June 28 Math 2306 sec. 53 Summer 2022

Section 8: Homogeneous Equations with Constant Coefficients

We are considering second order, linear, homogeneous ODEs with constant coefficients.

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0, \quad \text{with } a\neq 0.$$

We looked for solutions of the form $y=e^{mx}$ for constant m and arrived at the characteristic equation

$$am^2+bm+c=0.$$

If m is a solution of the characteristic equation, then $y=e^{mx}$ is a solution of the differential equation. The characteristic equation may have two distinct real roots, one repeated real root, or complex conjugate roots.

Case I: Two distinct real roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac > 0.$$

There are two different roots m_1 and m_2. A fundamental solution set consists of

$$y_1 = e^{m_1 x} \quad \text{and} \quad y_2 = e^{m_2 x}.$$

The general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

Case II: One repeated real root

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0$$

If the characteristic equation has one real repeated root m, then a fundamental solution set to the second order equation consists of

$$y_1 = e^{mx} \quad \text{and} \quad y_2 = xe^{mx}.$$

The general solution is

$$y = c_1 e^{mx} + c_2 xe^{mx}.$$

Case III: Complex conjugate roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac < 0$$

The two roots of the characteristic equation will be

$$m_1 = \alpha + i\beta \quad \text{and} \quad m_2 = \alpha - i\beta \quad \text{where} \quad i^2 = -1.$$ $\beta > 0$

We want our solutions in the form of ***real valued*** functions. We start by writing a pair of solutions

$$Y_1 = e^{(\alpha+i\beta)x} = e^{\alpha x}e^{i\beta x}, \quad \text{and} \quad Y_2 = e^{(\alpha-i\beta)x} = e^{\alpha x}e^{-i\beta x}.$$

We will use the **principle of superposition** to write solutions y_1 and y_2 that do not contain the complex number i.

Deriving the solutions Case III

Recall Euler's Formula[1] : $e^{i\theta} = \cos\theta + i\sin\theta$.

$$Y_1 = e^{\alpha x} e^{i\beta x} = e^{\alpha x}\left(\cos(\beta x) + i\sin(\beta x)\right)$$

$$= e^{\alpha x}\cos(\beta x) + i\, e^{\alpha x}\sin(\beta x)$$

$$Y_2 = e^{\alpha x} e^{-i\beta x} = e^{\alpha x}\left(\cos(\beta x) - i\sin(\beta x)\right)$$

$$= e^{\alpha x}\cos(\beta x) - i\, e^{\alpha x}\sin(\beta x)$$

Let $y_1 = \frac{1}{2}(Y_1 + Y_2) = \frac{1}{2}\left(2e^{\alpha x}\cos(\beta x)\right) = e^{\alpha x}\cos(\beta x)$

[1] As the sine is an odd function $e^{-i\theta} = \cos\theta - i\sin\theta$.

and $y_2 = \frac{1}{2i}(\psi_1 - \psi_2) = \frac{1}{2i}\left(2i\, e^{\alpha x} \sin(\beta x)\right) = e^{\alpha x} \sin(\beta x)$

The solutions we'll use are

$$y_1 = e^{\alpha x} \cos(\beta x) \quad \text{and} \quad y_2 = e^{\alpha x} \sin(\beta x)$$

Case III: Complex conjugate roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac < 0$$

Let α be the real part of the complex roots and β be the imaginary part of the complex roots. Then a fundamental solution set is

$$y_1 = e^{\alpha x}\cos(\beta x) \quad \text{and} \quad y_2 = e^{\alpha x}\sin(\beta x).$$

The general solution is

$$y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x).$$

Example

Constant coef.

Find the general solution of $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 6x = 0.$

Characteristic eqn $\quad m^2 + 4m + 6 = 0$

quadratic formula

$$m = \frac{-4 \pm \sqrt{4^2 - 4(1)(6)}}{2(1)} = \frac{-4 \pm \sqrt{-8}}{2}$$

$$= \frac{-4 \pm 2\sqrt{2}\, i}{2} = -2 \pm \sqrt{2}\, i$$

$$m = \alpha \pm i\beta \implies \alpha = -2, \quad \beta = \sqrt{2}$$

$$x_1 = e^{-2t}\cos(\sqrt{2}\,t), \quad x_2 = e^{-2t}\sin(\sqrt{2}\,t)$$

The general solution

$$\boxed{x = c_1 e^{-2t}\cos(\sqrt{2}\,t) + c_2 e^{-2t}\sin(\sqrt{2}\,t)}$$

Higer Order Linear Constant Coefficient ODEs

- The same approach applies. For an n^{th} order equation, we obtain an n^{th} degree polynomial.

