June 2 Math 2306 sec. 53 Summer 2022

Section 1: Concepts and Terminology

Suppose $y = \phi(x)$ is a differentiable function. We know that $dy/dx = \phi'(x)$ is another (related) function.

For example, if $y = \cos(2x)$, then y is differentiable on $(-\infty, \infty)$. In fact,

$$\frac{dy}{dx} = -2\sin(2x).$$

Even dy/dx is differentiable with $\dfrac{d^2y}{dx^2} = -4\cos(2x)$.

Suppose $y = \cos(2x)$

Note that $$\frac{d^2y}{dx^2} + 4y = 0.$$

← equation for function y that relates it to a rate of change

$y'' = -4\cos(2x)$

$4y = 4\cos(2x)$

$y'' + 4y = -4\cos(2x) + 4\cos(2x) = 0$

A *differential* equation

The equation $$\frac{d^2y}{dx^2} + 4y = 0$$

is an example of a **differential equation.**

Questions: If we only started with the equation, how could we determine that $\cos(2x)$ satisfies it?

Also, is $\cos(2x)$ the only possible function that y could be?

Definition

A **Differential Equation** is an equation containing the derivative(s) of one or more dependent variables, with respect to one or more indendent variables.

Solving a differential equation refers to determining the dependent variable(s)—as function(s).

Independent Variable: will appear as one that derivatives are taken with respect to.

Dependent Variable: will appear as one that derivatives are taken of.

In $y = f(x)$, the y is dependent and the x is independent.

Classifications (ODE versus PDE)

Type: An **ordinary differential equation (ODE)** has exactly one independent variable[1]. For example

$$\frac{dy}{dx} - y^2 = 3x, \quad \text{or} \quad \frac{dy}{dt} + 2\frac{dx}{dt} = t,$$

$$\text{or} \quad y'' + 4y = 0$$

[1] These are the subject of this course.

Classifications (ODE versus PDE)

A **partial differential equation (PDE)** has two or more independent variables. For example

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}, \quad \text{or} \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

The expression $\frac{\partial y}{\partial t}$ is read

the partial derivative of y with respect to t.

It's computed by taking the derivative of $y = f(x, t)$ while keeping the other variable, x, fixed.

Classifications (Order)

Order: The order of a differential equation is the same as the highest order derivative appearing anywhere in the equation.

$$\frac{dy}{dx} - y^2 = 3x$$

1st order ODE

$$y''' + (y')^4 = x^3$$

3rd order ODE

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$$

2nd order PDE

Notations and Symbols

We'll use standard derivative notations:

$$\text{Leibniz:} \quad \frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \ldots \quad \frac{d^ny}{dx^n}, \quad \text{or}$$

$$\text{Prime \& superscripts:} \quad y', \quad y'', \quad \ldots \quad y^{(n)}.$$

Newton's **dot notation** may be used if the independent variable is time. For example if s is a position function, then

$$\text{velocity is } \frac{ds}{dt} = \dot{s}, \quad \text{and acceleration is } \frac{d^2s}{dt^2} = \ddot{s}$$

Notations and Symbols

An n^{th} order ODE, with independent variable x and dependent variable y can always be expressed as an equation

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

where F is some real valued function of $n + 2$ variables.

Our equation $\dfrac{d^2y}{dx^2} + 4y = 0$ has this form where

$$F(x, y, y', y'') = \frac{d^2y}{dx^2} + 4y.$$

Notations and Symbols

Normal Form: If it is possible to isolate the highest derivative term, then we can write a **normal form** of the equation

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)}).$$

Our equation $\frac{d^2 y}{dx^2} + 4y = 0$ can be written in normal form.

$$\frac{d^2 y}{dx^2} = -4y \quad \text{note that} \quad f(x, y, y') = -4y$$

Classifications (Linear Equations)

Linearity: An n^{th} order differential equation is said to be **linear** if it can be written in the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

Example First Order:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Example Second Order:

$$a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Properties of a Linear ODE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

- Each of the coefficients $a_0, \ldots, a_n$ and the right hand side g may depend on the independent variable but not the dependent one.
- y, and its derivatives can only appear as themselves (not squared, square rooted, inside some other function).
- The characteristic structure of the left side is that

  $$y, \quad \frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}$$

  are multiplied by functions of the independent variable and added together.

