June 22 Math 3260 sec. 51 Summer 2023

Section 3.1: Introduction to Determinants

Definition: Determinant

For $n \geq 2$, the **determinant** of the $n \times n$ matrix $A = [a_{ij}]$ is the number

$$\det(A) = \sum_{j=1}^{n} a_{1j}C_{1j} = \sum_{j=1}^{n} (-1)^{1+j} a_{1j}M_{1j}$$

Remark: Here, M_{ij} and C_{ij} refer to the i, j^{th} *minor* and *cofactor*, respectively.

Remark: Note that the determinant is a number. At least for the 3×3 case, we defined it so that this number is nonzero if A is invertible.

Examples

Evaluate $\det(A)$. $A = \begin{bmatrix} -1 & 3 & 0 \\ -2 & 1 & 2 \\ 3 & 0 & 6 \end{bmatrix}$

Using the definition

$$\det(A) = -1 \begin{vmatrix} 1 & 2 \\ 0 & 6 \end{vmatrix} - 3 \begin{vmatrix} -2 & 2 \\ 3 & 6 \end{vmatrix} + 0 \begin{vmatrix} -2 & 1 \\ 3 & 0 \end{vmatrix} = 48$$

Find all values of x such that $\det(A) = 0$. $A = \begin{bmatrix} 3-x & 2 & 1 \\ 0 & 2-x & 4 \\ 0 & 3 & 1-x \end{bmatrix}$

We took the determinant using the definition.

$$\det(A) = (3-x) \begin{vmatrix} 2-x & 4 \\ 3 & 1-x \end{vmatrix} - 2 \begin{vmatrix} 0 & 4 \\ 0 & 1-x \end{vmatrix} + 1 \begin{vmatrix} 0 & 2-x \\ 0 & 3 \end{vmatrix}$$

General Cofactor Expansions

Theorem

The determinant of an $n \times n$ matrix can be computed by cofactor expansion across any row or down any column.

We can fix any row i of a matrix A and then

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

Or, we can fix any column j of a matrix A and then

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

Example

Evaluate $\det(A)$.

$$A = \begin{bmatrix} -1 & 3 & 4 & 0 \\ 0 & 0 & -3 & 0 \\ -2 & 1 & 2 & 2 \\ 3 & 0 & -1 & 6 \end{bmatrix}$$

Using row 2

$$\det(A) = \underset{0}{a_{21}} C_{21} + \underset{0}{a_{22}} C_{22} + a_{23} C_{23} + \underset{0}{a_{24}} C_{24}$$

$$\det(A) = a_{23} C_{23} = -(-3) \begin{vmatrix} -1 & 3 & 0 \\ -2 & 1 & 2 \\ 3 & 0 & 6 \end{vmatrix}$$

$$= 3(48) = 144$$

from last time

Triangular Matrices

Definition:

The $n \times n$ matrix $A = [a_{ij}]$ is said to be **upper triangular** if $a_{ij} = 0$ for all $i > j$.

It is said to be **lower triangular** if $a_{ij} = 0$ for all $j > i$. A matrix that is both upper and lower triangular is a **diagonal** matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

Upper Triangular **Lower Triangular**

Determinant of Triangular Matrix

Theorem:

For $n \geq 2$, the determinant of an $n \times n$ triangular matrix is the product of its diagonal entries. (i.e. if $A = [a_{ij}]$ is triangular, then $\det(A) = a_{11}a_{22}\cdots a_{nn}$.)