- Complex roots must appear in conjugate pairs (due to real coefficients) giving a pair of solutions $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$ for each pair of complex roots.

- It may require a computer algebra system to find the roots for a high degree polynomial.

Higer Order Linear Constant Coefficient ODEs: Repeated roots.

- For an n^{th} degree polynomial, m may be a root of multiplicity k where $1 \le k \le n$.

- If a real root m is repeated k times, we get k linearly independent solutions

$$e^{mx}, \quad xe^{mx}, \quad x^2 e^{mx}, \quad \ldots, \quad x^{k-1} e^{mx}$$

or in conjugate pairs cases $2k$ solutions

$$e^{\alpha x} \cos(\beta x),\ e^{\alpha x} \sin(\beta x), \quad xe^{\alpha x} \cos(\beta x),\ xe^{\alpha x} \sin(\beta x), \ldots,$$

$$x^{k-1} e^{\alpha x} \cos(\beta x),\ x^{k-1} e^{\alpha x} \sin(\beta x)$$

Example

Find the general solution of the ODE.

$y''' - 3y'' + 3y' - y = 0$ Const. coefs, homogeneous

The characteristic eqn is

$$m^3 - 3m^2 + 3m - 1 = 0 \implies (m-1)^3 = 0$$

$m = 1$, triple root.

$y_1 = e^x$, $y_2 = xe^x$ and $y_3 = x^2 e^x$

The general solution

$$\boxed{y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x}$$

Example

Find the general solution of the ODE.

$y^{(4)}+3y''-4y=0$ homogeneous, constant coef.

Characteristic equation

$m^4+3m^2-4=0$ factor

$(m^2+4)(m^2-1)=0$

$(m^2+4)(m-1)(m+1)=0$

$m-1=0 \Rightarrow m_1=1 \Rightarrow y_1=e^x$

$m+1=0 \Rightarrow m_2=-1 \Rightarrow y_2=e^{-x}$

$m^2+4=0 \Rightarrow m^2=-4 \quad m=\pm\sqrt{-4}=\pm 2i$

$m=\alpha\pm i\beta \Rightarrow \alpha=0,\ \beta=2$

$y_3=e^{0x}\cos(2x)=\cos(2x)$

$y_4=e^{0x}\sin(2x)=\sin(2x)$

The general solution

$$y=c_1e^{x}+c_2e^{-x}+c_3\cos(2x)+c_4\sin(2x)$$

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials, e^{kx}, k - constant
- sines and/or cosines, $\sin(kx)$, $\cos(kx)$
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Motivating Example[2]

Find a particular solution of the ODE

$$y'' - 4y' + 4y = 8x + 1$$

The left side is constant coefficient and the right side $g(x) = 8x+1$ is a polynomial.

Focusing on y_p, what kind of function might y_p be? We guess that y_p is also a 1st degree polynomial.

Set $y_p = Ax + B$ where A, B are constant.

[2] **We're only ignoring the y_c part to illustrate the process.**

Sub this into the ODE. We need

$$y_p'' - 4y_p' + 4y_p = 8x + 1$$

$$y_p = Ax + B, \quad y_p' = A, \quad y_p'' = 0$$

$$0 - 4(A) + 4(Ax + B) = 8x + 1$$

$$\underline{4A}x + (\underline{-4A + 4B}) = \underline{8}x + \underline{1}$$

Match like terms

$$4Ax = 8x \Rightarrow A = 2$$

$$-4A + 4B = 1 \Rightarrow 4B = 1 + 4A$$

$$B = \frac{1 + 4(2)}{4} = \frac{9}{4}$$

So $y_p = Ax+B$ is a solution

if $A=2$ and $B=\frac{9}{4}$.

That is, $y_p = 2x + \frac{9}{4}$.

The Method: Assume y_p has the same **form** as $g(x)$

$$y'' - 4y' + 4y = 6e^{-3x}$$

The left is constant coefficient, the right side $g(x) = 6e^{-3x}$ is an exponential.

g is a constant times e^{-3x}.

