & x & x \\ 0 & 0 & x \\ 0 & 0 & 0 \end{bmatrix}$

$-2R_1 + R_2 \to R_2$

$*$ $\quad -4 \quad -2 \quad -6$

$\quad\;\; 4 \quad\;\; 3 \quad\;\; 6$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 3 & 2 \end{bmatrix}$$

Choices: $-3R_2 + R_3 \to R_3$

~~$-3R_1 + R_3 \to R_3$~~

$-3R_2 + R_3 \to R_3$

we don't do the crossed out option because it would mess up the zero in column 1.

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

This is an ref

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

These are also row equivalent ref's. An ref isn't unique even though an rref is.

Example (rref)

Find the reduced echelon form for the following matrix.

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 6 \\ 0 & 3 & 2 \end{bmatrix}$$

We'll start with the ref already obtained.

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

$\frac{1}{2}R_3 \to R_3$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -3R_3 + R_1 \to R_1 \quad \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$-R_2 + R_1 \to R_1$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\frac{1}{2} R_1 \to R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is an rref.

Pivot Positions & Pivot Columns

Theorem

The reduced row echelon form of a matrix is unique.

That is, a given matrix is row equivalent to many different refs but to only ONE rref! This allows for the following unambiguous definitions.

Pivot Position & Pivot Column

Definition: A **pivot position** in a matrix A is a location that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position.

Identifying Pivot Positions and Columns

The following matrices are **row equivalent**. Identify the pivot positions and pivot columns of the matrix A.

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

B is an rref.

The pivot positions are row 1, column 1; row 2, column 2; and row 3, column 4.

Pivot columns are 1, 2, and 4.

Complete Row Reduction isn't needed to find Pivots

The following three matrices are row equivalent. (Note, B is an ref but not an rref, and C is an rref.)

$$A = \begin{bmatrix} 1 & 1 & 4 \\ -2 & 1 & -2 \\ 1 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Identify the pivot positions and pivot columns of the matrix A.

From B or C, we see that the pivot positions are row 1 column 1 and row 2 column 2. Columns 1 and 2 are pivot columns.

Row Reduction Algorithm

To obtain an echelon form, we work from left to right beginning with the top row working downward.

$$\begin{bmatrix} 0 & 3 & -6 & 4 & 6 \\ 3 & -7 & 8 & 8 & -5 \\ 3 & -9 & 12 & 6 & -9 \end{bmatrix}$$

Get nonzero in row 1 column 1 using

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 3 & -7 & 8 & 8 & -5 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

Step 1: The left most column is a pivot column. The top position is a pivot position.

Step 2: Get a nonzero entry in the top left position by row swapping if needed.

Row Reduction Algorithm

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 3 & -7 & 8 & 8 & -5 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

$-R_1 + R_2 \to R_2$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 2 & -4 & 2 & 4 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

Step 3: Use row operations to get zeros in all entries below the pivot.

Row Reduction Algorithm

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 2 & -4 & 2 & 4 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

$\frac{1}{2}R_2 \to R_2$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 3 & -6 & 4 & 6 \end{bmatrix}$$

$-3R_2 + R_3 \to R_3$

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$0 \quad -3 \quad 6 \quad -3 \quad -6$

$0 \quad 3 \quad -6 \quad 4 \quad 6$

Step 4: Ignore the row with a pivot, all rows above it, the pivot column, and all columns to its left, and repeat steps 1-3.

Row Reduction Algorithm

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Row Reduction Algorithm

To obtain a reduced row echelon form:

Step 5: Starting with the right most pivot and working up and to the left, use row operations to get a zero in each position above a pivot. Scale to make each pivot a 1.

$$\begin{bmatrix} 3 & -9 & 12 & 6 & -9 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$-R_3 + R_2 \to R_2$ and

$-6R_3 + R_1 \to R_1$

$$\begin{bmatrix} 3 & -9 & 12 & 0 & -9 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$\frac{1}{3} R_1 \to R_1$

Row Reduction Algorithm

$$\begin{bmatrix} 1 & -3 & 4 & 0 & -3 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$3R_2 + R_1 \to R_1$

$$\begin{matrix} 0 & 3 & -6 & 0 & 6 \\ 1 & -3 & 4 & 0 & -3 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 3 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Echelon Form & Solving a System

Recall: Row equivalent matrices correspond to equivalent systems.

