June 13 Math 3260 sec. 51 Summer 2023

Section 1.8: Intro to Linear Transformations

Recall that the product $A\mathbf{x}$ is a vector that is a linear combination of the columns of A.

If the columns of A are vectors in $\mathbb{R}^m$, and there are n of them, then

- A is an $m \times n$ matrix,
- the product $A\mathbf{x}$ is defined for $\mathbf{x}$ in $\mathbb{R}^n$, and
- the vector $\mathbf{b} = A\mathbf{x}$ is a vector in $\mathbb{R}^m$.

Remark: We can think of a matrix A as an **operator that acts** on vectors $\mathbf{x}$ in $\mathbb{R}^n$ (via the product $A\mathbf{x}$) to produce vectors $\mathbf{b}$ in $\mathbb{R}^m$.

Transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$

Definition

A transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

Remark

Another name for a *transformation* is a **function** or **mapping**. The parentheses notation $T(\cdot)$ is typical function notation. A transformation takes a vector as an input and spits out a vector as the output.

Transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$

Function Notation: If a transformation T takes a vector $\mathbf{x}$ in $\mathbb{R}^n$ and maps it to a vector $T(\mathbf{x})$ in $\mathbb{R}^m$, we can write

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

which reads "T maps $\mathbb{R}^n$ into $\mathbb{R}^m$."

And we can write

$$\mathbf{x} \mapsto T(\mathbf{x})$$

which reads "$\mathbf{x}$ maps to T of $\mathbf{x}$."

The following vertically stacked notation is often used:

$$\begin{aligned} T : \mathbb{R}^n &\longrightarrow \mathbb{R}^m \\ \mathbf{x} &\mapsto T(\mathbf{x}) \end{aligned}$$

Key Terms

For $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$,

- $\mathbb{R}^n$ is the **domain**, and
- $\mathbb{R}^m$ is called the **codomain.**
- For $\mathbf{x}$ in the domain, $T(\mathbf{x})$ is called the **image** of $\mathbf{x}$ under T. (We can call $\mathbf{x}$ a **pre-image** of $T(\mathbf{x})$.)
- The collection of all images is called the **range.**
- If $T(\mathbf{x})$ is defined by multiplication by the $m \times n$ matrix A, we may denote this by $\mathbf{x} \mapsto A\mathbf{x}$.

Matrix Transformation Example

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix}$. Define the transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ by the mapping $T(\mathbf{x}) = A\mathbf{x}$.

(a) Find the image of the vector $\mathbf{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ under T.

$$T(\vec{u}) = A\vec{u}$$

$$= \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + 3(-3) \\ 2\cdot 1 + 4(-3) \\ 0\cdot 1 + (-2)(-3) \end{bmatrix} = \begin{bmatrix} -8 \\ -10 \\ 6 \end{bmatrix}$$

Example Continued...

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix}, \qquad \begin{aligned} T : \mathbb{R}^n &\longrightarrow \mathbb{R}^m \\ x &\mapsto A\mathbf{x} \end{aligned}$$

(b) Determine a vector $\mathbf{x}$ in $\mathbb{R}^2$ whose image under T is $\begin{bmatrix} -4 \\ -4 \\ 4 \end{bmatrix}$.

For what $\vec{x}$ is

$T(\vec{x}) = \begin{bmatrix} -4 \\ -4 \\ 4 \end{bmatrix}$? This is equivalent

to the matrix equation $A\vec{x} = \begin{bmatrix} -4 \\ -4 \\ 4 \end{bmatrix}$

i.e., $\begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \\ 4 \end{bmatrix}$.

We can use an augmented matrix

$$\begin{bmatrix} 1 & 3 & -4 \\ 2 & 4 & -4 \\ 0 & -2 & 4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} x_1 = 2 \\ x_2 = -2 \end{array}$$

The solution is

$$\vec{x} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

Example Continued...

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & -2 \end{bmatrix}, \qquad \begin{array}{c} T : \mathbb{R}^n \longrightarrow \mathbb{R}^m \\ x \mapsto A\mathbf{x} \end{array}$$

(c) Determine if $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is in the range of T.

Is there a vector $\vec{x}$ such that

$T(\vec{x}) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$?

$A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ has augmented matrix

$$\begin{bmatrix} 1 & 3 & 1 \\ 3 & 4 & 0 \\ 0 & -2 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is inconsistent

hence $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is <u>not</u> in the range of T.

Linear Transformations

Definition

A transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, is **linear** provided

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for every $\mathbf{u}, \mathbf{v}$ in the domain of T, and

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for every scalar c and vector $\mathbf{u}$ in the domain of T.

