June 21 Math 2306 sec. 53 Summer 2022

Section 6: Linear Equations Theory and Terminology

Recall that an n^{th} order linear IVP consists of an equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

to solve subject to conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}.$$

The problem is called **homogeneous** if $g(x) \equiv 0$. Otherwise it is called **nonhomogeneous**.

Theorem: Existence & Uniqueness

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}.$$

Theorem: If $a_0, \ldots, a_n$ and g are continuous on an interval I, $a_n(x) \neq 0$ for each x in I, and x_0 is any point in I, then for any choice of constants $y_0, \ldots, y_{n-1}$, the IVP has a unique solution $y(x)$ on I.

Put differently, we're guaranteed to have a solution exist, and it is the only one there is!

The Principle of Superposition (homogeneous ode)

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0$$

Assume a_i are continuous and $a_n(x)\neq 0$ for all x in I.

Theorem: If y_1, $y_2, \ldots, y_k$ are all solutions of this homogeneous equation on an interval I, then the *linear combination*

$$y(x)=c_1y_1(x)+c_2y_2(x)+\cdots+c_ky_k(x)$$

is also a solution on I for any choice of constants $c_1,\ldots,c_k$.

Corollaries

(i) If y_1 solves the homogeneous equation, the any constant multiple $y = cy_1$ is also a solution.

(ii) The solution $y = 0$ (called the trivial solution) is always a solution to a homogeneous equation.

Big Questions:

- Does an equation have any **nontrivial** solution(s), and
- since y_1 and cy_1 aren't truly *different* solutions, what criteria will be used to call solutions distinct?

Linear Dependence

Definition: A set of functions $f_1(x), f_2(x), \dots, f_n(x)$ are said to be **linearly dependent** on an interval I if there exists a set of constants $c_1, c_2, \dots, c_n$ with at least one of them being nonzero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all} \quad x \text{ in } I. \tag{1}$$

A set of functions that is not linearly dependent on I is said to be **linearly independent** on I.

NOTE: Taking all of the c's to be zero will **always** satisfy equation (1). The set of functions is linearly **independent** if taking all of the c's equal to zero is the **only** way to make the equation true.

Example: A linearly Independent Set

The functions $f_1(x) = \sin x$ and $f_2(x) = \cos x$ are linearly independent on $I = (-\infty, \infty)$.

Suppose there are numbers c_1 and c_2 such that

$$c_1 f_1(x) + c_2 f_2(x) = 0 \quad \text{for all } x$$

That is,

$$c_1 \sin x + c_2 \cos x = 0$$

Since this is true for all real x, it's true when $x=0$. When $x=0$, the equation is

$$c_1 \sin(0) + c_2 \cos(0) = 0$$

$$\Longrightarrow \quad c_1(0) + c_2(1) = 0 \quad \Longrightarrow \quad c_2 = 0.$$

The equation is also true when $x = \frac{\pi}{2}$.

When $x = \frac{\pi}{2}$, the equation is

$$c_1 \sin\left(\frac{\pi}{2}\right) + 0 \cdot \cos\left(\frac{\pi}{2}\right) = 0$$

$$c_1(1) + 0 = 0 \quad \Longrightarrow \quad c_1 = 0$$

Both $c_1 = 0$ and $c_2 = 0$. The set $\{f_1(x), f_2(x)\}$ is linearly independent.

Side note: If we have two functions

f_1 and f_2, the equation

$$c_1 f_1(x) + c_2 f_2(x) = 0$$

can be rearranged to $c_1 f_1(x) = -c_2 f_2(x)$

If $c_1 \neq 0$, this becomes

$$f_1(x) = \frac{-c_2}{c_1} f_2(x)$$

The functions are dependent if one is a multiple of the other.

Determine if the set is Linearly Dependent or Independent on $(-\infty, \infty)$

$$f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2$$

Note that f_3 can be built as a linear combination of f_1 and f_2.

$$f_3(x) = \frac{1}{4} f_2(x) - f_1(x)$$

$$x - x^2 \stackrel{?}{=} \frac{1}{4}(4x) + (-1)\,x^2 \quad \checkmark$$

We can rearrange to get

$$f_1(x) - \frac{1}{4} f_2(x) + f_3(x) = 0$$

Note $x^2 - \frac{1}{4}(4x) + x - x^2 =$

$x^2 - x^2 - x + x = 0$ for all real x.

We have $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$

with $c_1 = 1$, $c_2 = \frac{-1}{4}$, $c_3 = 1$ Not All zero

The set $\{f_1, f_2, f_3\}$ is linearly dependent.

Linear Dependence Relation

An equation with at least one c nonzero, such as

$$f_1(x) - \frac{1}{4}f_2(x) + f_3(x) = 0$$

from this last example is called a **linear dependence relation** for the functions $\{f_1, f_2, f_3\}$.

Definition of Wronskian

Definition: Let $f_1, f_2, \ldots, f_n$ posses at least $n-1$ continuous derivatives on an interval I. The **Wronskian** of this set of functions is the determinant

$$W(f_1, f_2, \ldots, f_n)(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

(Note that, in general, this Wronskian is a function of the independent variable x.)