Example: Evaluate $\det(A)$.

$$A = \begin{bmatrix} 7 & 0 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 4 & 2 & 2 & 2 \end{bmatrix}$$

$\det(A) = 7(6)(2)(2)$

$= 42(4) = 168$

Example

Evaluate $\det(A)$

$$A = \begin{bmatrix} -1 & 3 & 4 & 0 & 2 \\ 0 & 2 & -3 & 0 & -4 \\ 0 & 0 & 3 & 7 & 5 \\ 0 & 0 & 0 & -4 & 6 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

$$\det(A) = (-1)(2)(3)(-4)(6)$$

$$= 6(24) = 144$$

Section 3.2: Properties of Determinants

Theorem:

Let A be an $n \times n$ matrix, and suppose the matrix B is obtained from A by performing a single elementary row operation[a]. Then

(i) If B is obtained by adding a multiple of a row of A to another row of A (row replacement), then

$$\det(B) = \det(A).$$

(ii) If B is obtained from A by swapping any pair of rows (row swap) , then

$$\det(B) = -\det(A).$$

(iii) If B is obtained from A by scaling any row by the constant k (scaling), then

$$\det(B) = k\det(A).$$

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[a] If "row" is replaced by "column" in any of the operations, the conclusions still follow.

Example

Compute the determinant by first performing row operations to obtain a triangular matrix and recording the effect of each row operation.

$$\begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{vmatrix}$$

Let $A = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{bmatrix}$

$R_2 + R_4 \to R_4$

$$\begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{vmatrix}$$

$R_2 \leftrightarrow R_1$

Row op

1. replacement
2. swap (-1)
3. replacement
4. swap (-1)

$$\begin{vmatrix} 2 & 5 & -7 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{vmatrix}$$

$$-3R_2 + R_3 \to R_3$$

$$\begin{vmatrix} 2 & 5 & -7 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & -3 & 1 \end{vmatrix}$$

$$R_3 \leftrightarrow R_4$$

$$\begin{vmatrix} 2 & 5 & -7 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 5 \end{vmatrix} = 2(1)(-3)(5) = -30$$

$$\det(A) = \frac{-30}{(-1)(-1)} = -30$$

Row replacement

$$kR_i + R_j \to R_j$$

What about

$$R_i + kR_j \to R_j$$

This is a scale by k plus a replacement

Results on Determinants

Theorem

The $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$.

Theorem

For $n \times n$ matrix A, $\det(A^T) = \det(A)$.

Theorem

For $n \times n$ matrices A and B, $\det(AB) = \det(A)\det(B)$.

Example

Show that if A is an $n \times n$ invertible matrix, then

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Note that $AA^{-1} = I$. So

$$\det(AA^{-1}) = \det(I) = 1$$

By the previous theorem

$$\det(AA^{-1}) = \det(A)\det(A^{-1}) = 1$$

$$\Rightarrow \quad \det(A^{-1}) = \frac{1}{\det(A)}.$$

Example

Let A be an $n \times n$ matrix, and suppose there exists invertible matrix P such that[1]

$$B = P^{-1}AP.$$

Show that

$$\det(B) = \det(A).$$

$$\det(B) = \det(P^{-1}AP)$$

$$= \det(P^{-1})\det(A)\det(P)$$

these are scalars

$$= \det(A)\det(P^{-1})\det(P)$$

[1]The process of multiplying by P^{-1} on the left and P on the right is called a *similarly transform*. The matrices A and B are said to be *similar*.

$$= \det(A) \frac{1}{\det(P)} \det(P)$$

$$= \det(A).$$

Section 3.3: Cramer's Rule, Volume, and Linear Transformations

Cramer's Rule is a method for solving some small linear systems of equations.

Notation:

For $n \times n$ matrix A and $\mathbf{b}$ in $\mathbb{R}^n$, let $A_i(\mathbf{b})$ be the matrix obtained from A by replacing the i^{th} column with the vector $\mathbf{b}$. That is

$$A_i(\mathbf{b}) = [\mathbf{a}_1 \cdots \mathbf{a}_{i-1}\ \mathbf{b}\ \mathbf{a}_{i+1} \cdots \mathbf{a}_n]$$

Example Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, then

$$A_3(\mathbf{b}) = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}$$

Cram er's Rule

Theorem:

Let A be an $n \times n$ nonsingular matrix. Then for any vector $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution of the system $A\mathbf{x} = \mathbf{b}$ is given by $\mathbf{x}$ where

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \quad i = 1, \ldots, n$$

Remark: The condition $\det(A) \neq 0$ is necessary for Cramer's rule to be a viable method. This allows for the solution to be given in terms of ratios of determinants.

