June 23 Math 2306 sec. 53 Summer 2022

Section 7: Reduction of Order

We'll focus on second order, linear, homogeneous equations. Recall that such an equation has the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

Let us assume that $a_2(x) \neq 0$ on the interval of interest. We will write our equation in **standard form**

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

where $P = a_1/a_2$ and $Q = a_0/a_2$.

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$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Some things to keep in mind:

- Every fundamental solution set has two linearly independent solutions y₁ and y₂,
- The general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x).$$

Suppose we know one solution $y_1(x)$. This section is about a process called **Reduction of order**. Reduction of order is a method for finding a second solution by assuming that

 $y_2(x) = u(x)y_1(x).$

The goal is to find the unknown function *u*.

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Context

We start with a second order, linear, homogeneous ODE in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0.$$

- We know one solution y₁. (Keep in mind that y₁ is a known!)
- We know there is a second linearly independent solution (section 6 theory says so).
- We try to find y_2 by guessing that it can be found in the form

$$y_2(x) = u(x)y_1(x)$$

where the goal becomes finding *u*.

Due to linear independence, we know that u cannot be constant.

Example

Find the general solution to the ODE $x^2y'' - xy' + y = 0$ for x > 0 given that $y_1(x) = x$ is one solution.

The ODE in standard for ~ 15 $y'' - \frac{1}{x}y' + \frac{1}{x^2}y = 0$, $y_1 = x$ con Que V Suppose $y_z = u(x)y_i(x) = x u$ $y_2 = \chi u$ $y_{z}' = Xu' + u$ $y_{z}'' = x u'' + u' + u' = x u'' + Z u''$

Sub into the ODE

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$$y_{z}" - \frac{1}{x} y_{z}' + \frac{1}{x^{2}} y_{z} = 0$$

$$xu'' + zu' - \frac{1}{x} (xu' + u) + \frac{1}{x^{2}} (xu) = 0$$

$$xu'' + 2u' - u' - \frac{1}{x} u + \frac{1}{x} u = 0$$

$$xu'' + u' = 0 \quad \text{and} \quad \text{opti for } u^{(0)}$$

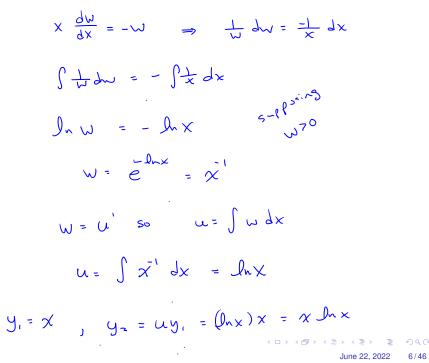
$$Let \quad w = u', \quad \text{then} \quad w' = u''. \quad \text{The ODE for } w$$
is
$$xw' + w = 0$$

$$This is a 1st order linear and separable$$

$$ODE.$$

$$Let's \quad separate \quad \text{the variables}$$

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The general solution

$$y = C_1 \times + C_2 \times ln \times$$

Generalization

Consider the equation **in standard form** with one known solution. Determine a second linearly independent solution.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \quad y_1(x) - \text{is known.}$$
Suppose
$$y_z = u y_1, \quad y_2' = u' y_1 + u y_1', \quad y_2'' = u' y_1 + u' y_1' + u' y_1' + u y_1''$$

$$= u'' y_1 + 2u' y_1' + u y_1''$$

We know that $y_1'' + P(x)y_1' + Q(x)y_1 = 0.$

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$$y_{z}'' + P(x)y_{z}' + Q(x)y_{z} = 0$$

$$u''y_{1} + zu'y_{1}' + uy_{1}'' + P(x)(u'y_{1} + uy_{1}') + Q(x)(uy_{1}) = 0$$
Collect $u'', u'_{1}u$

$$y_{1}u'' + (Zy_{1}' + P(x)y_{1})u' + (Y_{1}'' + P(x)y_{1}' + Q(x)y_{1})u = 0$$

$$u''y_{1}u'' + (Zy_{1}' + P(x)y_{1})u' + (Y_{1}'' + Q(x)y_{1})u = 0$$
The ODE for u is
$$y_{1}u'' + (Zy_{1}' + P(x)y_{1})u' = 0$$
Divide by y_{1}

$$u''y_{1}u'' + (Zy_{1}' + P(x)y_{1})u' = 0$$

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$$u'' + \left(\frac{ay'}{y} + P(x)\right)u' = 0$$
Let $w = u' \quad s_0 \quad w' = u'' \quad a_d \quad u = \int w \, dx$

$$w \quad \text{solves'} \quad \text{the} \quad 1^{s+} \text{ order} \quad \text{direct} \quad \text{and} \quad \text{separable}$$

$$op \in \frac{dw}{dx} + \left(\frac{zy'}{y} + P(x)\right)w = 0$$
Separating variables

Separating variables

$$\frac{dW}{dX} = -\left(\frac{2y_1^2}{y_1} + P(x_1)\right)W$$

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$$\frac{1}{\sqrt{2}} dw = -2 \frac{dy_1}{y_1} dx - P(x) dx$$

$$\int \frac{1}{\sqrt{2}} dw = -2 \int \frac{dy_1}{y_1} - \int P(x) dx$$

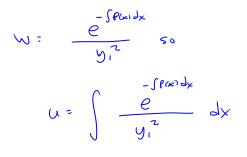
$$\int w = -2 \int y_1 - \int P(x) dx$$

$$-2hy_1 - \int P(x) dx$$

$$= e^{Ay_1^2} e^{-\int P(x) dx}$$

$$= y_1^2 e^{\int P(x) dx}$$
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Reduction of Order Formula