Suppose the matrix on the left is the augmented matrix for a linear system of equations in the variables x_1, x_2, x_3 and x_4. Use the rref to characterize the solution set to the linear system.

$$\begin{bmatrix} 0 & 3 & -6 & 4 & 6 \\ 3 & -7 & 8 & 8 & -5 \\ 3 & -9 & 12 & 6 & -9 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & -2 & 0 & 3 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$x_1 \quad -2x_3 \quad = 3$

$x_2 - 2x_3 \quad = 2$

$x_4 = 0$

$x_1 = 3 + 2x_3$

$x_2 = 2 + 2x_3$

$x_4 = 0$

x_3 is any real #

Basic & Free Variables

Definition

Suppose a system has m equations and n variables, $x_1, x_2, \ldots, x_n$. The first n columns of the augmented matrix correspond to the n variables.

- If the i^{th} column is a pivot column, then x_i is called a **basic variable**.
- If the i^{th} column is NOT a pivot column, then x_i is called a **free variable**.

Basic & Free Variables

Consider the system of equations along with its augmented matrix.

$$\begin{array}{rcrcrcrcr} & & 3x_2 & - & 6x_3 & + & 4x_4 & = & 6\\ 3x_1 & - & 7x_2 & + & 8x_3 & + & 8x_4 & = & -5\\ 3x_1 & - & 9x_2 & + & 12x_3 & + & 6x_4 & = & -9 \end{array} \qquad \left[\begin{array}{ccccc} 0 & 3 & -6 & 4 & 6\\ 3 & -7 & 8 & 8 & -5\\ 3 & -9 & 12 & 6 & -9 \end{array}\right]$$

We determined that the matrix was row equivalent to the rref

$$\left[\begin{array}{ccccc} 1 & 0 & -2 & 0 & 3\\ 0 & 1 & -2 & 0 & 2\\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Hence the **basic** variables are x_1, x_2, and x_4, and the **free** variable is x_3.

Expressing Solutions

To avoid confusion, i.e., in the interest of clarity, we will **always** write solution sets by expressing basic variables in terms of free variables. We will not write free variables in terms of basic. That is, the solution set to the system whose augmented matrix is row equivalent to

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 3 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

will be written

$$\begin{aligned} x_1 &= 3 + 2x_3 \\ x_2 &= 2 + 3x_3 \\ x_3 & \text{ is free} \\ x_4 &= 0 \end{aligned}$$

This is called a *parametric* form or description of the solution set.

Consistent versus Inconsistent Systems

Consider each rref. Determine if the underlying system (the one with this as its augmented matrix) is consistent or inconsistent.

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Consistent

one free variable

Consistent

Inconsistent

Last row says

$0 = 1$

An Existence and Uniqueness Theorem

Theorem

A linear system is consistent if and only if the right most column of the augmented matrix is **NOT** a pivot column. That is, if and only if each echelon form **DOES NOT** have a row of the form

$$[0\ 0\ \cdots\ 0\ b], \quad \text{for some nonzero } b.$$

Moreover, if a linear system is consistent, then it has

(i) exactly one solution if there are no free variables, and
(ii) infinitely many solutions if there is at least one free variable.

Section 1.3: Vector Equations

Definition: A matrix that consists of one column is called a **column vector** or simply a **vector**.

When we give a vector a name (i.e. use a variable to denote a vector), the convention

- in typesetting is to use bold face

$$\mathbf{u} \quad \text{and} \quad \mathbf{x}$$

- in handwriting is to place a little arrow over the variable

$$\vec{u} \quad \text{and} \quad \vec{x}$$

Set of Real Ordered Pairs: $\mathbb{R}^2$

The set of vectors of the form $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ with x_1 and x_2 any real numbers is denoted by

$$\mathbb{R}^2$$

(read "R two"). It's the set of all real ordered pairs.

