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Section 2.2: Inverse of a Matrix

In this section, we will consider square, a.k.a. $n \times n$, matrices, and we are looking for something analogous to the reciprocal of a real number.

If A is an $n \times n$ matrix, we ask whether there is another $n \times n$ matrix A^{-1} with the property

$$A^{-1}A = AA^{-1} = I_n.$$

- If such matrix A^{-1} exists, we'll say that A is **nonsingular** or **invertible.**
- Otherwise, we'll say that A is **singular.**

2×2 case

Theorem

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is singular.

Determinant

The quantity $ad - bc$ is called the **determinant** of A and may be denoted in several ways

$$\det(A) \ = \ |A| \ = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Examples

(a) $A = \begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix}$

We determined that $\det(A) = 17$ making A nonsingular, and

$$A^{-1} = \frac{1}{17}\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$$

(b) $A = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$

For this example, $\det(A) = 0$ making A singular. Being **singular** means that there is no inverse.

Theorem

Theorem

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Proof: Let's show that $A^{-1}\vec{b}$ is a solution; Substitute $\vec{x} = A^{-1}\vec{b}$ into the system:

$$
\begin{aligned}
A \cdot \vec{x} &= A(A^{-1}\vec{b}) \\
&= (AA^{-1})\vec{b} \\
&= I\vec{b} = \vec{b}
\end{aligned}
$$

Hence $A^{-1}\vec{b}$ is a solution. Now, suppose $\vec{x}$ is any solution. So

$$A\vec{x} = \vec{b}.$$

Since A is invertible A^{-1} exists.

Now, multiply on the left by A^{-1}.

$$A^{-1}(A\vec{x}) = A^{-1}\vec{b}$$

$$(A^{-1}A)\vec{x} = A^{-1}\vec{b}$$

$$I\vec{x} = A^{-1}\vec{b} \Rightarrow \vec{x} = A^{-1}\vec{b}.$$

So $A^{-1}\vec{b}$ is the unique solution

Example

Use a matrix inverse to solve the system.

$$\begin{array}{rcrcr} 3x_1 & + & 2x_2 & = & -1 \\ -x_1 & + & 5x_2 & = & 4 \end{array}$$

as a matrix equation

$$\begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

Let $A = \begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

We know that $A^{-1} = \frac{1}{17} \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$

$$\vec{x} = A^{-1}\vec{b} = \frac{1}{17}\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$= \frac{1}{17}\begin{bmatrix} -13 \\ 11 \end{bmatrix} = \begin{bmatrix} \frac{-13}{17} \\ \frac{11}{17} \end{bmatrix}$$

$$x_1 = \frac{-13}{17}, \quad x_2 = \frac{11}{17}$$

Inverses, Products, & Transposes

Theorem

(i) If A is invertible, then A^{-1} is also invertible and
$$\left(A^{-1}\right)^{-1} = A.$$

(ii) If A and B are invertible $n \times n$ matrices, then the product AB is also invertible[a] with
$$(AB)^{-1} = B^{-1}A^{-1}.$$

(iii) If A is invertible, then so is A^T. Moreover
$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T.$$

[a] This can generalize to the product of k invertible matrices.

Elementary Matrices

Definition:

An **elementary** matrix is a square matrix obtained from the identity by performing one elementary row operation.

Examples[1]:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$3R_2 \to R_2$ $\qquad$ $2R_1 + R_3 \to R_3$ $\qquad$ $R_1 \leftrightarrow R_2$

[1] There's nothing standard about the subscripts used here, although using E to denote an elementary matrix is common.

Action of Elementary Matrices

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, and compute the following products

$$E_1A, \quad E_2A, \quad \text{and} \quad E_3A.$$

$$E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$= \begin{bmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{bmatrix}.$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$E_2 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$= \begin{bmatrix} a & b & c \\ d & e & f \\ 2a+g & 2b+h & 2c+i \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$E_3 A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$= \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Remarks

Remarks

1. Elementary row operations can be equated with matrix multiplication (multiply on the left by an elementary matrix),
2. Each elementary matrix is invertible where the inverse *undoes* the row operation,
3. Reduction to rref is a sequence of row operations, so it is a sequence of matrix multiplications

$$\text{rref}(A) = E_k \cdots E_2 E_1 A.$$

Matrix Inverses

Theorem

An $n \times n$ matrix A is invertible if and only if it is row equivalent to the identity matrix I_n. Moreover, if

$$\text{rref}(A) = E_k \cdots E_2 E_1 A = I_n, \quad \text{then} \quad A = (E_k \cdots E_2 E_1)^{-1} I_n.$$

That is,

$$A^{-1} = \left[(E_k \cdots E_2 E_1)^{-1} \right]^{-1} = E_k \cdots E_2 E_1.$$

The sequence of operations that reduces A to I_n, transforms I_n into A^{-1}.