Remark: If $\det(A) = 0$, the system may be consistent, but another method is required to make a determination.

Example

Determine whether Cramer's rule can be used to solve the system. If so, use it to solve the system.

$$\begin{array}{rcrcr} 2x_1 & + & x_2 & = & 9 \\ -x_1 & + & 7x_2 & = & -3 \end{array} \longrightarrow A\vec{x} = \vec{b}$$

$$\underbrace{\begin{bmatrix} 2 & 1 \\ -1 & 7 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 9 \\ -3 \end{bmatrix}}_{\vec{b}}$$

$$\det(A) = 2(7) - (-1)(1) = 14 + 1 = 15$$

$\det(A) \neq 0$ so Cramer's rule applies

$$A_1(\vec{b}) = \begin{bmatrix} 9 & 1 \\ -3 & 7 \end{bmatrix} \qquad A_2(\vec{b}) = \begin{bmatrix} 2 & 9 \\ -1 & -3 \end{bmatrix}$$

$$\det(A_1(\vec{b})) = 9(7) - (-3)(1)$$
$$= 63 + 3 = 66$$

$$\det(A_2(\vec{b})) = 2(-3) - (-1)9$$
$$= -6 + 9 = 3$$

$$\det(A) = 15$$

$$x_1 = \frac{66}{15} = \frac{22}{5} \quad \text{and} \quad x_2 = \frac{3}{15} = \frac{1}{5}$$

$$x_1 = \frac{22}{5}, \quad x_2 = \frac{1}{5}$$

Application: Laplace Transforms

In various engineering applications, electrical or mechanical components are often chosen to try to control the long term behavior of a system (e.g. adding a damper to kill off oscillatory behavior). Using *Laplace Transforms*, differential equations are converted into algebraic equations containing a parameter s. These give rise to systems of the form

$$\begin{array}{rcrcl} 3sX & - & 2Y & = & 4 \\ -6X & + & sY & = & 1 \end{array}$$

Determine the values of s for which the system is uniquely solvable. For such s, find the solution (X, Y) using Crammer's rule.

$$\begin{aligned} 3sX &- 2Y = 4 \\ -6X &+ sY = 1 \end{aligned}$$

$$\underbrace{\begin{bmatrix} 3s & -2 \\ -6 & s \end{bmatrix}}_{A} \begin{bmatrix} X \\ Y \end{bmatrix} = \underbrace{\begin{bmatrix} 4 \\ 1 \end{bmatrix}}_{\vec{b}}$$

$\det(A) = 3s(s) - (-6)(-2) = 3s^2 - 12$

$\det(A) = 0 \Rightarrow 3(s^2-4) = 0$

if $s = 2$ or $s = -2$

The system is uniquely solvable if

$s \neq \pm 2$.

$A_1(\vec{b}) = \begin{bmatrix} 4 & -2 \\ 1 & s \end{bmatrix}$, $A_2(\vec{b}) = \begin{bmatrix} 3s & 4 \\ -6 & 1 \end{bmatrix}$

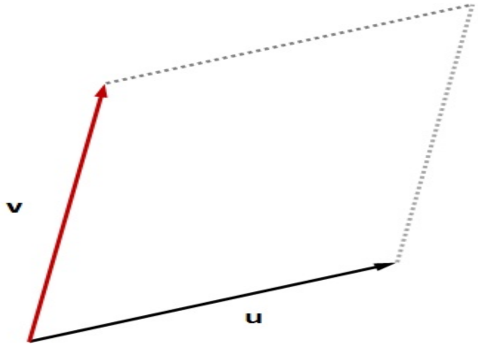
$$\det(A_1(\vec{b})) = 4s + 2, \quad \det(A_2(\vec{b})) = 3s + 24$$

$$\det(A) = 3(s^2 - 4)$$

For $s \neq \pm 2$, the solution

$$x = \frac{4s+2}{3(s^2-4)} \quad \text{and} \quad y = \frac{3s+24}{3(s^2-4)} = \frac{s+8}{s^2-4}$$