Remark 1: These were the two properties (that I claimed were a *big deal*) of the product $A\mathbf{x}$ from section 1.4.

Remark 2: Every matrix transformation (e.g. $\mathbf{x} \mapsto A\mathbf{x}$) is a linear transformation. And every linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ can be stated in terms of a matrix.

A Theorem About Linear Transformations:

Theorem:

If T is a linear transformation, then

(i) $T(\mathbf{0}) = \mathbf{0}$, and

(ii) $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$

for any scalars c, and d and vectors $\mathbf{u}$ and $\mathbf{v}$.

Remark: This second statement says:

> *The image of a linear combination is the linear combination of the images.*

It can be generalized to an arbitrary linear combination[1]

$$T(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k) = c_1T(\mathbf{u}_1) + c_2T(\mathbf{u}_2) + \cdots + c_kT(\mathbf{u}_k).$$

[1] This is called the *principle of superposition*.

An Example on $\mathbb{R}^2$

Let $r > 0$ be a scalar and consider the transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$T(\mathbf{x}) = r\mathbf{x}.$$

This transformation is called a **dilation** if $r > 1$ and a **contraction** if $0 < r < 1$.

Exercise: Show that T is a linear transformation.

We have to show that

$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ and

$T(c\vec{u}) = cT(\vec{u})$

for every $\vec{u}, \vec{v} \in \mathbb{R}^2$ and $c \in \mathbb{R}$.

Let $\vec{u}, \vec{v} \in \mathbb{R}^2$,

$$T(\vec{u}+\vec{v}) = r(\vec{u}+\vec{v}) = r\vec{u} + r\vec{v}$$

$$= T(\vec{u}) + T(\vec{v})$$

by prop. of scaler mult. property (v)

Let $c \in \mathbb{R}$

$$T(c\vec{u}) = r(c\vec{u}) = c(r\vec{u})$$

$$= c\,T(\vec{u})$$

property (vii)

See slides June 1

The Geometry of Dilation/Contraction

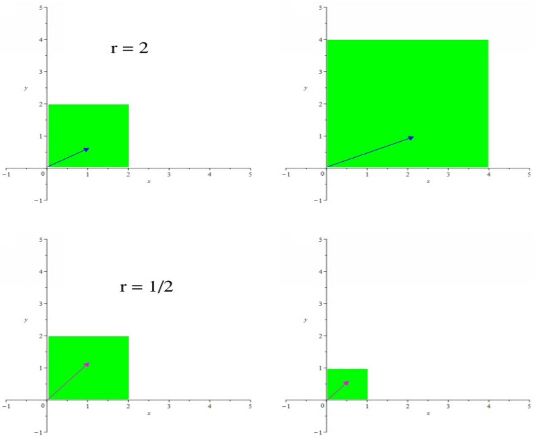


Figure: The 2×2 square in the plane under the dilation $\mathbf{x} \mapsto 2\mathbf{x}$ (top) and the contraction $\mathbf{x} \mapsto \frac{1}{2}\mathbf{x}$ (bottom). Each includes an example of a single vector and its image.

Section 1.9: The Matrix for a Linear Transformation

Recall Linear Transformation

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a **linear transformation** provided for every vector $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ and every scalar c

$$\begin{aligned} T(\mathbf{u}+\mathbf{v}) &= T(\mathbf{u})+T(\mathbf{v}), \quad \text{and} \\ T(c\mathbf{u}) &= cT(\mathbf{u}). \end{aligned}$$

Two Remarks

1. Any mapping defined by matrix multiplication, $\mathbf{x} \mapsto A\mathbf{x}$, is a linear transformation.
2. Every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be realized in terms of matrix multiplication.

Elementary Vectors

Definition: Elementary Vectors

We'll use the notation $\mathbf{e}_i$ to denote the vector in $\mathbb{R}^n$ having a 1 in the i^{th} position and zero everywhere else. The vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are called **elementary** vectors.

For example, the elementary vectors in $\mathbb{R}^2$ are

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The elementary vectors in $\mathbb{R}^3$ are

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Elementary Vectors

Remark:

In general, the elementary vectors are the columns of the $n \times n$ identity matrix.