Remark: This last observation—operations that take A to I_n also take I_n to A^{-1}—gives us a method for computing an inverse!

Algorithm for finding A^{-1}

Inverse Matrix Algorithm

To find the inverse of a given matrix A:

- Form the $n \times 2n$ augmented matrix $[A \quad I]$.
- Perform whatever row operations are needed to get the first n columns (the A part) to rref.
- If rref(A) is I, then $[A \quad I]$ is row equivalent to $[I \quad A^{-1}]$, and the inverse A^{-1} will be the last n columns of the reduced matrix.
- If rref(A) is NOT I, then A is not invertible.

Remarks: We don't need to know ahead of time if A is invertible to use this algorithm. If A is singular, we can stop as soon as it's clear that rref$(A) \neq I$.

Examples: Find the Inverse if Possible

(a) $\begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 3 \\ -2 & -6 & 4 \end{bmatrix} = A$ set up $[A \ I]$

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ -4 & -7 & 3 & 0 & 1 & 0 \\ -2 & -6 & 4 & 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} 4R_1 + R_2 \to R_2 \\ 2R_1 + R_3 \to R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 4 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{bmatrix} \quad 2R_2 + R_3 \to R_3$$

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 10 & 2 & 1 \end{bmatrix}.$$

A has only two pivot columns,

A^{-1} doesn't exist.

Examples: Find the Inverse if Possible

(b) $\begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & 1 \\ 5 & 6 & 0 \end{bmatrix} = A$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{array}\right] \qquad \begin{array}{l} R_1 + R_2 \to R_2 \\ -5R_1 + R_3 \to R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 1 & 0 \\ 0 & -4 & -15 & -5 & 0 & 1 \end{array}\right] \qquad 4R_2 + R_3 \to R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 4 & 1 \end{array}\right]$$

$-3R_3 + R_1 \to R_1$

$-4R_3 + R_2 \to R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & -12 & -3 \\ 0 & 1 & 0 & 5 & -15 & -4 \\ 0 & 0 & 1 & -1 & 4 & 1 \end{array}\right]$$

$-2R_2 + R_1 \to R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -6 & 18 & 5 \\ 0 & 1 & 0 & 5 & -15 & -4 \\ 0 & 0 & 1 & -1 & 4 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -6 & 18 & 5 \\ 5 & -15 & -4 \\ -1 & 4 & 1 \end{bmatrix}$$

Section 2.3: Characterization of Invertible Matrices

Given an $n \times n$ matrix A, we can think of...

- A matrix equation $A\mathbf{x} = \mathbf{b}$;
- A linear system that has A as its coefficient matrix;
- A linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$;
- Not to mention things like its **pivots**, its **rref**, the linear dependence/independence of its columns, blah blah blah...

Question: Is this stuff related to being singular/nonsingular? How?

The Invertible Matrix Theorem

Suppose A is $n \times n$. The following are equivalent. [a]

(a) A is invertible.

(b) A is row equivalent to I_n.

(c) A has n pivot positions.

(d) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(e) The columns of A are linearly independent.

(f) The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one to one.

(g) $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^n$.

(h) The columns of A span $\mathbb{R}^n$.

(i) The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto.

(j) There exists an $n \times n$ matrix C such that $CA = I$.

(k) There exists an $n \times n$ matrix D such that $AD = I$.

(l) A^T is invertible.

[a] Meaning all are true or none are true.

The Inverse of a Matrix is Unique

Theorem

Let A and B be $n \times n$ matrices. If $AB = I$, then A and B are both invertible with $A^{-1} = B$ and $B^{-1} = A$.

Proof: Assume $AB = I$. We'll show that B is invertible and $B^{-1} = A$. Consider the homogeneous system $B\vec{x} = \vec{0}$. We wish to show that $\vec{x}$ must be $\vec{0}$. From $B\vec{x} = \vec{0}$ multiply on the left by A.

$$B\vec{x} = \vec{0}$$

$$AB\vec{x} = A\vec{0}$$

$$I\vec{x} = \vec{0}$$

$$\vec{x} = \vec{0}.$$

$B\vec{x} = \vec{0}$ has only the trivial solution, hence B is invertible. There exist a matrix B^{-1}. From $AB = I$, multiply on the right by B^{-1}.

$$AB = I$$

$$ABB^{-1} = IB^{-1}$$

$$AI = B^{-1}$$

$$A = B^{-1}$$

Since B is invertible B^{-1} is also invertible., i.e., A is invertible and

$$(A)^{-1} = (B^{-1})^{-1} = B$$

That is, $B = A^{-1}$.