Examples of Linear ODEs

$$y'' + 4y = 0$$

$a_0(x) = 4$

$a_1(x) = 0$

$a_2(x) = 1$

$g(x) = 0$

$$t^2 \frac{d^2x}{dt^2} + 2t\frac{dx}{dt} - x = e^t$$

$a_0(t) = -1$

$a_1(t) = 2t$

$a_2(t) = t^2$

$g(t) = e^t$

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Examples of Nonlinear ODEs

$$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 = x^3 \qquad\qquad u'' + u' = \cos u$$

$(y')^4$ is a nonlinear term

u is dependent so $\cos u$ is a nonlinear term.

Example: Classification

Identify the independent and dependent variables. Determine the order of the equation. State whether it is linear or nonlinear.

(a) $y'' + 2ty' = \cos t + y - y'''$

$$y''' + y'' + 2ty' - y = \cos t$$

Independent: t

dependent: y

order: 3

It is linear

(b) $\ddot{\theta} + \dfrac{g}{\ell}\sin\theta = 0$ g and ℓ are constant

Independent variable time

dependent θ

order 2

nonlinear due to $\sin\theta$ term.

Solution of $F(x, y, y', \ldots, y^{(n)}) = 0$ (*)

Solution or Explicit Solution

Definition: A function ϕ defined on an interval[2] I and possessing at least n continuous derivatives on I is a **solution** of (*) on I if upon substitution (i.e. setting $y = \phi(x)$) the equation reduces to an identity.

Example: $\phi(x) = \cos(2x)$ is a **solution** of $y'' + 4y = 0$ on $(-\infty, \infty)$ because

- it is twice differentiable, and
- when we set $y = \cos(2x)$ in the equation, it gives a true statement (namely, $0 = 0$).

[2] The interval is called the *domain of the solution* or the *interval of definition*.

Examples:

Verify that the given function is an solution of the ODE on the indicated interval. The c_1 and c_2 are constants.

$$\phi(x) = c_1 x + \frac{c_2}{x}, \quad I = (0, \infty), \quad x^2 y'' + xy' - y = 0$$

• Note ϕ is twice continuously diff'ble

Do the substitution, set $y = \phi(x) = c_1 x + c_2 x^{-1}$

$$y' = c_1 - c_2 x^{-2}$$

$$y'' = 2 c_2 x^{-3}$$

$$x^2 y'' + x y' - y \stackrel{?}{=} 0$$

$$x^2(2c_2x^{-3}) + x(c_1 - c_2x^{-2}) - (c_1x + c_2x^{-1}) \overset{?}{=} 0$$

$$2c_2x^{-1} + c_1x - c_2x^{-1} - c_1x - c_2x^{-1} \overset{?}{=} 0$$

$$x(c_1 - c_1) + x^{-1}(2c_2 - c_2 - c_2) = 0$$

$$0 = 0$$

Solution of $F(x, y, y', \dots, y^{(n)}) = 0$ (*)

Implicit Solution

Definition: An **implicit solution** of (*) is a relation $G(x, y) = 0$ provided there exists at least one function $y = \phi$ that satisfies both the differential equation (*) and this relation.

Recall that a **relation** is an equation in the two variables x and y. Something like

$$x^2 + y^2 = 4, \quad \text{or} \quad xy = e^y$$

would be examples of relations.

Example: Implicitly Defined Solution(s)

Verify that the relation(left) defines and implicit solution of the differential equation (right).

$$y^2 - 2x^2y = 1, \qquad \frac{dy}{dx} = \frac{2xy}{y - x^2}$$

We assume the relation. That is, suppose

$$y^2 - 2x^2 y = 1$$

We'll use implicit differentiation to show

$$\frac{dy}{dx} = \frac{2xy}{y-x^2}$$

$$y^2 - 2x^2y = 1 \qquad \text{take } \frac{d}{dx}$$

It may not be possible to clearly identify the domain of definition of an implicit solution.

$$2y\frac{dy}{dx} \div 2\left(2xy + x^2\frac{dy}{dx}\right) = 0$$

Isolate $\frac{dy}{dx}$:

$$y\frac{dy}{dx} - 2xy - x^2\frac{dy}{dx} = 0$$

$$(y - x^2)\frac{dy}{dx} = 2xy$$

$$\Longrightarrow \quad \frac{dy}{dx} = \frac{2xy}{y - x^2}$$

assuming $y \neq x^2$

Function vs Solution

The interval of defintion has to be an interval.

Consider the ODE

$$\frac{dy}{dx} = -y^2.$$

The function $y = \dfrac{1}{x}$ is a solution. The domain of $f(x) = \dfrac{1}{x}$

- as a **function** could be stated as $(-\infty, 0) \cup (0, \infty)$.
- as a **solution** to an ODE could be stated as $(0, \infty)$, or as $(-\infty, 0)$.

In the absence of additional information, we'll usually take the interval of definition to be the largest possible one (or one of the largest possible ones).

