June 15 Math 3260 sec. 51 Summer 2023

Section 1.9: The Matrix for a Linear Transformation

Recall Linear Transformation

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a **linear transformation** provided for every vector $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ and every scalar c

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}), \quad \text{and}$$

$$T(c\mathbf{u}) = cT(\mathbf{u}).$$

Recall Elementary Vectors

We use the notation $\mathbf{e}_i$ to denote the vector in $\mathbb{R}^n$ having 1 in the i^{th} position and zero everywhere else. There are n such vectors in $\mathbb{R}^n$, and they are the columns of the $n \times n$ identity matrix.

Standard Matrix of a Linear Transformation

Theorem

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. There exists a unique $m \times n$ matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for every} \quad \mathbf{x} \in \mathbb{R}^n.$$

Moreover, the j^{th} column of the matrix A is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the j^{th} column of the $n \times n$ identity matrix I_n. That is,

$$A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \cdots \quad T(\mathbf{e}_n)].$$

The matrix A is called the **standard matrix** for the linear transformation T.

Onto and One to One

Definition

A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each **b** in $\mathbb{R}^m$ is the image of at least one **x** in $\mathbb{R}^n$—i.e. if the range of T is all of the codomain.

Definition

A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be **one to one** if each **b** in $\mathbb{R}^m$ is the image of **at most one x** in $\mathbb{R}^n$.

Some Theorems about *Onto* and *One to One*

Theorem:

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then T is one to one if and only if the homogeneous equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Theorem:

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then

(i) T is onto if and only if the columns of A span $\mathbb{R}^m$, and

(ii) T is one to one if and only if the columns of A are linearly independent.

Remarks

Suppose $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation and A is the standard matrix for T.

- If T is **onto**, then
  - the range of T is $\mathbb{R}^m$,
  - the equation $T(\mathbf{x}) = \mathbf{b}$ is always solvable,
  - the system $A\mathbf{x} = \mathbf{b}$ is always consistent.
- If T is **one to one**, then
  - $T(\mathbf{x}) = T(\mathbf{y})$ implies that $\mathbf{x} = \mathbf{y}$,
  - $A\mathbf{x} = \mathbf{0}$ has no free variables.

Example

Consider the linear transformation
$$T : \mathbb{R}^3 \to \mathbb{R}^2$$
$$(x_1, x_2, x_3) \mapsto (x_3, x_1 + x_2)$$

Determine the set of all preimages[1] of **0**. State the solution as a span.

We want to characterize all $\vec{x}$ in $\mathbb{R}^3$ such that $T(\vec{x}) = \vec{0}$. Let's find the standard matrix A. Let's find $T(\vec{e}_1)$, $T(\vec{e}_2)$, $T(\vec{e}_3)$.

$T(\vec{e}_1) = T(1,0,0) = (0, 1+0) = (0,1)$

$T(\vec{e}_2) = T(0,1,0) = (0, 0+1) = (0,1)$

[1] This actually has a special name. The set of all preimages of the zero vector is called the *kernel* of T.

$T(\vec{e}_3) = T(0,0,1) = (1, 0+0) = (1,0)$

$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. We need the solutions

to $A\vec{x} = \vec{0}$. The augmented matrix is

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ rref}$$

$x_1 = -x_2$

x_2 is free

$x_3 = 0$

solutions $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ 0 \end{bmatrix}$

$= x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

The set is $\text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Section 2.1: Matrix Operations

Recall the convenient notaton for a matrix A

$$A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Here each column is a vector $\mathbf{a}_j$ in $\mathbb{R}^m$. We'll use the additional convenient notation to refer to A by entries

$$A = [a_{ij}].$$

a_{ij} is the entry in **row** i and **column** j.

Main Diagonal & Diagonal Matrices

The **main diagonal** of a matrix consist of the entries a_{ii}.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{22} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}.$$

A **diagonal matrix** is a square matrix, $m = n$, for which all entries **not** on the main diagonal are zero.

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}.$$

Matrix Equality

Matrix Equality:

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal provided they are of the same size, $m \times n$, and

$$a_{ij} = b_{ij} \quad \text{for every} \quad i = 1, \ldots, m \quad \text{and} \quad j = 1, \ldots, n.$$

In this case, we can write

$$A = B.$$

Scalar Multiplication & Matrix Addition

We have two initial operations we can perform on matrices.

Scalar Multiplication:

For $m \times n$ matrix $A = [a_{ij}]$ and scalar c

$$cA = [ca_{ij}].$$

Matrix Addition:

For $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$

$$A + B = [a_{ij} + b_{ij}].$$

Note: The sum of two matrices is only defined if they are of the same size.

Example

Consider the following matrices.

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 4 \\ 7 & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix}$$

Evaluate each expression or state why it fails to exist.

(a) 3B $= \begin{bmatrix} 3(-2) & 3(4) \\ 3(7) & 3(0) \end{bmatrix} = \begin{bmatrix} -6 & 12 \\ 21 & 0 \end{bmatrix}$

Evaluate each expression or state why it fails to exist.