Geometry

Each vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ corresponds to a point in the Cartesian plane. We can equate them with ordered pairs written in the traditional format

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (x_1, x_2).$$

This is **not to be confused with a row matrix.**

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq [x_1 \;\; x_2]$$

We can identify vectors with points or with directed line segments emanating from the origin (little arrows).

Geometry

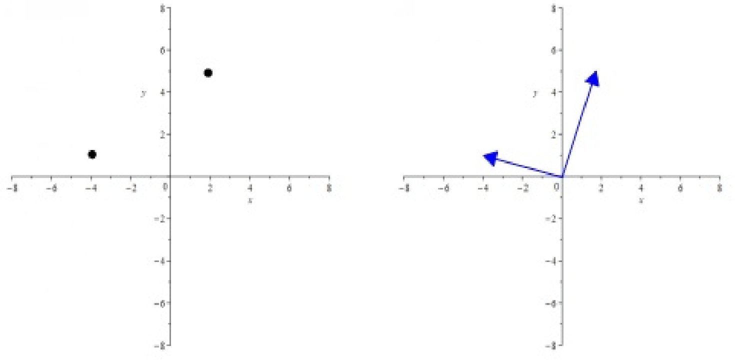

Figure: Vectors characterized as points, and vectors characterized as directed line segments.

$$\begin{bmatrix} -4 \\ 1 \end{bmatrix} = (-4, 1), \quad \begin{bmatrix} 2 \\ 5 \end{bmatrix} = (2, 5)$$

Vector Equality

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, and c be a scalar*.

Vector Equivalence: Equality of vectors is defined by

$$\mathbf{u} = \mathbf{v} \quad \text{if and only if} \quad u_1 = v_1 \quad \text{and} \quad u_2 = v_2.$$

*A **scalar** is an element of the set from which u_1 and u_2 come. For our purposes, a scalar is a *real* number.

Algebraic Operations

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, and c be a scalar.

Scalar Multiplication: The scalar multiple of $\mathbf{u}$

$$c\mathbf{u} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}.$$

Vector Addition: The sum of vectors $\mathbf{u}$ and $\mathbf{v}$

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$

Examples

$$\text{Let} \quad \mathbf{u} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}, \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} -3 \\ \frac{3}{2} \end{bmatrix}$$

Evaluate

(a) $-2\mathbf{u}$

$$-2\vec{u} = \begin{bmatrix} -2(4) \\ -2(-2) \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$$

Examples

$$\text{Let} \quad \mathbf{u} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}, \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} -3 \\ \frac{3}{2} \end{bmatrix}$$

Evaluate

(b) $-2\mathbf{u}+3\mathbf{v}$

$$-2\vec{u} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}, \quad 3\vec{v} = \begin{bmatrix} -3 \\ 21 \end{bmatrix}$$

$$-2\vec{u} + 3\vec{v} = \begin{bmatrix} -8+(-3) \\ 4+21 \end{bmatrix} = \begin{bmatrix} -11 \\ 25 \end{bmatrix}$$

Examples

Let $\quad \mathbf{u} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}, \quad$ and $\quad \mathbf{w} = \begin{bmatrix} -3 \\ \frac{3}{2} \end{bmatrix}$

(c) Is it true that $\mathbf{w} = -\frac{3}{4}\mathbf{u}$?

$$-\frac{3}{4}\vec{u} = \begin{bmatrix} -3 \\ \frac{3}{2} \end{bmatrix}$$

Note, they have the same 1st and 2nd entry, so

$$\vec{w} = -\frac{3}{4}\vec{u}.$$

Geometry of Algebra with Vectors

Scalar Multiplication: stretches or compresses a vector but can only change direction by an angle of 0 (if $c > 0$) or π (if $c < 0$). We'll see that $0\mathbf{u} = (0, 0)$ for any vector $\mathbf{u}$.