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$I_n = [\mathbf{e}_1\ \mathbf{e}_2\ \cdots\ \mathbf{e}_n]$$

Matrix of Linear Transformation: an Example

Suppose $T:\mathbb{R}^2\longrightarrow\mathbb{R}^4$ is a linear transformation, and that

$$T(\mathbf{e}_1)=\begin{bmatrix}0\\1\\-2\\4\end{bmatrix},\quad\text{and}\quad T(\mathbf{e}_2)=\begin{bmatrix}1\\1\\-1\\6\end{bmatrix}.$$

Use the fact that T is linear, and the fact that for each $\mathbf{x}$ in $\mathbb{R}^2$ we have

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\end{bmatrix}=x_1\begin{bmatrix}1\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\end{bmatrix}=x_1\mathbf{e}_1+x_2\mathbf{e}_2$$

to find a matrix A such that

$$T(\mathbf{x})=A\mathbf{x}\quad\text{for every}\quad\mathbf{x}\in\mathbb{R}^2.$$

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 4 \end{bmatrix}, \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 6 \end{bmatrix}$$

$$T(\vec{x}) = T(x_1 \vec{e}_1 + x_2 \vec{e}_2)$$

$$= x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2)$$

$$= x_1 \begin{bmatrix} 0 \\ 1 \\ -2 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ -1 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ -2 & -1 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

This holds for any $\vec{x} \in \mathbb{R}^2$, so

$$T(\vec{x}) = A\vec{x} \quad \text{if}$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ -2 & -1 \\ 4 & 6 \end{bmatrix}$$

Standard Matrix of a Linear Transformation

Theorem

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. There exists a unique $m \times n$ matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for every} \quad \mathbf{x} \in \mathbb{R}^n.$$

Moreover, the j^{th} column of the matrix A is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the j^{th} column of the $n \times n$ identity matrix I_n. That is,

$$A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \cdots \quad T(\mathbf{e}_n)].$$

The matrix A is called the **standard matrix** for the linear transformation T.

Example

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the scaling transformation (contraction or dilation for $r > 0$) defined by

$$T(\mathbf{x}) = r\mathbf{x}, \quad \text{for positive scalar } r.$$

Find the standard matrix for T.

The domain is $\mathbb{R}^2$, so we have two elementary vectors, $\vec{e}_1$ and $\vec{e}_2$.

$$T(\vec{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = r\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}$$

$$T(\vec{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = r\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}$$

Calling the standard matrix A

$$A = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}.$$

A Shear Transformation on $\mathbb{R}^2$

Find the standard matrix for the linear transformation from $\mathbb{R}^2 \to \mathbb{R}^2$ that maps $\mathbf{e}_2$ to $\mathbf{e}_2 - \frac{1}{2}\mathbf{e}_1$ and leaves $\mathbf{e}_1$ unchanged.

Calling it T,

$$T(\vec{e}_1) = \vec{e}_1 \qquad T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix}$$

$$T(\vec{e}_2) = \vec{e}_2 - \frac{1}{2}\vec{e}_1 = \begin{bmatrix}0\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-1/2\\1\end{bmatrix}$$

Calling the matrix A $\qquad A = \begin{bmatrix}1 & -1/2\\0 & 1\end{bmatrix}.$

A Shear Transformation on $\mathbb{R}^2$

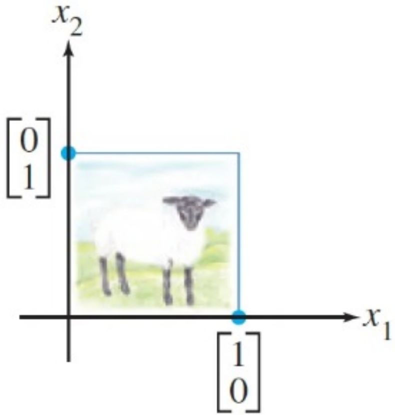
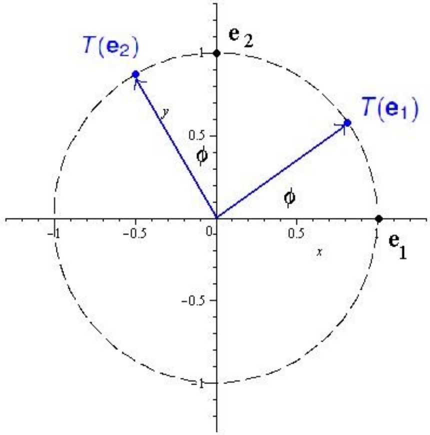


Figure: The unit square under the transformation $\mathbf{x} \mapsto \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} \mathbf{x}$.

A Rotation on $\mathbb{R}^2$

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the rotation transformation that rotates each point in $\mathbb{R}^2$ counter clockwise about the origin through an angle ϕ. Find the standard matrix for T.