Determinants

If A is a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then its determinant

$$\det(A) = ad - bc.$$

If A is a 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then its determinant

$$\det(A) = a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Determine the Wronskian of the Functions

$$f_1(x) = \sin x, \quad f_2(x) = \cos x$$

2 functions $\Rightarrow$ matrix will be 2×2.

$$W(f_1, f_2)(x) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix};$$

$f_1(x) = \sin x$

$f_1'(x) = \cos x$

$f_2(x) = \cos x$

$f_2'(x) = -\sin x$

$$W(f_1, f_2)(x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$$= \sin x\,(-\sin x) - \cos x\,(\cos x)$$

$$= -\sin^2 x - \cos^2 x$$

$$= -(\sin^2 x + \cos^2 x)$$

$$= -1$$

$$W(\sin x, \cos x)(x) = -1$$

Determine the Wronskian of the Functions

$$f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2$$

3 functions $\Rightarrow$ matrix will be 3×3.

$$W(f_1, f_2, f_3)(x) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}$$

$$= \begin{vmatrix} x^2 & 4x & x - x^2 \\ 2x & 4 & 1 - 2x \\ 2 & 0 & -2 \end{vmatrix}$$

$$= x^2 \begin{vmatrix} 4 & 1-2x \\ 0 & -2 \end{vmatrix} - 4x \begin{vmatrix} 2x & 1-2x \\ 2 & -2 \end{vmatrix} + (x-x^2) \begin{vmatrix} 2x & 4 \\ 2 & 0 \end{vmatrix}$$

$$= x^2(-8-0) - 4x\left(-4x - 2(1-2x)\right) + (x-x^2)(0-8)$$

$$= -8x^2 - 4x\left(-\cancel{4x} - 2 + \cancel{4x}\right) - 8x + 8x^2$$

$$= -8x^2 + 8x - 8x + 8x^2$$

$$= 0$$

$$W(x^2, 4x, x-x^2)(x) = 0$$

Theorem (a test for linear independence)

Theorem: Let f_1, $f_2, \ldots, f_n$ be $n-1$ times continuously differentiable on an interval I. If there exists x_0 in I such that $W(f_1, f_2, \ldots, f_n)(x_0) \neq 0$, then the functions are **linearly independent** on I.

If $W \neq 0$ the functions are lin. Independent

Alternative Version

If $y_1, y_2, \ldots, y_n$ are n solutions of the linear homogeneous n^{th} order equation on an interval I, then the solutions are **linearly independent** on I if and only if $W(y_1, y_2, \ldots, y_n)(x) \neq 0$ for[1] each x in I.

[1] For solutions of one linear homogeneous ODE, the Wronskian is either always zero or is never zero.

Determine if the functions are linearly dependent or independent:

$$y_1 = e^x, \quad y_2 = e^{-2x} \quad I = (-\infty, \infty)$$

We can use the Wronskian.

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^x & e^{-2x} \\ e^x & -2e^{-2x} \end{vmatrix} = e^x(-2e^{-2x}) - e^x(e^{-2x})$$

$$= -2e^{-x} - e^{-x} = -3e^{-x}$$

$$W(e^{x}, e^{-2x})(x) = -3e^{-x}$$

Since this is not zero, y_1 and y_2

are <u>linearly independent.</u>

Fundamental Solution Set

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

Assume a_i are continuous and $a_n(x) \neq 0$ for all x in I.

Definition: A set of functions $y_1, y_2, \ldots, y_n$ is a **fundamental solution set** of the n^{th} order homogeneous equation provided they

(i) are solutions of the equation,

(ii) there are n of them, and

(iii) they are linearly independent.

Theorem: Under the assumed conditions, the equation has a fundamental solution set.

General Solution of n^{th} order Linear Homogeneous Equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

Assume a_i are continuous and $a_n(x) \neq 0$ for all x in I.

Definition Let $y_1, y_2, \ldots, y_n$ be a fundamental solution set of the n^{th} order linear homogeneous equation. Then the **general solution** of the equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Example

Verify that $y_1 = x^2$ and $y_2 = x^3$ form a fundamental solution set of the ODE

$$x^2y'' - 4xy' + 6y = 0 \quad \text{on} \quad (0, \infty),$$

and determine the general solution.

The ODE is 2^{nd} order, so we need two linearly independent solutions.

Verify they are solutions:

$y_1 = x^2, \quad y_1' = 2x, \quad y_1'' = 2$

$$x^2 y_1'' - 4x y_1' + 6y_1 \overset{?}{=} 0$$

$$x^2(2) - 4x(2x) + 6(x^2) \overset{?}{=} 0$$

$$2x^2 - 8x^2 + 6x^2 = 0 \checkmark$$

y_1 is a solution.

$$y_2 = x^3, \quad y_2' = 3x^2, \quad y_2'' = 6x$$

$$x^2 y_2'' - 4x y_2' + 6y_2 \stackrel{?}{=} 0$$

$$x^2(6x) - 4x(3x^2) + 6x^3 \stackrel{?}{=} 0$$

$$6x^3 - 12x^3 + 6x^3 = 0 \quad \checkmark$$

y_2 is a solution

Are they linearly independent?

