June 30 Math 2306 sec. 53 Summer 2022

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Method of Undetermined Coefficients

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

- The left must be constant coefficient
- the right must be from one of the given classes of functions (polynomial, exponential, sine/cosine)
- we assume that a particular solution y_p is of the same general type as g.
- the general solution will be $y_c + y_p$

Examples of Forms of y_p based on g (Trial Guesses)

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

(a) $g(x) = 1$ (or really any nonzero constant)

$$y_p = A$$

(b) $g(x) = x - 7$

$$y_p = Ax + B$$

(c) $g(x) = 5x^2$

$$y_p = Ax^2 + Bx + C$$

(d) $g(x) = 3x^3 - 5$

$$y_p = Ax^3 + Bx^2 + Cx + D$$

Examples of Forms of y_p based on g (Trial Guesses)

(e) $g(x) = xe^{3x}$

$$y_p = (Ax + B)e^{3x}$$

(f) $g(x) = \cos(7x)$

$$y_p = A\cos(7x) + B\sin(7x)$$

(g) $g(x) = \sin(2x) - \cos(4x)$

$$y_p = A\sin(2x) + B\cos(2x) + C\sin(4x) + D\cos(4x)$$

(h) $g(x) = x^2\sin(3x)$

$$y_p = (Ax^2 + Bx + C)\sin(3x) + (Dx^2 + Ex + F)\cos(3x)$$

Examples of Forms of y_p based on g (Trial Guesses)

(i) $g(x) = e^x \cos(2x)$ Linear combo of $e^x \cos(2x)$ and $e^x \sin(2x)$

$$y_p = Ae^x \cos(2x) + Be^x \sin(2x)$$

(j) $g(x) = xe^{-x} \sin(\pi x)$
Think of the x factor as a first degree polynomial. This is a linear combo of first degree polynomials times exponentials e^{-x} times sines and cosines of πx.

$$y_p = (Ax + B)e^{-x} \sin(\pi x) + (Cx + D)e^{-x} \cos(\pi x)$$

The Superposition Principle

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x) + \ldots + g_k(x)$$

The principle of superposition for nonhomogeneous equations tells us that we can find y_p by considering separate problems

$$y_{p_1} \text{ solves} \quad a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x)$$

$$y_{p_2} \text{ solves} \quad a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_2(x),$$

and so forth.

Then $y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_k}$.

The Superposition Principle

Example: Determine the correct form of the particular solution using the method of undetermined coefficients for the ODE

$$y'' - 4y' + 4y = 6e^{-3x} + 16x^2$$

Consider two problems:

Find y_{p_1} that solves

$$y'' - 4y' + 4y = 6e^{-3x}$$

$$y_{p_1} = Ae^{-3x}$$

And find y_{p_2} that solves

$$y'' - 4y' + 4y = 16x^2$$

$$y_{p_2} = Bx^2 + Cx + D$$

The form for y_p is

$$y_p = Ae^{-3x} + Bx^2 + Cx + D$$

A Glitch!

What happens if the assumed form for y_p is part[1] of y_c? Consider applying the process to find a particular solution to the ODE

$$y'' - y' = 3e^x$$

Constant coef left and exponential right.

$g(x) = 3e^x$, guess $y_p = Ae^x$

Let's try to determine A.

$y_p' = Ae^x$, $y_p'' = Ae^x$

$$y_p'' - y_p' = 3e^x$$

[1] A term in $g(x)$ is contained in a fundamental solution set of the associated homogeneous equation.

$$Ae^x - Ae^x = 3e^x$$

$$0 = 3e^x$$

need $0 = 3$

this is always false.

The guess $y_p = Ae^x$ is a solution to the associated homogeneous equation . .

* Multiplying by x, setting

$$y_p = Axe^x$$

will result in finding y_p

Cases: Comparing y_p to y_c

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x) + \ldots + g_k(x)$$

Consider one of the g's, say $g_i(x)$. We write out the guess for y_{p_i} and compare it to $y_c(x)$.

Case I: The guess for y_{p_i} **DOES NOT** have any like terms in common with y_c.

Then our guess for y_{p_i} will work as written. We do the substitution to find the A, B, etc.

Cases: Comparing y_p to y_c

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x) + \ldots + g_k(x)$$

Consider one of the g's, say $g_i(x)$. We write out the guess for y_{p_i} and compare it to $y_c(x)$.

Case II: The guess for y_{p_i} **DOES** have a like term in common with y_c.

Then we multiply our guess at y_{p_i} by x^n where n is the smallest positive integer such that our new guess $x^n y_{p_i}$ does not have any like terms in common with y_c. Then we take this new guess and substitute to find the A, B, etc.