Invertible Linear Transformations

Definition:

A linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to be **invertible** if there exists a function $S : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that both

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{and} \quad T(S(\mathbf{x})) = \mathbf{x}$$

for every $\mathbf{x}$ in $\mathbb{R}^n$.

If such a function exists, we typically denote it by

$$S = T^{-1}.$$

Invertability of a Transformation and its Matrix

Theorem

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation and A its standard matrix. Then T is invertible if and only if A is invertible. Moreover, if T is invertible, then

$$T^{-1}(\mathbf{x}) = A^{-1}\mathbf{x}$$

for every $\mathbf{x}$ in $\mathbb{R}^n$.

Remark: This indicates that we can determine if a linear transformation is invertible and identify the inverse transform using the standard matrix.

Example

Use the standard matrix to determine if the linear transformation is invertible. If it is invertible, characterize the inverse transformation.

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad \text{given by} \quad T(x_1, x_2) = (3x_1 - x_2, 4x_2).$$

Calling the standard matrix A,

$$A = \left[T(\vec{e}_1) \quad T(\vec{e}_2) \right].$$

$$T(\vec{e}_1) = T(1,0) = (3-0, 0) = (3,0)$$

$$T(\vec{e}_2) = T(0,1) = (0-1, 4) = (-1,4)$$

$$A = \begin{bmatrix} 3 & -1 \\ 0 & 4 \end{bmatrix}, \quad \det(A) = 3\cdot 4 - 0(-1) = 12$$

A^{-1} exists and $A^{-1} = \frac{1}{12}\begin{bmatrix} 4 & 1 \\ 0 & 3 \end{bmatrix}$.

T^{-1} exists and $T^{-1}(\vec{x}) = A^{-1}\vec{x}$

$$T^{-1}(\vec{x}) = \frac{1}{12}\begin{bmatrix} 4 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 4x_1 + x_2 \\ 3x_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3}x_1 + \frac{1}{12}x_2 \\ \frac{1}{4}x_2 \end{bmatrix}$$

$$\boxed{T^{-1}(x_1, x_2) = \left(\frac{1}{3}x_1 + \frac{1}{12}x_2,\ \frac{1}{4}x_2\right)}$$

Example

Suppose $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a one to one linear transformation. Can we determine whether T is onto? Why (or why not)?

If we let A be the standard matrix, $T(\vec{x}) = A\vec{x}$, by the invertible matrix Theorem, if $\vec{x} \mapsto A\vec{x}$ is one to one, then $\vec{x} \mapsto A\vec{x}$ is also onto.

$(f) \Longleftrightarrow (i)$ from the theorem.

Section 3.1: Introduction to Determinants

We defined a number, called a **determinant**, for a 2×2 matrix. And that number was related to whether the matrix was invertible.

For $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, we said that the determinant

$$\det(A) = a_{11}a_{22} - a_{21}a_{12}.$$

And we had the critical relationship that A is nonsingular (a.k.a. invertible) if and only if $\det(A)$ is nonzero.

Here, we want to extend the concept of **determinant** to all $n \times n$ matrices and do it in such a way that for any square matrix A,

$$\boxed{\textbf{A is nonsinguar if and only if } \det(A) \neq 0.}$$

Determinant: 3×3 Matrix

Let's assume that $A = [a_{ij}]$ is an **invertible** 3×3 matrix, and suppose that $a_{11} \neq 0$. We can start the row reduction process to obtain zeros below the left most pivot position.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{array}{c} a_{11}R_2 \to R_2 \\ a_{11}R_3 \to R_3 \end{array} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{bmatrix}$$

$$\begin{array}{c} -a_{21}R_1 + R_2 \to R_2 \\ -a_{31}R_1 + R_3 \to R_3 \end{array} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \end{bmatrix}$$

Determinant: 3×3 Matrix

If $A \sim I$, one of the entries in the 2, 2 or the 3, 2 position must be nonzero. Let's assume it is the 2, 2 entry and continue the reduction[2]

$$\begin{array}{c} b_{22}R_3 \to R_3 \\ -b_{32}R_2 + R_3 \to R_3 \end{array} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & a_{11}\Delta \end{bmatrix},$$

where Δ is an expression involving the entries of A. We can state the following:

> If A is invertible, it must be that the bottom right entry is nonzero. That is
>
> $$\Delta \neq 0.$$
>
> Note that if $\Delta = 0$, then A would not be row equivalent to I making A singular. We will define the determinant to be Δ.

[2] The factors shown here are $b_{22} = a_{11}a_{22} - a_{12}a_{21}$ and $b_{32} = a_{11}a_{32} - a_{12}a_{31}$.