Check the Wronskian.

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = x^2(3x^2) - 2x(x^3)$$

$$= 3x^4 - 2x^4 = x^4$$

$W \neq 0$, y_1 and y_2 are linearly independent.

We have a fundamental solution set.

The general solution

$$y = c_1 y_1 + c_2 y_2$$

$$\boxed{y = c_1 x^2 + c_2 x^3}$$

Nonhomogeneous Equations

Now we will consider the equation

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

where g is not the zero function. We'll continue to assume that a_n doesn't vanish and that a_i and g are continuous.

The **associated homogeneous equation** is

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0.$$

Theorem: General Solution of Nonhomogeneous Equation

Theorem: Let y_p be any solution of the nonhomogeneous equation, and let $y_1, y_2, \dots, y_n$ be any fundamental solution set of the associated homogeneous equation.
Then the general solution of the nonhomogeneous equation is

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x)$$

where $c_1, c_2, \dots, c_n$ are arbitrary constants.

$y_c = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$

Note the form of the solution $y_c + y_p$!
(complementary plus particular)

Superposition Principle (for nonhomogeneous eqns.)

Consider the nonhomogeneous equation

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g_1(x)+g_2(x) \quad (2)$$

Theorem: If y_{p_1} is a particular solution for

$$a_n(x)\frac{d^ny}{dx^n}+\cdots+a_0(x)y=g_1(x),$$

and y_{p_2} is a particular solution for

$$a_n(x)\frac{d^ny}{dx^n}+\cdots+a_0(x)y=g_2(x),$$

then

$$y_p=y_{p_1}+y_{p_2}$$

is a particular solution for the nonhomogeneous equation (2).

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

We will construct the general solution by considering sub-problems.

(a) **Part 1** Verify that

$g_1(x) = 36, \quad g_2(x) = -14x$

our right side is $g_1(x) + g_2(x)$

$$y_{p_1} = 6 \quad \text{solves} \quad x^2y'' - 4xy' + 6y = 36.$$

$y_{p_1} = 6, \quad y_{p_1}' = 0, \quad y_{p_1}'' = 0$

$$x^2 y_{p_1}'' - 4xy_{p_1}' + 6y_{p_1} \stackrel{?}{=} 36$$

$$x^2(0) - 4x(0) + 6(6) \stackrel{?}{=} 36$$

$$36 = 36$$

✓ $y_{p_1} = 6$ solves this ODE

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(b) **Part 2** Verify that

$$y_{p_2} = -7x \quad \text{solves} \quad x^2y'' - 4xy' + 6y = -14x.$$

$y_{p_2} = -7x, \quad y_{p_2}' = -7, \quad y_{p_2}'' = 0$

$x^2 y_{p_2}'' - 4x\, y_{p_2}' + 6\, y_{p_2} \stackrel{?}{=} -14x$

$x^2(0) - 4x(-7) + 6(-7x) \stackrel{?}{=} -14x$

$28x - 42x \stackrel{?}{=} -14x$

$-14x = -14x \quad \checkmark$

y_{p_2} does solve this ODE

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(c) **Part 3** We already know that $y_1 = x^2$ and $y_2 = x^3$ is a fundamental solution set of

$$x^2y'' - 4xy' + 6y = 0.$$

Use this along with results (a) and (b) to write the general solution of $x^2y'' - 4xy' + 6y = 36 - 14x$.

The general solution $y = y_c + y_p$

$y_c = c_1 x^2 + c_2 x^3$

From (a) and (b) $y_p = y_{p_1} + y_{p_2} = 6 - 7x$

The general solution

$$\boxed{y = c_1 x^2 + c_2 x^3 + 6 - 7x}$$

Solve the IVP

$$x^2y'' - 4xy' + 6y = 36 - 14x, \quad y(1) = 0, \quad y'(1) = -5$$

The general solution is

$$y = c_1 x^2 + c_2 x^3 + 6 - 7x$$

we need to find the c' values such that
$y(1) = 0$ and $y'(1) = -5$.

$$y' = 2c_1 x + 3c_2 x^2 - 7$$

$$y(1) = c_1(1^2) + c_2(1^3) + 6 - 7(1) = 0$$

$$c_1 + c_2 - 1 = 0$$

$$c_1 + c_2 = 1$$

$$y'(1) = 2c_1(1) + 3c_2(1^2) - 7 = -5$$

$$2c_1 + 3c_2 - 7 = -5$$

$$2c_1 + 3c_2 = 2$$

Solve

$$c_1 + c_2 = 1$$
$$2c_1 + 3c_2 = 2$$

$\Rightarrow$

$$2c_1 + 2c_2 = 2$$
$$2c_1 + 3c_2 = 2$$

$-$

$$-c_2 = 0$$

$$\Rightarrow \quad c_2 = 0$$

$$\Rightarrow \quad c_1 = 1$$

The solution to the IVP

$$y = x^2 + 6 - 7x$$