Using some basic trigonometry, the points on the unit circle

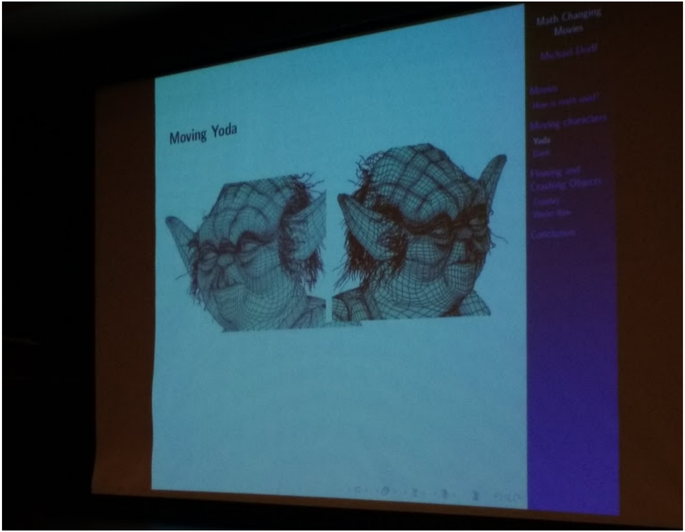
$$T(\mathbf{e}_1) = (\cos\phi, \sin\phi)$$

$$T(\mathbf{e}_2) = (\cos(90^\circ + \phi), \sin(90^\circ + \phi))$$

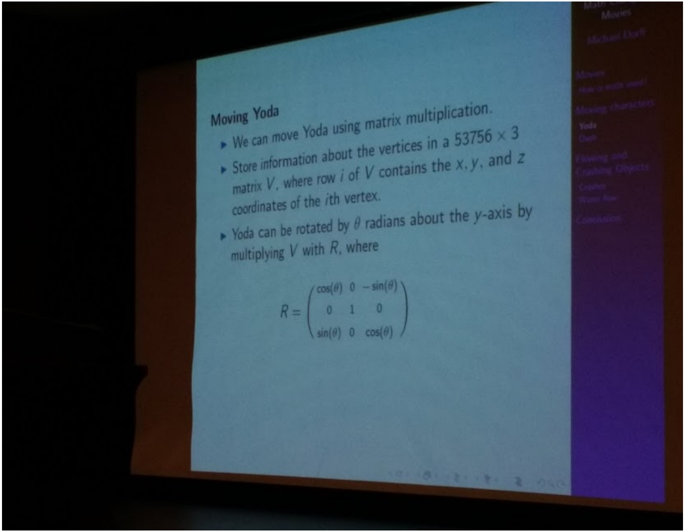
$$= (-\sin\phi, \cos\phi)$$

$$\text{So } A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}.$$

Rotation in Animation



Rotation in Animation



Rotation in Curve Generation

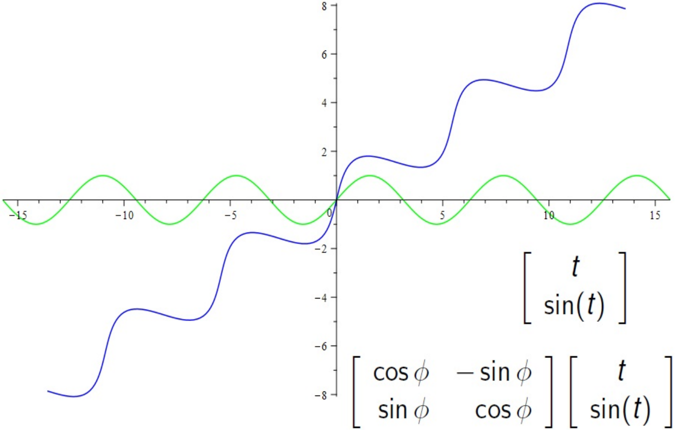


Figure: The curve $y = \sin(x)$ plotted as a vector valued function along with a version rotated through and angle $\phi = \frac{\pi}{6}$.

Example[2]

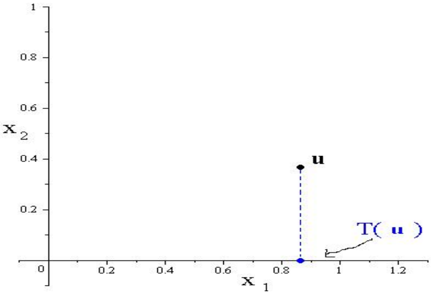
Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the projection transformation that projects each point onto the x_1 axis

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}.$$

Find the standard matrix for T.

$$T(\vec{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T(\vec{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



[2]See pages 77–80 in Lay for matrices associated with other geometric tranformation on $\mathbb{R}^2$

Calling the matrix A,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The Property **Onto**

Definition

A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one $\mathbf{x}$ in $\mathbb{R}^n$—i.e. if the range of T is all of the codomain.