For the second order, homogeneous equation in standard form with one known solution y_1 , a second linearly independent solution y_2 is given by

$$y_2 = y_1(x) \int \frac{e^{-\int P(x) \, dx}}{(y_1(x))^2} \, dx$$

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Example

Find the solution of the IVP where one solution of the ODE is given.

$$y'' + 4y' + 4y = 0 \quad y_1 = e^{-2x}, \quad y(0) = 1, \quad y'(0) = 1$$

we need y_2 . Using reduction of order
 $y_2 = y_1 u \quad u = \int \frac{e^{-\int f(x) dx}}{(y_1)^2} dx$
 $P(x) = u_1 - \int P(x) dx = -\int u dx = -ux$
The numerator is e^{-ux}
The denominator $(y_1)^2 = (e^{2x})^2 = e^{-ux}$

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So
$$u = \int \frac{e^{-u_X}}{e^{-u_X}} dx = \int dx = x$$

$$y_2 = y_1 u = \chi e^{-2\chi}$$

The general solution y= Ciy, + Coy? $y = C_1 e^{2x} + C_2 x e^{-2x}$ Apply y(0)=1 and y)(0)=1 $y' = -2C, e^{-2x} + C_2 e^{-2x} - 2C_2 \times e^{-2x}$ $V(0 = C, e^{\circ} + C_2 \cdot 0 e^{\circ} =) \implies C_1 = 1$ $y'(0) = -2C_1 e^{0} + C_2 e^{0} - 2C_2 \cdot 0 \cdot e^{0} =)$

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 $-2C_1 + C_2 = 1 \implies (z = 1 + 2C_1 = 1 + 2 = 3)$ The solution to the 1VP $y = e^{2x} + 3x e^{-2x}$

Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order¹, linear, homogeneous equation with constant coefficients

$$arac{d^2y}{dx^2}+brac{dy}{dx}+cy=0, \quad ext{with } a
eq 0.$$

If we put this in normal form, we get

$$\frac{d^2y}{dx^2} = -\frac{b}{a}\frac{dy}{dx} - \frac{c}{a}y.$$

Question: What sorts of functions *y* could be expected to satisfy y'' = (constant) y' + (constant) y?

¹We'll extend the result to higher order at the end of this section.

We look for solutions of the form $y = e^{mx}$ with *m* constant.

Sub in
$$y = e^{Mx}$$
.
 $y' = me^{Kx}$, $y'' = m^2 e^{Mx}$

$$a(m^2 e^{mx}) + b(m e^{mx}) + C e^{mx} = 0$$

$$e^{mx}\left(am^{2}+bm+c\right)=0$$

$$am^2 + bm + C = 0$$

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This is a quadratic equation for m.

Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

There are three cases:

 $b^2 - 4ac > 0$ and there are two distinct real roots $m_1 \neq m_2$

II $b^2 - 4ac = 0$ and there is one repeated real root $m_1 = m_2 = m$

III $b^2 - 4ac < 0$ and there are two roots that are complex conjugates $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$.

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Case I: Two distinct real roots

$$ay'' + by' + cy = 0$$
, where $b^2 - 4ac > 0$.

There are two different roots m_1 and m_2 . A fundamental solution set consists of

$$y_1 = e^{m_1 x}$$
 and $y_2 = e^{m_2 x}$.

The general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

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Example

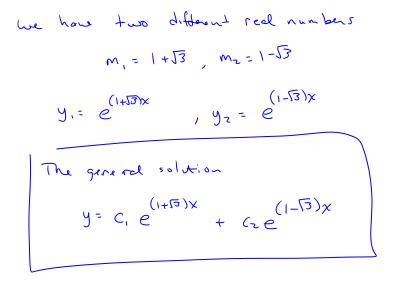
Find the general solution of the ODE.

$$y'' - 2y' - 2y = 0$$

The characteristic equation is
$$m^{2} - 2m - 2 = 0.$$

Find the costr. Completing the square
$$m^{2} - 2m + 1 = 2 + 1$$
$$(m - 1)^{2} = 3$$
$$m - 1 = \pm \sqrt{3} \implies m = 1 \pm \sqrt{3}$$

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Case II: One repeated real root

$$ay'' + by' + cy = 0$$
, where $b^2 - 4ac = 0$

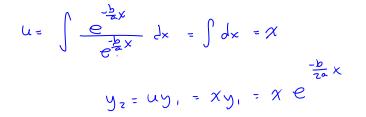
There is only one real, double root, $m = \frac{-b}{2a}$.

Use reduction of order to find the second solution to the equation (in standard form)

 $y'' + \frac{b}{a}y' + \frac{c}{a}y = 0 \quad \text{given one solution} \quad y_1 = e^{-\frac{b}{2a}x}$ $y_2 = y_1 u_2, \quad u = \int \frac{e^{-\int f x d dx}}{(y_0)^2} dx$ $P(x) = \frac{b}{a} \quad y_1 = \int P(x) dx = -\int \frac{b}{a} dx = -\frac{b}{a} x$

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The numerator is
$$e^{-\frac{b}{a}x}$$
.
The denominator is $(y_i)^2 = (e^{-\frac{b}{2a}x})^2 = e^{-\frac{b}{a}x}$



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Case II: One repeated real root

$$ay'' + by' + cy = 0$$
, where $b^2 - 4ac = 0$

If the characteristic equation has one real repeated root *m*, then a fundamental solution set to the second order equation consists of

$$y_1 = e^{mx}$$
 and $y_2 = xe^{mx}$.

The general solution is

$$y=c_1e^{mx}+c_2xe^{mx}.$$

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