Area & Volume (Video)

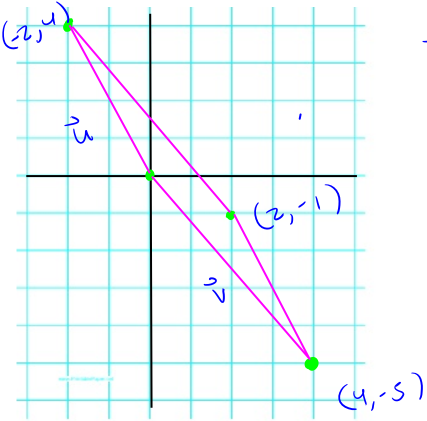


Theorem:

If **u** and **v** are nonzero, nonparallel vectors in $\mathbb{R}^2$, then the area of the parallelogram determined by these vectors is $|\det(A)|$ where $A = [\mathbf{u}\ \mathbf{v}]$.

Example

Find the area of the parallelogram with vertices $(0,0)$, $(-2,4)$, $(4,-5)$, and $(2,-1)$.



$$\vec{u} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$$

Let $A = \begin{bmatrix} \vec{u} & \vec{v} \end{bmatrix}$

$$= \begin{bmatrix} -2 & 4 \\ 4 & -5 \end{bmatrix}$$

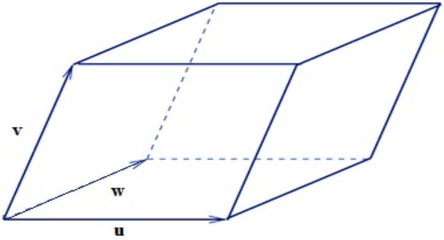
$$\text{Area} = |\det(A)|$$

$$= \left| \begin{vmatrix} -2 & 4 \\ 4 & -5 \end{vmatrix} \right|$$

$$= |-2(-5) - 4(4)|$$

$$= |10 - 16| = 6$$

Volume of a Parallelepiped



Theorem:

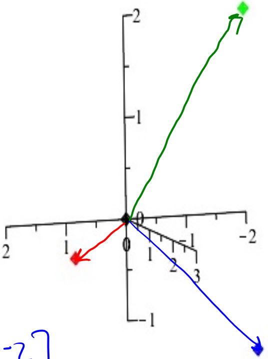
If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are nonzero, non-collinear vectors in $\mathbb{R}^3$, then the volume of the parallelepiped determined by these vectors is $|\det(A)|$ where $A = [\mathbf{u}\ \mathbf{v}\ \mathbf{w}]$.

Example

Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(2,3,0)$, $(-2,0,2)$ and $(-1,3,-1)$.

$$\vec{u} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \qquad \vec{w} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$$



Let $A = [\vec{u}\ \vec{v}\ \vec{w}] = \begin{bmatrix} 2 & -1 & -2 \\ 3 & 3 & 0 \\ 0 & -1 & 2 \end{bmatrix}$

Let's do cofactor expansion across row 3.

$$\det(A) = \overset{0}{\cancel{a_{31}}} C_{31} + a_{32} C_{32} + a_{33} C_{33}$$