Determinant: 3×3 Matrix

We can rearrange the term Δ and state the determinant in an easy to remember way.

$$\Delta = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Notice that each expression in parentheses is a *product minus product*, i.e., they look like determinants of 2×2 matrices! We can restate these as determinants and arrive at the following formula for the determinant of a 3×3 matrix.

3×3 Determinant

For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, the determinant

$$\det(A) = a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Minors & Cofactors

Some Notation

Let $n \geq 2$. For an $n \times n$ matrix A, let A_{ij} denote the $(n-1)\times(n-1)$ matrix obtained from A by deleting the i^{th} row and the j^{th} column of A.

For example, if

$$A = \begin{bmatrix} -1 & 3 & 2 & 0 \\ 4 & 4 & 0 & -3 \\ -2 & 1 & 7 & 2 \\ 3 & 0 & -1 & 6 \end{bmatrix} \quad \text{then} \quad A_{23} = \begin{bmatrix} -1 & 3 & 0 \\ -2 & 1 & 2 \\ 3 & 0 & 6 \end{bmatrix}.$$

Minors & Cofactors

Suppose A is an $n \times n$ matrix for some $n \geq 2$.

Definition: Minor

The i, j^{th} **minor** of the $n \times n$ matrix A is the number

$$M_{ij} = \det(A_{ij}).$$

Definition: Cofactor

Let A be an $n \times n$ matrix with $n \geq 2$. The i, j^{th} **cofactor** of A is the number

$$C_{ij} = (-1)^{i+j} M_{ij}.$$

Minors & Cofactors

Find the three minors M_{11}, M_{12}, M_{13} and find the 3 cofactors C_{11}, C_{12}, C_{13} of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

$$M_{11} = \det(A_{11})$$

$$= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$= a_{22}a_{33} - a_{32}a_{23}$$

$$C_{11} = (-1)^{1+1} M_{11} = M_{11} = a_{22}a_{33} - a_{32}a_{23}$$

$$M_{12} = \det(A_{12}) = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{31}a_{23}$$

$$C_{12} = (-1)^{1+2} M_{12} = -\left(a_{21}a_{33} - a_{31}a_{23}\right)$$

$$M_{13} = \det(A_{13}) = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{21}a_{32} - a_{31}a_{22}$$

$$C_{13} = (-1)^{1+3} \cdot M_{13} = a_{21}a_{32} - a_{31}a_{22}$$

Observation:

Recall that the determinant of the 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

was given by

$$\det(A) = a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Cofactor Expansion

Note that we can write

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

An expression of this form is called a *cofactor expansion*.

The Determinant

Definition: Determinant

For $n \geq 2$, the **determinant** of the $n \times n$ matrix $A = [a_{ij}]$ is the number

$$\begin{aligned} \det(A) &= a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} M_{1j} \end{aligned}$$

Remark: Note that this definition defines determinants iteratively via the minors. The determinant of a 3×3 matrix is given in terms of the determinants of three 2×2 matrices. The determinant of a 4×4 matrix is given in terms of the determinants of four 3×3 matrices, and so forth.

Example

Evaluate $\det(A)$.

$$A = \begin{bmatrix} -1 & 3 & 0 \\ -2 & 1 & 2 \\ 3 & 0 & 6 \end{bmatrix}$$

$C_{12} = (-1)^3 M_{12}$

$\det(A) = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$

$\det(A) = (-1) \begin{vmatrix} 1 & 2 \\ 0 & 6 \end{vmatrix} - 3 \begin{vmatrix} -2 & 2 \\ 3 & 6 \end{vmatrix} + 0 \begin{vmatrix} -2 & 1 \\ 3 & 0 \end{vmatrix}$

$= -(6-0) - 3(-12-6) = -6 + 54 = 48$

Example

Find all values of x such that[3] $\det(A) = 0$.

$$A = \begin{bmatrix} 3-x & 2 & 1 \\ 0 & 2-x & 4 \\ 0 & 3 & 1-x \end{bmatrix}$$

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\det(A) = (3-x)\begin{vmatrix} 2-x & 4 \\ 3 & 1-x \end{vmatrix} - 2\underbrace{\begin{vmatrix} 0 & 4 \\ 0 & 1-x \end{vmatrix}}_{=0} + 1\underbrace{\begin{vmatrix} 0 & 2-x \\ 0 & 3 \end{vmatrix}}_{=0}$$

$$= (3-x)\left((2-x)(1-x) - 12\right)$$

[3] In the next section, we'll state that a matrix is **singular** when its determinant is zero.

$$= (3-x)\left(x^2 - 3x - 10\right)$$

$$= (3-x)(x-5)(x+2)$$

$\det(A) = 0$ if $x=3$, $x=5$ or

$x=-2$.