June 9 Math 2306 sec. 53 Summer 2022

Section 3: Separation of Variables

Recall that a first order ODE is called **separable** if it has the form

$$\frac{dy}{dx} = g(x)h(y).$$

We solve the equation by separating the variables and integrating:

$$\int \frac{1}{h(y)}\,dy = \int g(x)\,dx.$$

This does assume that dividing by $h(y)$ is acceptable (i.e., $h(y) \neq 0$ on the domain of definition).

Find an explicit solution to the IVP[1]

$$\frac{dQ}{dt} = -2(Q-70), \quad Q(0) = 180$$

$$\frac{dQ}{dt} = g(t)\,h(Q)$$

$$g(t) = -2\ ,\ h(Q) = Q-70$$

Separate

$$\frac{1}{Q-70}\,dQ = -2\,dt$$

$$\int \frac{1}{Q-70}\,dQ = \int -2\,dt \Rightarrow \ln|Q-70| = -2t + C$$

Let's solve for Q: $e^{\ln|Q-70|} = e^{-2t+C} = e^{C} e^{-2t}$

$$|Q-70| = e^{c} e^{-2t}$$

[1]Recall IVP stands for *initial value problem*.

Let $k = \pm e^{c}$ or $k = 0$

$$Q - 70 = k e^{-2t} \Rightarrow Q = k e^{-2t} + 70$$

Apply the initial condition $Q(0) = 180$

$$Q(0) = 180 = k e^{-2(0)} + 70 = k + 70$$

$$\Longrightarrow k = 110$$

The solution to the IVP is

$$\boxed{Q(t) = 110 e^{-2t} + 70}$$

Consider $\lim_{t\to\infty} Q(t) = \lim_{t\to\infty}\left(110e^{-2t}+70\right)$

$= 70$

In Class Exercise:

Take a few minutes to find a family of explicit solutions to the ODE

$$\frac{dy}{dx} = 4x\sqrt{y}.$$

$$y = x^4 + Cx + \frac{C^2}{4} = (x^2 + C/2)^2$$

$$2\sqrt{y} = 2x^2 + C$$

$$y = (x^2 + k)^2$$

Solve the IVP $\quad y' = 4x\sqrt{y}, \quad y(0) = 0$

$$y = (x^2 + k)^2$$

Apply $y(0) = 0$

$$y(0) = 0 = (0^2 + k)^2 \Rightarrow \quad 0 = k^2 \Rightarrow k = 0$$

The soln to the IVP is

$$y = x^4$$

Missed Solution

We made an assumption about being able to divide by $h(y)$ when solving $\frac{dy}{dx} = g(x)h(y)$. This may cause us to not find valid solutions.

The IVP $\frac{dy}{dx} = 4x\sqrt{y}, \quad y(0) = 0$ has two distinct solutions

$$y = x^4, \quad \text{and} \quad y(x) = 0.$$

The second solution **CANNOT** be found by separation of variables. **Why?**

$$\frac{1}{\sqrt{y}}\,dy = 4x\,dx$$

Missed Solutions $\frac{dy}{dx} = g(x)h(y)$.

Theorem: If the number c is a zero of the function h, i.e. $h(c) = 0$, then the constant function $y(x) = c$ is a solution to the differential equation $\frac{dy}{dx} = g(x)h(y)$.

Remark: Such a constant solution may or may not be recovered by separation of variables. We can always look for such solutions in addition to separation of variables.

Solutions Defined by Integrals

The Fundamental Theorem of Calculus tells us that: If g and $\frac{dy}{dx}$ are continuous on an interval $[x_0, b)$ and x is in this interval, then

$$\frac{d}{dx}\int_{x_0}^{x} g(t)\,dt = g(x) \quad \text{and} \quad \int_{x_0}^{x} \frac{dy}{dt}\,dt = y(x) - y(x_0).$$

Theorem: If g is continuous on some interval containing x_0, then the function

$$y = y_0 + \int_{x_0}^{x} g(t)\,dt$$

is a solution of the initial value problem

$$\frac{dy}{dx} = g(x), \quad y(x_0) = y_0$$

Example

Express the solution of the IVP in terms of an integral.

$$\frac{dy}{dx} = \sin(x^2), \quad y(\sqrt{\pi}) = 1$$

$\frac{dy}{dx} = g(x)$, $y(x_0) = y_0$

$y = y_0 + \int_{x_0}^{x} g(t)\, dt$

Here, $g(x) = \sin(x^2)$

$x_0 = \sqrt{\pi}$, $y_0 = 1$

$\boxed{\text{The solution } y = 1 + \int_{\sqrt{\pi}}^{x} \sin(t^2)\, dt}$

Let's verify:

Show $y(\sqrt{\pi}) = 1$

$$y(\sqrt{\pi}) = 1 + \int_{\sqrt{\pi}}^{\sqrt{\pi}} \sin(t^2)\,dt = 1 + 0 = 1$$

Show $\frac{dy}{dx} = \sin(x^2)$

$$\frac{dy}{dx} = \frac{d}{dx}\left(1 + \int_{\sqrt{\pi}}^{x} \sin(t^2)\,dt\right)$$

$$= \frac{d}{dx}(1) + \frac{d}{dx}\int_{\sqrt{\pi}}^{x} \sin(t^2)\,dt$$

$$= 0 + \frac{d}{dx}\int_{\sqrt{\pi}}^{x} \sin(t^2)\,dt = \sin(x^2)$$

y solves the ODE and the initial condition.