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 4 \\ 7 & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix}$$

(b) A + B

$$= \begin{bmatrix} 1-2 & -3+4 \\ -2+7 & 2+0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 5 & 2 \end{bmatrix}$$

(c) C + A

This is not defined. The dimensions of C and A are different.

Zero Matrix

The $m \times n$ **zero matrix** has a zero in each entry. We'll denote this matrix as O (or $O_{m,n}$ if the size is not clear from the context).

Theorem: Algebraic Properties of Scalar Mult. and Matrix Add.

Let A, B, and C be matrices of the same size and r and s be scalars. Then

(i) $A + B = B + A$

(ii) $(A + B) + C = A + (B + C)$

(iii) $A + O = A$

(iv)[a] $A + (-A) = O$

(v) $r(A + B) = rA + rB$

(vi) $(r + s)A = rA + sA$

(vii) $r(sA) = s(rA) = (rs)A$

(viii) $1A = A$

[a] The term $-A$ denotes $(-1)A$.

Matrix Multiplication

We know that for any $m \times n$ matrix A, the operation **"multiply vectors in $\mathbb{R}^n$ by A"** defines a linear transformation (from $\mathbb{R}^n$ to $\mathbb{R}^m$).

We wish to define matrix multiplication in such a way as to correspond to **function composition.** Thus if

$$S(\mathbf{x}) = B\mathbf{x}, \quad \text{and} \quad T(\mathbf{v}) = A\mathbf{v},$$

then

$$(T \circ S)(\mathbf{x}) = T(S(\mathbf{x})) = A(B\mathbf{x}) = (AB)\mathbf{x}.$$

Matrix Multiplication

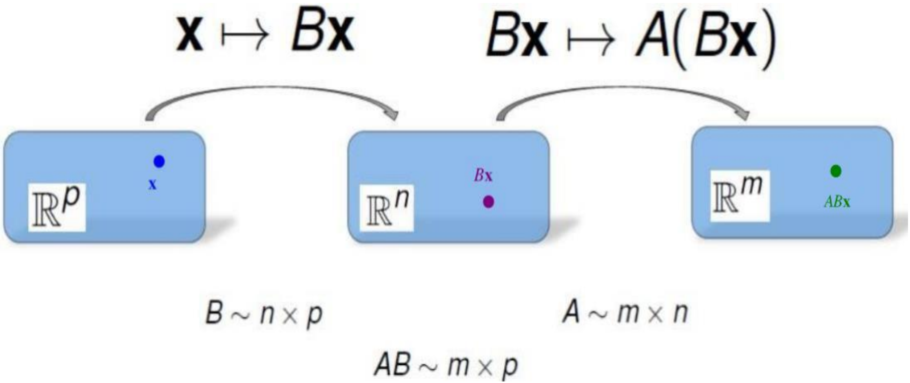

Figure: $\mathbf{x}$ is mapped from $\mathbb{R}^p$ to $B\mathbf{x}$ in $\mathbb{R}^n$. Then $B\mathbf{x}$ in $\mathbb{R}^n$ is mapped to $AB\mathbf{x}$ in $\mathbb{R}^m$. The composition is a mapping $\mathbb{R}^p \to \mathbb{R}^m$. This is only defined if the number of rows of the matrix B is equal to the number of columns of the matrix A.

Matrix Multiplication

$$S : \mathbb{R}^p \longrightarrow \mathbb{R}^n \quad \Longrightarrow \quad B \sim n \times p$$

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad \Longrightarrow \quad A \sim m \times n$$

$$T \circ S : \mathbb{R}^p \longrightarrow \mathbb{R}^m \quad \Longrightarrow \quad AB \sim m \times p$$

$$B\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_p\mathbf{b}_p \Longrightarrow$$

$$A(B\mathbf{x}) = x_1A\mathbf{b}_1 + x_2A\mathbf{b}_2 + \cdots + x_pA\mathbf{b}_p \Longrightarrow$$

$$AB = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_p]$$

The j^{th} column of AB is A times the j^{th} column of B.

Product of Matrices

The product AB is only defined if the number of columns of A (the left matrix) matches the number of rows of B (the right matrix).

$$A\,B$$

$$m \times n \quad n \times p$$

$$\downarrow$$

$$m \times p$$

Example

Compute the product AB where

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix}$$

AB: 2×2, 2×3 → 2×3

$$\vec{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \vec{b}_2 = \begin{bmatrix} 0 \\ -4 \end{bmatrix}, \quad \vec{b}_3 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

$$A\vec{b}_1 = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$A\vec{b}_2 = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ -4 \end{bmatrix} = \begin{bmatrix} 12 \\ -8 \end{bmatrix}$$

$$A\vec{b}_3 = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} -16 \\ 8 \end{bmatrix}$$

$$AB = \begin{bmatrix} -1 & 12 & -16 \\ -2 & -8 & 8 \end{bmatrix}$$

Row-Column Rule for Computing the Matrix Product

If $AB = C = [c_{ij}]$, then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

(The ij^{th} entry of the product is the *dot product* of i^{th} row of A with the j^{th} column of B.)