Remark: If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is an **onto** transformation, then the equation

$$T(\mathbf{x}) = \mathbf{b}$$

is always solvable. If T is a linear transformation with standard matrix A, then this is equivalent to saying $A\mathbf{x} = \mathbf{b}$ is always consistent.

Determine if the transformation is onto.

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \mathbf{x}. \quad \leftarrow A$$

Note $T: \mathbb{R}^3 \to \mathbb{R}^2$. We can ask whether $T(\vec{x}) = \vec{b}$ is always solvable. Is $A\vec{x} = \vec{b}$ always consistent?

We can use an augmented matrix

$$[A \; \vec{b}] = \begin{bmatrix} 1 & 0 & 2 & b_1 \\ 0 & 1 & 3 & b_2 \end{bmatrix}$$

It's an rref, and the last column cannot be a pivot column.

$A\vec{x} = \vec{b}$ is always consistent,

T is onto $\mathbb{R}^2$.

The Property **One to One**

Definition

A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **one to one** if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of **at most one x** in $\mathbb{R}^n$.

Remark 1: If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a **one to one** transformation, then the equation

$$T(\mathbf{x}) = T(\mathbf{y}) \quad \text{is only true when} \quad \mathbf{x} = \mathbf{y}.$$

Remark 2: If T is a linear transformation with standard matrix A, being one to one would mean that

whenever $A\mathbf{x} = \mathbf{b}$ is consistent, there is exactly one solution.

Determine if the transformation is one to one.

$T: \mathbb{R}^3 \to \mathbb{R}^2$

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \mathbf{x}.$$

If $A\vec{x} = \vec{b}$ is consistent, is there a unique solution? We know that $A\vec{x} = \vec{b}$ is always consistent.

We had the augmented matrix

$$\begin{bmatrix} 1 & 0 & 2 & b_1 \\ 0 & 1 & 3 & b_2 \end{bmatrix}$$

We see that x_1 and x_2 are basic and x_3 is a free variable,

$A\vec{x} = \vec{b}$ has infinitely many solutions

T is not one to one.

Two Distinct Properties

Note: We considered the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$\mathbf{x} \mapsto \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \mathbf{x}$$

and found that

- it **IS** onto, but
- it **IS NOT** one to one.

This illustrates that, in general, these are distinct properties. Any given transformation may be onto, one to one, neither of these, or both.

Some Theorems about *Onto* and *One to One*

Theorem:

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then T is one to one if and only if the homogeneous equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Theorem:

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then

(i) T is onto if and only if the columns of A span $\mathbb{R}^m$, and

(ii) T is one to one if and only if the columns of A are linearly independent.

Example: Consider $T(x_1, x_2) = (x_1, 2x_1 - x_2, 3x_2)$.

(a) Verify that T is one to one.

$T: \mathbb{R}^2 \to \mathbb{R}^3$ we can find the standard matrix A. We need $T(\vec{e}_1)$ and $T(\vec{e}_2)$.

$$T(1,0) = (1, 2\cdot 0, 3\cdot 0) = (1,2,0)$$

$$T(0,1) = (0, 2\cdot 0 - 1, 3\cdot 1) = (0,-1,3)$$

$$A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 3 \end{bmatrix}$$

We can show that $A\vec{x}=\vec{0}$ has only the trivial solution.

$A\vec{x}=\vec{0}$ has augmented matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 3 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$\Rightarrow \vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Hence T is one to one.

Example Continued... $T(x_1, x_2) = (x_1, 2x_1 - x_2, 3x_2)$

(b) Determine whether T is onto

$$A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 3 \end{bmatrix}$$ we could ask if

$A\vec{x} = \vec{b}$ is always solvable consider the augmented matrix

$$\begin{bmatrix} 1 & 0 & b_1 \\ 2 & -1 & b_2 \\ 0 & 3 & b_3 \end{bmatrix} \xrightarrow{-2R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 0 & b_1 \\ 0 & -1 & b_2 - 2b_1 \\ 0 & 3 & b_3 \end{bmatrix}$$

$3R_2 + R_3 \to R_3$

$$\begin{bmatrix} 1 & 0 & b_1 \\ 0 & -1 & b_2 - 2b_1 \\ 0 & 0 & b_3 + 3b_2 - 6b_1 \end{bmatrix}$$

The last column will be a pivot column for some vectors **b** in $\mathbb{R}^3$. So A**x** = **b** is not always consistent and T is NOT onto.