June 23 Math 2306 sec. 53 Summer 2022

Section 7: Reduction of Order

We'll focus on second order, linear, homogeneous equations. Recall that such an equation has the form

$$a_2(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=0.$$

Let us assume that $a_2(x)\neq 0$ on the interval of interest. We will write our equation in **standard form**

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0$$

where $P=a_1/a_2$ and $Q=a_0/a_2$.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Some things to keep in mind:

- Every fundamental solution set has two linearly independent solutions y_1 and y_2,
- The general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x).$$

Suppose we know one solution $y_1(x)$. This section is about a process called **Reduction of order.** Reduction of order is a method for finding a second solution by assuming that

$$y_2(x) = u(x)y_1(x).$$

The goal is to find the unknown function u.

Context

- We start with a second order, linear, homogeneous ODE in standard form
$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0.$$
- We know one solution y_1. (Keep in mind that y_1 is a known!)
- We know there is a second linearly independent solution (section 6 theory says so).
- We try to find y_2 by guessing that it can be found in the form
$$y_2(x) = u(x)y_1(x)$$
where the goal becomes finding u.
- **Due to linear independence, we know that u <u>cannot</u> be constant.**

Example

Find the general solution to the ODE $x^2y'' - xy' + y = 0$ for $x > 0$ given that $y_1(x) = x$ is one solution.

The ODE in standard form is

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = 0 \quad , \quad y_1 = x$$

Suppose

$$y_2 = u(x)y_1(x) = xu$$

can we find u?

$$y_2 = xu$$

$$y_2' = xu' + u$$

$$y_2'' = xu'' + u' + u' = xu'' + 2u'$$

Sub into the ODE

$$y_2'' - \frac{1}{x} y_2' + \frac{1}{x^2} y_2 = 0$$

$$x u'' + 2u' - \frac{1}{x}\left(x u' + u \right) + \frac{1}{x^2}(x u) = 0$$

$$x u'' + 2u' - u' - \cancel{\frac{1}{x} u} + \cancel{\frac{1}{x} u} = 0$$

$$x u'' + u' = 0 \quad \text{and ODE for } u(x)$$

Let $w = u'$, then $w' = u''$. The ODE for w is

$$x w' + w = 0$$

This is a 1st order linear and separable ODE.

Let's separate the variables

$$x\frac{dw}{dx} = -w \quad \Rightarrow \quad \frac{1}{w}\,dw = \frac{-1}{x}\,dx$$

$$\int \frac{1}{w}\,dw = -\int \frac{1}{x}\,dx$$

$$\ln w = -\ln x$$ supposing $w > 0$

$$w = e^{-\ln x} = x^{-1}$$

$$w = u' \quad \text{so} \quad u = \int w\,dx$$

$$u = \int x^{-1}\,dx = \ln x$$

$$y_1 = x, \quad y_2 = u y_1 = (\ln x)\,x = x\ln x$$

The general solution

$$\boxed{y = c_1 x + c_2 x \ln x}$$

Generalization

Consider the equation **in standard form** with one known solution. Determine a second linearly independent solution.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \quad y_1(x) - \text{is known.}$$

Suppose

$$y_2 = uy_1$$

$$y_2' = u'y_1 + uy_1'$$

$$y_2'' = u''y_1 + u'y_1' + u'y_1' + uy_1''$$

$$= u''y_1 + 2u'y_1' + uy_1''$$

$$\boxed{\text{We know that} \quad y_1'' + P(x)y_1' + Q(x)y_1 = 0.}$$

Sub into the ODE

$$y_2'' + P(x)\,y_2' + Q(x)\,y_2 = 0$$

$$u''y_1 + 2u'y_1' + uy_1'' + P(x)\left(u'y_1 + uy_1'\right) + Q(x)\left(uy_1\right) = 0$$

Collect u'', u', u

$$y_1 u'' + \left(2y_1' + P(x)\,y_1\right)u' + \underbrace{\left(y_1'' + P(x)y_1' + Q(x)y_1\right)}_{=\,0}u = 0$$

since y_1 solves the ODE

The ODE for u is

$$y_1 u'' + \left(2y_1' + P(x)y_1\right)u' = 0$$

Divide by y_1

$$u'' + \left(\frac{2y_1'}{y_1} + P(x) \right) u' = 0$$

Let $w = u'$ so $w' = u''$ and $u = \int w \, dx$

w solves the 1^{st} order linear and separable ODE

$$\frac{dw}{dx} + \left(\frac{2y_1'}{y_1} + P(x) \right) w = 0$$

Separating variables

$$\frac{dw}{dx} = -\left(\frac{2y_1'}{y_1} + P(x) \right) w$$

$$\frac{1}{w}\,dw = -2\,\frac{\frac{dy_1}{dx}}{y_1}\,dx - P(x)\,dx$$

$$\int \frac{1}{w}\,dw = -2\int \frac{dy_1}{y_1} - \int P(x)\,dx$$

$$\ln w = -2\ln y_1 - \int P(x)\,dx$$

$$w = e^{-2\ln y_1 - \int P(x)dx}$$

$$= e^{\ln y_1^{-2}} \cdot e^{-\int P(x)dx}$$

$$= y_1^{-2}\, e^{-\int P(x)dx}$$

$$W = \frac{e^{-\int P(x)\,dx}}{y_1^2} \quad \text{so}$$

$$u = \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$$

$$y_2 = u\,y_1$$

Reduction of Order Formula

For the second order, homogeneous equation **in standard form** with one known solution y_1, a second linearly independent solution y_2 is given by

$$y_2 = y_1(x)\int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx$$

Example

Find the solution of the IVP where one solution of the ODE is given.