For example, if A is 2×2 and B is 2×3, then $n = 2$. The entry in row 2 column 3 of AB would be

$$c_{23} = \sum_{k=1}^{2} a_{2k} b_{k3} = a_{21} b_{13} + a_{22} b_{23}.$$

Example

For example: $AB = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix} =$

A B

2×2 ✓ 2×3

2×3

$$\begin{bmatrix} -1 & 12 & -16 \\ -2 & -8 & 8 \end{bmatrix}$$

Theorem: Properties of the Matrix Product

Let A be an $m \times n$ matrix. Let r be a scalar and B and C be matrices for which the indicated sums and products are defined. Then

(i) $A(BC) = (AB)C$

(ii) $A(B + C) = AB + AC$

(iii) $(B + C)A = BA + CA$

(iv) $r(AB) = (rA)B = A(rB)$, and

(v) $I_m A = A = AI_n$

Critical Remarks

Caveats

1. Matrix multiplication **does not commute**! That is, in general $AB \neq BA$. In fact, the validity of AB does not even imply that BA is defined.

2. The zero product property **does not** hold! That is, if $AB = O$, one **cannot** conclude that one of the matrices A or B is a zero matrix.

3. There is **No** *cancelation law*. That is, $AB = CB$ **does not** imply that A and C are equal.

Compute AB and BA where $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$.

AB: $2\times 2 \quad 2\times 2$ ✓ → 2×2

BA: $2\times 2 \quad 2\times 2$ ✓ → 2×2

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -3 & 6 \end{bmatrix}$$

$$BA = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 11 \\ -1 & 4 \end{bmatrix}$$

$$AB \neq BA$$

Compute the products AB, CB, and BB where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \qquad AB = CB$$

$$CB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{but } A \neq C$$

$$BB = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad BB = O \text{ but } B \neq O$$

Matrix Powers

Positive Integer Powers:

If A is square—meaning A is an $n \times n$ matrix for some $n \geq 2$, then the product AA is defined. For positive integer k, we'll define

$$A^k = AA^{k-1}.$$

Zero Power: We define $A^0 = I_n$, where I_n is the $n \times n$ identity matrix.

Transpose

Definition: Matrix Transpose

Let $A = [a_{ij}]$ be an $m \times n$ matrix. The **transpose** of A is the $n \times m$ matrix denoted and defined by

$$A^T = [a_{ji}].$$

For example, if

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}, \quad \text{then} \quad A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}.$$

Example

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix}$$

Compute A^T, B^T, the transpose of the product $(AB)^T$, and the product $B^T A^T$.

We already computed $AB = \begin{bmatrix} -1 & 12 & -16 \\ -2 & -8 & 8 \end{bmatrix}$ in a previous example.

$$A^T = \begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix} \quad B^T = \begin{bmatrix} 2 & 1 \\ 0 & -4 \\ 2 & 6 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} -1 & -2 \\ 12 & -8 \\ -16 & 8 \end{bmatrix} \qquad B^T A^T$$

$3\times 2 \quad 2\times 2$ ✓

3×2

$$B^T A^T = \begin{bmatrix} 2 & 1 \\ 0 & -4 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 12 & -8 \\ -16 & 8 \end{bmatrix}$$

Properties-Matrix Transposition

Theorem

Let A and B be matrices such that the appropriate sums and products are defined, and let r be a scalar. Then

(i) $(A^T)^T = A$

(ii) $(A+B)^T = A^T + B^T$

(iii) $(rA)^T = rA^T$

(iv) $(AB)^T = B^T A^T$

$(ABC)^T = C^T B^T A^T$

Section 2.2: Inverse of a Matrix

Consider the scalar equation $ax = b$. Provided $a \neq 0$, we can solve this explicity

$$x = a^{-1}b$$

where a^{-1} is the unique number such that $aa^{-1} = a^{-1}a = 1$.

If A is an $n \times n$ matrix, we seek an analog A^{-1} that satisfies the condition

$$A^{-1}A = AA^{-1} = I_n.$$

- If such matrix A^{-1} exists, we'll say that A is **nonsingular** or **invertible**.
- Otherwise, we'll say that A is **singular**.

2×2 case

Theorem

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is singular.

Determinant

The quantity $ad - bc$ is called the **determinant** of A and may be denoted in several ways

$$\det(A) \;=\; |A| \;=\; \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Find the inverse if possible

(a) $A = \begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix}$ $\det(A) = 3(5) - (-1)(2) = 17$

$\det(A) \neq 0 \Rightarrow A^{-1}$ exists

A is nonsingular.

$A^{-1} = \frac{1}{17}\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 5/17 & -2/17 \\ 1/17 & 3/17 \end{bmatrix}$

Check: $A^{-1}A = \frac{1}{17}\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 17 & 0 \\ 0 & 17 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Find the inverse if possible

(b) $A = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$

$\det(A) = 3(4) - 6(2) = 0$

A is singular, i.e., A^{-1} doesn't exist.