Section 4: First Order Equations: Linear

A first order linear equation has the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

If $g(x) = 0$ the equation is called **homogeneous.** Otherwise it is called **nonhomogeneous.**

Provided $a_1(x) \neq 0$ on the interval I of definition of a solution, we can write the **standard form** of the equation

$$P(x) = \frac{a_0(x)}{a_1(x)}$$

$$\frac{dy}{dx} + P(x)y = f(x).$$

$$f(x) = \frac{g(x)}{a_1(x)}$$

We'll be interested in equations (and intervals I) for which P and f are continuous on I.

Solutions (the General Solution)

$$\frac{dy}{dx} + P(x)y = f(x).$$

It turns out the solution will always have a basic form of $y = y_c + y_p$ where

- y_c is called the **complementary** solution and would solve the equation
$$\frac{dy}{dx} + P(x)y = 0$$
(called the associated homogeneous equation), and
- y_p is called the **particular** solution, and is heavily influenced by the function $f(x)$.

The cool thing is that our solution method will get both parts in one process—we won't get this benefit with higher order equations!

Motivating Example

$$x^2\frac{dy}{dx}+2xy = e^x$$

This is not in standard form.

If you look at the left carefully, you might notice that it's a derivative of a product.

$$x^2\frac{dy}{dx}+2xy = \frac{d}{dx}\left(x^2y\right)$$

The ODE is actually

$$\frac{d}{dx}\left(x^2y\right) = e^x$$

To find y, integrate and divide by x^2.

$$\int \frac{d}{dx}(x^2 y)\,dx = \int e^x\,dx$$

$$x^2 y = e^x + C$$

$$\boxed{\text{The solutions are} \quad y = \frac{e^x + C}{x^2}}$$

1 parameter family of solutions

$$y = \underbrace{\frac{C}{x^2}}_{y_c} + \underbrace{\frac{e^x}{x^2}}_{y_p}$$

Derivation of Solution via Integrating Factor

Solve the equation in standard form

$$\frac{dy}{dx} + P(x)y = f(x)$$

We'll find a function $\mu(x)$ such that when we multiply both sides by $\mu(x)$, the left side becomes $\frac{d}{dx}(\mu(x)y)$.

Suppose μ exists. μ times the left side is

$$\mu(x)\left(\frac{dy}{dx} + P(x)y\right) = \mu\frac{dy}{dx} + \mu P(x)y$$

Note $\frac{d}{dx}(\mu y) = \mu \frac{dy}{dx} + \frac{d\mu}{dx} y$

We require

$$\mu \frac{dy}{dx} + \mu P(x) y = \mu \frac{dy}{dx} + \frac{d\mu}{dx} y$$

Subtract $\mu \frac{dy}{dx}$

$$\mu P(x) y = \frac{d\mu}{dx} y$$

Cancel y to get a separable equation for μ

$$\frac{d\mu}{dx} = \mu P(x)$$

$$\Rightarrow \quad \frac{1}{\mu}\, d\mu = P(x)\, dx$$

$$\int \frac{1}{\mu}\, dx = \int P(x)\, dx$$

$$\ln|\mu| = \int P(x)\, dx$$

Let's assume $\mu(x) > 0$

$$\boxed{\mu = e^{\int P(x)\, dx}}$$

General Solution of First Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\, dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)f(x)\, dx = e^{-\int P(x)\, dx}\left(\int e^{\int P(x)\, dx} f(x)\, dx + C\right)$$

Solve the IVP

$$x\frac{dy}{dx} - y = 2x^2, \ x > 0 \quad y(1) = 5$$

Standard form:

$$\frac{dy}{dx} - \frac{1}{x} y = 2x$$

Identify $P(x)$: $P(x) = \frac{-1}{x}$

Get μ : $\mu = e^{\int P(x)\,dx} = e^{\int \frac{-1}{x} dx} = e^{-\ln x}$

$= e^{\ln x^{-1}} = x^{-1}$

$\mu = x^{-1}$

$$x^{-1}\left(\frac{dy}{dx} - \frac{1}{x}y\right) = x^{-1}(2x)$$

$$\frac{d}{dx}\left(x^{-1}y\right) = 2$$

Integrate and divide by μ.

$$\int \frac{d}{dx}\left(x^{-1}y\right)dx = \int 2\,dx$$

$$x^{-1}y = 2x + C$$

$$\Longrightarrow y = \frac{2x+C}{x^{-1}} = 2x^2 + Cx$$

The general solution to the ODE is

$$y = 2x^2 + Cx$$

Apply $y(1) = 5$

$$y(1) = 2(1^2) + C(1) = 5$$

$$2 + C = 5 \Rightarrow C = 3$$

$$\boxed{\text{The solution to the IVP is } \quad y = 2x^2 + 3x}$$