$$y'' + 4y' + 4y = 0 \quad y_1 = e^{-2x}, \quad y(0) = 1, \quad y'(0) = 1$$

we need y_2. Using reduction of order

$y_2 = y_1 u \qquad u = \int \frac{e^{-\int P(x)dx}}{(y_1)^2} dx$

$P(x) = 4, \quad -\int P(x)dx = -\int 4\,dx = -4x$

The numerator is e^{-4x}

The denominator $(y_1)^2 = (e^{-2x})^2 = e^{-4x}$

So $u = \int \frac{e^{-4x}}{e^{-4x}}\,dx = \int dx = x$

$y_2 = y_1 u = x e^{-2x}$

The general solution $y = c_1 y_1 + c_2 y_2$

$y = c_1 e^{-2x} + c_2 x e^{-2x}$

Apply $y(0) = 1$ and $y'(0) = 1$

$y' = -2c_1 e^{-2x} + c_2 e^{-2x} - 2c_2 x e^{-2x}$

$y(0) = c_1 e^{0} + c_2 \cdot 0\, e^{0} = 1 \implies c_1 = 1$

$y'(0) = -2c_1 e^{0} + c_2 e^{0} - 2c_2 \cdot 0 \cdot e^{0} = 1$

$$-2C_1 + C_2 = 1 \implies C_2 = 1 + 2C_1 = 1 + 2 = 3$$

The solution to the IVP

$$\boxed{y = e^{-2x} + 3x e^{-2x}}$$

Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order[1], linear, homogeneous equation with constant coefficients

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0,\quad \text{with } a\neq 0.$$

If we put this in normal form, we get

$$\frac{d^2y}{dx^2}=-\frac{b}{a}\frac{dy}{dx}-\frac{c}{a}y.$$

Question: What sorts of functions y could be expected to satisfy

$$y''=(\text{constant})\,y'+(\text{constant})\,y?$$

$y=e^{mx}$, Sine + Cosines

[1] We'll extend the result to higher order at the end of this section.

We look for solutions of the form $y = e^{mx}$ with m constant.

$$ay'' + by' + cy = 0$$

Sub in $y = e^{mx}$.

$$y' = me^{mx}, \quad y'' = m^2 e^{mx}$$

$$a(m^2 e^{mx}) + b(me^{mx}) + ce^{mx} = 0$$

$$e^{mx}(am^2 + bm + c) = 0$$

This holds if m satisfies

$$am^2 + bm + c = 0$$

This is a quadratic equation for m.

Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

There are three cases:

I $b^2 - 4ac > 0$ and there are two distinct real roots $m_1 \neq m_2$

II $b^2 - 4ac = 0$ and there is one repeated real root $m_1 = m_2 = m$

III $b^2 - 4ac < 0$ and there are two roots that are complex conjugates $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$.

Case I: Two distinct real roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac > 0.$$

There are two different roots m_1 and m_2. A fundamental solution set consists of

$$y_1 = e^{m_1 x} \quad \text{and} \quad y_2 = e^{m_2 x}.$$

The general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

Example

Find the general solution of the ODE.

$$y'' - 2y' - 2y = 0$$

The characteristic equation is

$$m^2 - 2m - 2 = 0.$$

Find the roots. Completing the square

$$m^2 - 2m + 1 = 2 + 1$$

$$(m-1)^2 = 3$$

$$m - 1 = \pm\sqrt{3} \implies m = 1 \pm \sqrt{3}$$

We have two different real numbers

$$m_1 = 1+\sqrt{3}, \quad m_2 = 1-\sqrt{3}$$

$$y_1 = e^{(1+\sqrt{3})x}, \quad y_2 = e^{(1-\sqrt{3})x}$$

The general solution

$$y = c_1 e^{(1+\sqrt{3})x} + c_2 e^{(1-\sqrt{3})x}$$

Case II: One repeated real root

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0$$

There is only one real, double root, $m = \dfrac{-b}{2a}$.

Use reduction of order to find the second solution to the equation (in standard form)

$$y'' + \frac{b}{a}y' + \frac{c}{a}y = 0 \quad \text{given one solution} \quad y_1 = e^{-\frac{b}{2a}x}$$

$y_2 = y_1 u$, $\quad u = \displaystyle\int \frac{e^{-\int P(x)dx}}{(y_1)^2}\,dx$

$P(x) = \frac{b}{a}$, $\quad -\int P(x)dx = -\int \frac{b}{a}\,dx = -\frac{b}{a}x$

The numerator is $e^{-\frac{b}{a}x}$.

The denominator is $(y_1)^2 = \left(e^{\frac{-b}{2a}x}\right)^2 = e^{\frac{-b}{a}x}$

$$u = \int \frac{e^{\frac{-b}{a}x}}{e^{\frac{-b}{a}x}}\,dx = \int dx = x$$

$$y_2 = u y_1 = x y_1 = x\, e^{\frac{-b}{2a}x}$$

Case II: One repeated real root

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0$$

If the characteristic equation has one real repeated root m, then a fundamental solution set to the second order equation consists of

$$y_1 = e^{mx} \quad \text{and} \quad y_2 = xe^{mx}.$$

The general solution is

$$y = c_1 e^{mx} + c_2 xe^{mx}.$$