Case II Examples

Find the general solution of the ODE.

$$y'' - 2y' + y = -4e^x$$

Constant coef left and exponential right.

Find y_c: y_c solves $y'' - 2y' + y = 0$

Characteristic eqn $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0 \Rightarrow m = 1$ double root

$y_1 = e^x$, $y_2 = xe^x$, $y_c = c_1 e^x + c_2 x e^x$

Find y_p: $g(x) = -4e^x$ guess $y_p = Ae^x$ ✗ duplicates y_1

try again $\quad y_p = Axe^x$

again $\quad y_p = Ax^2e^x$

The correct form is $y_p = Ax^2e^x$.

$$y_p'' - 2y_p' + y_p = -4e^x$$

$$y_p = Ax^2e^x$$

$$y_p' = Ax^2e^x + 2Axe^x$$

$$y_p'' = Ax^2e^x + 4Axe^x + 2Ae^x$$

$$Ax^2e^x + 4Axe^x + 2Ae^x - 2(Ax^2e^x + 2Axe^x) + Ax^2e^x = -4e^x$$

Collect like terms

$$x^2e^x(A-2A+A)+xe^x(4A-4A)+e^x(2A)=-4e^x$$

$$2Ae^x=-4e^x$$

$$2A=-4 \implies A=-2$$

$$y_p=-2x^2e^x$$

The general solution

$$y=c_1e^x+c_2xe^x-2x^2e^x$$

Find the form of the particular soluition

$$y'' - 4y' + 4y = \sin(4x) + xe^{2x}$$

Find y_c: $m^2 - 4m + 4 = 0 \implies (m-2)^2 = 0$

$m = 2$ double root

$$\boxed{y_1 = e^{2x}, \quad y_2 = xe^{2x}}$$

Look for $y_p = y_{p_1} + y_{p_2}$ where y_{p_1}

solves $y'' - 4y' + 4y = \sin(4x)$ and y_{p_2}

solves $y'' - 4y' + 4y = xe^{2x}$

$g_1(x) = \sin(4x)$

set $y_{p_1} = A\sin(4x) + B\cos(4x)$ ✓

no like terms in common w/ y_c

$$\boxed{y_1 = e^{2x}, \quad y_2 = xe^{2x}}$$

$g_2(x) = xe^{2x}$, $\quad y_{p_2} = (Cx+D)e^{2x}x^2$

$= Cx^3e^{2x} + Dx^2e^{2x}$ ✓

$$y_p = A\sin(4x) + B\cos(4x) + Cx^3e^{2x} + Dx^2e^{2x}$$

Solve the IVP

$$y'' - y = 4e^{-x} \quad y(0) = -1, \quad y'(0) = 1$$

The ODE is constant coef, non homogeneous with exponential right hand side.

Find y_c: $m^2 - 1 = 0 \implies (m-1)(m+1) = 0$

$m = 1$ or $m = -1$

$y_1 = e^x$, $y_2 = e^{-x}$ so $y_c = c_1 e^x + c_2 e^{-x}$

Find y_p: $g(x) = 4e^{-x}$, $y_p = Ae^{-x}x = Axe^{-x}$

The correct form is $y_p = Axe^{-x}$.

$$y_p' = Ae^{-x} - Axe^{-x}$$

$$y_p'' = -Ae^{-x} - Ae^{-x} + Axe^{-x} = -2Ae^{-x} + Axe^{-x}$$

$$y_p'' - y_p = 4e^{-x}$$

$$-2Ae^{-x} + Axe^{-x} - Axe^{-x} = 4e^{-x}$$

$$-2Ae^{-x} = 4e^{-x} \Rightarrow A = -2$$

Hence $y_p = -2xe^{-x}$

The general solution is

$$y = C_1 e^{x} + C_2 e^{-x} - 2xe^{-x}$$

Now, apply $y(0)=-1$, $y'(0)=1$

$$y' = c_1 e^{x} - c_2 e^{-x} - 2e^{-x} + 2x e^{-x}$$

$$y(0) = c_1 e^0 + c_2 e^0 - 2\cdot 0 e^0 = -1$$

$$c_1 + c_2 = -1$$

$$y'(0) = c_1 e^0 - c_2 e^0 - 2e^0 + 2\cdot 0\cdot e^0 = 1$$

$$c_1 - c_2 - 2 = 1$$

$$c_1 - c_2 = 3$$

Solve

$$\begin{array}{l} c_1 + c_2 = -1 \\ c_1 - c_2 = 3 \\ \hline \end{array}$$

$2c_1 = 2$ $\quad c_1 = 1$

$2c_2 = -4$ $\quad c_2 = -2$

$$\boxed{\text{The solution to the IVP} \quad y = e^{x} - 2e^{-x} - 2xe^{-x}}$$