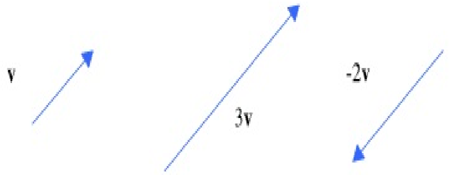

Figure: Scaled vectors are parallel. For nonzero vector $\mathbf{v}$, $c\mathbf{v}$ is stretched or compressed by a factor $|c|$ and flips 180° if c is negative.

Geometry of Algebra with Vectors

Vector Addition: The sum $\mathbf{u}+\mathbf{v}$ of two nonparallel vectors (each different from $(0,0)$) is the the fourth vertex of a parallelogram whose other three vertices are (u_1, u_2), (v_1, v_2), and $(0,0)$.

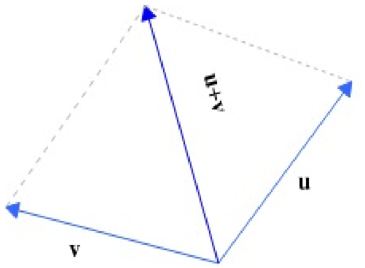

Figure: If $\mathbf{u}$ and $\mathbf{v}$ are nonzero and not parallel, they determine a paralelogram. The sum $\mathbf{u}+\mathbf{v}$ is a diagonal. (Note, the difference $\mathbf{u}-\mathbf{v}$ is the other diagonal.)

Geometry of Algebra with Vectors

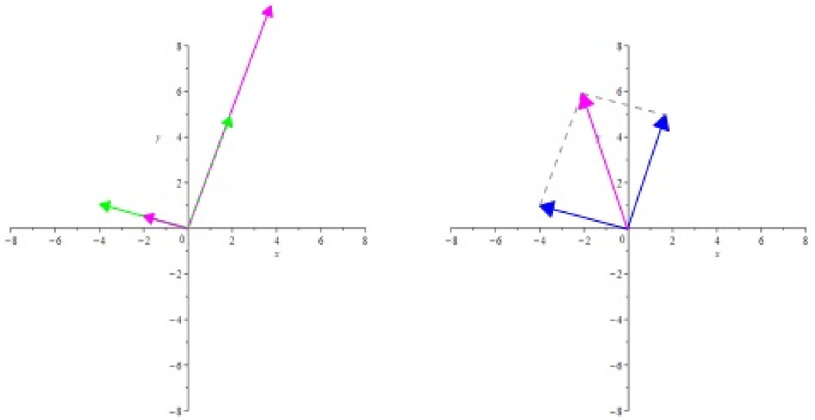

Figure: Left: Scalar multiplication *scales* vectors. $\frac{1}{2}(-4, 1) = (-2, \frac{1}{2})$ and $2(2, 5) = (4, 10)$.
Right: Addition gives the diagonal of a paralellogram.
$(-4, 1) + (2, 5) = (-2, 6)$

Vectors in $\mathbb{R}^3$ (R three)

A vector in $\mathbb{R}^3$ is a 3×1 column matrix. For example

$$\mathbf{a} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \quad \text{or} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Similar to vectors in $\mathbb{R}^2$, vectors in $\mathbb{R}^3$ are ordered triples.

$$\mathbf{a} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = (1, 3, -1).$$

Vectors in $\mathbb{R}^n$ (R n)

A vector in $\mathbb{R}^n$ for $n \geq 2$ is a $n \times 1$ column matrix. These are ordered n-tuples. For example

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

The Zero Vector: is the vector whose entries are all zeros. It will be denoted by $\mathbf{0}$ or $\vec{0}$ and is not to be confused with the scalar 0.

Scalar multiplication and vector addition are defined as they are in R^2.

Algebraic Properties on $\mathbb{R}^n$

For every **u**, **v**, and **w** in $\mathbb{R}^n$ and scalars c and d[1]

(i) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(ii) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$

(iii) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$

(iv) $\mathbf{u}+(-\mathbf{u})=-\mathbf{u}+\mathbf{u}=\mathbf{0}$

(v) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$

(vi) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$

(vii) $c(d\mathbf{u})=d(c\mathbf{u})=(cd)\mathbf{u}$

(viii) $1\mathbf{u}=\mathbf{u}$

[1] The term $-\mathbf{u}$ denotes $(-1)\mathbf{u}$.