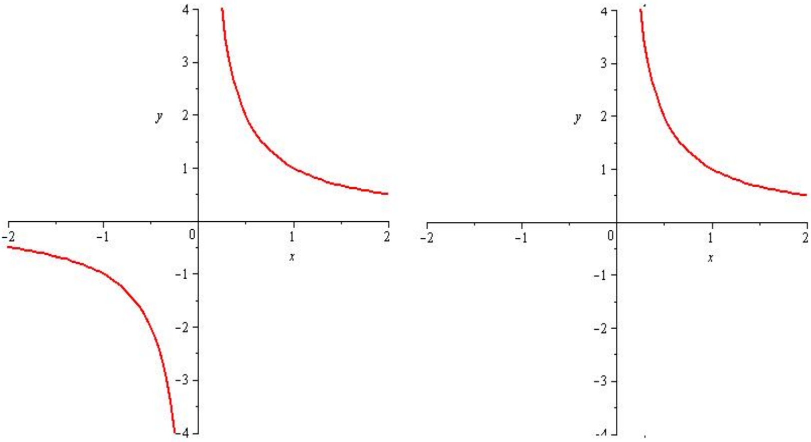

Figure: Left: Plot of $f(x) = \frac{1}{x}$ as a **function**. Right: Plot of $f(x) = \frac{1}{x}$ as a possible **solution** of an ODE.

Systems of ODEs

Sometimes we want to consider two or more dependent variables that are functions of the same independent variable. The ODEs for the dependent variables can depend on one another. Some examples of relevant situations are

- predator and prey
- competing species
- two or more masses attached to a system of springs
- two or more composite fluids in attached tank systems

Such systems can be **linear** or **nonlinear**.

Example of Nonlinear System

$$
\begin{aligned}
\frac{dx}{dt} &= -\alpha x + \beta xy \\
\frac{dy}{dt} &= \gamma y - \delta xy
\end{aligned}
$$

This is known as the **Lotka-Volterra** predator-prey model. $x(t)$ is the population (density) of predators, and $y(t)$ is the population of prey. The numbers α, β, γ and δ are nonnegative constants.
This model is built on the assumptions that

- in the absence of predation, prey increase exponentially
- in the absence of predation, predators decrease exponentially,
- predator-prey interactions increase the predator population and decrease the prey population.

Example of a Linear System

$$\frac{di_2}{dt} = -2i_2 - 2i_3 + 60$$
$$\frac{di_3}{dt} = -2i_2 - 5i_3 + 60$$

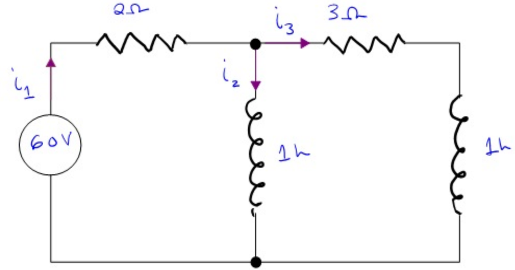

Figure: Electrical Network of resistors and inductors showing currents i_2 and i_3 modeled by this system of equations.

Solution of a System

When we talk about a **solution** to a system of ODEs, we mean a set of functions, one for each dependent variable. For example, a solution to

$$\begin{aligned}\frac{di_2}{dt} &= -2i_2 - 2i_3 + 60\\ \frac{di_3}{dt} &= -2i_2 - 5i_3 + 60\end{aligned}$$

would have to include functions for both of i_2 and i_3.

A fun exercise is to show that

$$\begin{aligned}i_2(t) &= 30 - 24e^{-t} - 6e^{-t}\\ i_3(t) &= 12e^{-t} - 12e^{-6t}\end{aligned}$$

gives a solution. This is what you get if you assume that the initial currents are all zero.

Systems of ODEs

There are various approaches to solving a system of differential equations. These can include

- elimination (try to eliminate a dependent variable),
- matrix techniques,
- Laplace transforms[3]
- numerical approximation techniques

[3]We'll consider this later.

Some Terms

- A **parameter** is an unspecified constant (such as c_1 and c_2 in the last example with $\phi(x) = c_1 x + \frac{c_2}{x}$).
- A **family of solutions** is a collection of solution functions that only differ by a parameter.
- An ***n*-parameter family of solutions** is one containing n parameters (e.g. $\phi(x) = c_1 x + \frac{c_2}{x}$ is a 2 parameter family).
- A **particular solution** is one with no arbitrary constants in it.
- The **trivial solution** is the simple constant function $y = 0$.
- An **integral curve** is the graph of one solution (perhaps from a family).