Set $y_p = Ae^{-3x}$

Sub in $y_p' = -3Ae^{-3x}$, $y_p'' = 9Ae^{-3x}$

$$y_p'' - 4y_p' + 4y_p = 6e^{-3x}$$

$$9Ae^{-3x} - 4(-3Ae^{-3x}) + 4(Ae^{-3x}) = 6e^{-3x}$$

$$e^{-3x}(9A + 12A + 4A) = 6e^{-3x}$$

$$25Ae^{-3x} = 6e^{-3x}$$

Matching like terms

$$25A = 6 \Rightarrow A = \frac{6}{25}$$

The particular solution

$$\boxed{y_p = \frac{6}{25}e^{-3x}}$$

Make the form general

$$y'' - 4y' + 4y = 16x^2$$

Constant coef. left and $g(x) = 16x^2$ is a polynomial.

Suppose we see $g(x)$ as a constant times x^2. Set

$$y_p = Ax^2$$

$$y_p' = 2Ax$$

$$y_p'' = 2A.$$

$$y_p'' - 4y_p' + 4y_p = 16x^2$$

$$2A - 4(2Ax) + 4(Ax^2) = 16x^2$$

$$4Ax^2 - 8Ax + 2A = 16x^2 + 0x + 0$$

Match like terms.

$4A = 16 \implies A = 4$

$-8A = 0$ and $2A = 0 \implies A = 0$

not possible!

The guess for y_p is wrong.

$g(x) = 16x^2$ is a 2^{nd} degree polynomial.

Set $y_p = Ax^2 + Bx + C$

Try again. $y_p' = 2Ax + B, \quad y_p'' = 2A$

$$y_p'' - 4y_p' + 4y_p = 16x^2$$

$$2A - 4(2Ax+B) + 4(Ax^2+Bx+C) = 16x^2$$

$$4Ax^2 + (-8A+4B)x + (2A-4B+4C) = 16x^2 + 0x + 0$$

Match like terms

$$4A = 16 \implies A = 4$$

$$-8A + 4B = 0 \implies 4B = 8A \implies B = 2A = 8$$

$$2A - 4B + 4C = 0 \implies 4C = -2A + 4B$$

$$4C = -2(4) + 4(8) = 24$$

$$C = 6$$

The particular solution is

$$y_p = 4x^2 + 8x + 6$$

General Form: sines and cosines

$$y'' - y' = 20\sin(2x)$$

If we assume that $y_p = A\sin(2x)$, taking two derivatives would lead to the equation

$$-4A\sin(2x) - 2A\cos(2x) = 20\sin(2x).$$

This would require (matching coefficients of sines and cosines)

$$-4A = 20 \quad \text{and} \quad -2A = 0.$$

This is impossible as it would require $-5 = 0$!

General Form: sines and cosines

We must think of our equation $y'' - y' = 20\sin(2x)$ as

$$y'' - y' = 20\sin(2x) + 0\cos(2x).$$

The correct format for y_p is

$$y_p = A\sin(2x) + B\cos(2x).$$

Examples of Forms of y_p based on g (Trial Guesses)

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

(a) $g(x) = 1$ (or really any constant)

Constant a.k.a. zero degree polynomial

$y_p = A$

(b) $g(x) = x - 7$ 1st degree polynomial

$y_p = Ax + B$

Examples of Forms of y_p based on g (Trial Guesses)

(c) $g(x) = 5x^2$ 2nd degree poly.

$$y_p = Ax^2 + Bx + C$$

(d) $g(x) = 3x^3 - 5$ 3rd degree poly

$$y_p = Ax^3 + Bx^2 + Cx + D$$

Examples of Forms of y_p based on g (Trial Guesses)

(e) $g(x) = xe^{3x}$ 1st degree poly times e^{3x}

$$y_p = (Ax+B)e^{3x}$$

(f) $g(x) = \cos(7x)$ linear combo of $\cos(7x)$ and $\sin(7x)$.

$$y_p = A\cos(7x) + B\sin(7x).$$

Examples of Forms of y_p based on g (Trial Guesses)

(g) $g(x) = \sin(2x) - \cos(4x)$

linear combos of sine | cos and | 2x 4x

$y_p = A\sin(2x) + B\cos(2x) + C\cos(4x) + D\sin(4x)$

(h) $g(x) = x^2 \sin(3x)$

2nd degree polys times sin(3x) and cos(3x).

$y_p = (Ax^2 + Bx + C)\sin(3x) + (Dx^2 + Ex + F)\cos(3x)$