Linear Combination

Definition

A **linear combination** of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ is a vector $\mathbf{y}$ of the form

$$\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

where the scalars $c_1, \ldots, c_p$ are often called weights.

For example, suppose we have two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Some linear combinations include

$$3\mathbf{v}_1, \quad -2\mathbf{v}_1 + 4\mathbf{v}_2, \quad \frac{1}{3}\mathbf{v}_2 + \sqrt{2}\mathbf{v}_1, \quad \text{and} \quad \mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2.$$

Example

Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix}$. Determine if $\mathbf{b}$ can be written as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

Can we find numbers (weights) c_1, c_2 such that: $\vec{b} = c_1\vec{a}_1 + c_2\vec{a}_2$.

$$c_1 \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ -2c_1 \\ -c_1 \end{bmatrix} + \begin{bmatrix} 3c_2 \\ 0 \\ 2c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} c_1 + 3c_2 \\ -2c_1 \\ -c_1 + 2c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix}.$$

$$\begin{aligned} c_1 + 3c_2 &= -2 \\ -2c_1 &= -2 \\ -c_1 + 2c_2 &= -3 \end{aligned}$$

We can solve this using an augmented matrix.

$$\begin{bmatrix} 1 & 3 & -2 \\ -2 & 0 & -2 \\ -1 & 2 & -3 \end{bmatrix}$$

Using TI 92

$$\text{rref} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} c_1 &= 1 \\ c_2 &= -1 \end{aligned}$$

Yes, $\vec{b}$ is a linear combination of $\vec{a}_1$ and $\vec{a}_2$

In fact, $\vec{b} = \vec{a}_1 - \vec{a}_2$

Some Convenient Notation

Letting $\mathbf{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}$, and in general $\mathbf{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$, for $j = 1, ..., n$, we can denote the $m \times n$ matrix whose columns are these vectors by

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Note that each vector $\mathbf{a}_j$ is a vector in $\mathbb{R}^m$.

Vector and Matrix Equations

Vector & Matrix Equations

The vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}. \tag{1}$$

In particular, $\mathbf{b}$ is a linear combination of the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ if and only if the linear system whose augmented matrix is given in (1) is consistent.

Span

Definition

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set of vectors in $\mathbb{R}^n$. The set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by

$$\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = \text{Span}(S).$$

It is called the **subset of $\mathbb{R}^n$ spanned by (a.k.a. generated by)** the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

Remark: To say that a vector $\mathbf{b}$ is in $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ means that there exists a set of scalars $c_1, \ldots, c_p$ such that

$$\mathbf{b} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p.$$

Equivalent Statements

Suppose $\mathbf{b}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are vectors in $\mathbb{R}^m$. The following are equivalent:

- $\mathbf{b}$ is in $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$,
- $\mathbf{b} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$ for some scalars $c_1, \ldots, c_p$,
- the vector equation $x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{b}$ has a solution,
- the linear system of equations whose augmented matrix is $[\mathbf{v}_1 \ \cdots \ \mathbf{v}_p \ \mathbf{b}]$ is consistent.

Examples

Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, and $\mathbf{a}_2 = \begin{bmatrix} -1 \\ 4 \\ -2 \end{bmatrix}$.

(a) Determine if $\mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ is in Span$\{\mathbf{a}_1, \mathbf{a}_2\}$.

We can see if the system w/ augmented matrix $[\vec{a}_1\ \vec{a}_2\ \vec{b}]$ is consistent.

$$[\vec{a}_1\ \vec{a}_2\ \vec{b}] = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & 2 \\ 2 & -2 & 1 \end{bmatrix}$$

rref $\longrightarrow$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

↑ pivot column

The system is inconsistent.

Hence $\vec{b}$ is not in

$\text{Span}\{\vec{a}_1, \vec{a}_2\}$.

Recall, the system is consistent if and only if the last column is NOT a pivot column.