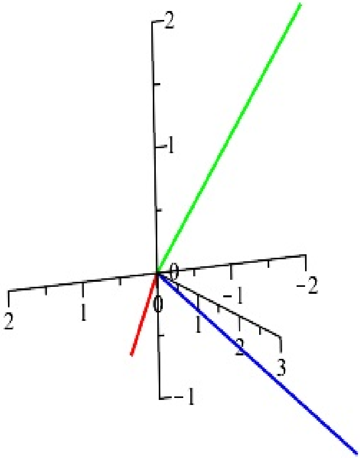
$$= -(-1)\begin{vmatrix} 2 & -2 \\ 3 & 0 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ 3 & 3 \end{vmatrix}$$

$$= (0-(-6)) + 2(6+3)$$

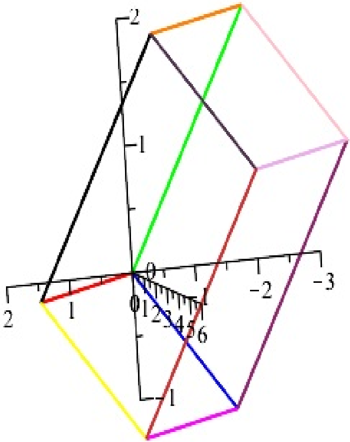
$$= 6 + 18$$

$$= 24$$

The volume is 24



2
1
0
-1
-2
2
1
0
-1
-2
1
2
3



2
1
0
-1
2
1
0
-1
-2
-3

Section 4.1: Vector Spaces and Subspaces

Recall that we had defined $\mathbb{R}^n$ as the set of all n-tuples of real numbers. We defined two operations, vector addition and scalar multiplication, and said that the following algebraic properties hold:

For every $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^n$ and scalars c and d

(i) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(ii) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$

(iii) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$

(iv) $\mathbf{u}+(-\mathbf{u})=-\mathbf{u}+\mathbf{u}=\mathbf{0}$

(v) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$

(vi) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$

(vii) $c(d\mathbf{u})=d(c\mathbf{u})=(cd)\mathbf{u}$

(viii) $1\mathbf{u}=\mathbf{u}$

We later saw that a set of $m\times n$ matrices with scalar multiplication and matrix addition satisfies the same set of properties.

Question: Are there other sets of objects with operations that share this same structure?

Definition: Vector Space

A **vector space** is a nonempty set V of objects called *vectors* together with two operations called *vector addition* and *scalar multiplication* that satisfy the following ten axioms:

For all $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in V, and for any scalars c and d

1. The sum $\mathbf{u} + \mathbf{v}$ is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
4. There exists a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each vector $\mathbf{u}$ there exists a vector $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. For each scalar c, $c\mathbf{u}$ is in V.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $c(d\mathbf{u}) = d(c\mathbf{u}) = (cd)\mathbf{u}$.
10. $1\mathbf{u} = \mathbf{u}$

Remarks:

- V is more accurately called a *real vector space* when we assume that the relevant scalars are the real numbers.
- Property 1., $\mathbf{u} + \mathbf{v} \in V$, is called being **closed** under (or *with respect to*) vector addition.
- Property 6., $c\mathbf{u} \in V$, is called being **closed** under (or *with respect to*) scalar multiplication.
- A vector space has the same basic *algebraic structure* as $\mathbb{R}^n$
- These are **axioms**. That means they are assumed, not proven. However, we can use them to prove or disprove that some set with operations is actually a vector space.

An Example of a Vector Space

For an integer $n \geq 0$, let $\mathbb{P}_n$ denote the set of all polynomials with real coefficients of degree at most n.

$$\mathbb{P}_n = \{\mathbf{p}(t) = p_0 + p_1 t + \cdots + p_n t^n \mid p_0, p_1, \ldots, p_n \in \mathbb{R}\},$$

where addition[2] and scalar multiplication are defined by

$$(\mathbf{p} + \mathbf{q})(t) = \mathbf{p}(t) + \mathbf{q}(t) = (p_0 + q_0) + (p_1 + q_1)t + \cdots + (p_n + q_n)t^n,$$

$$(c\mathbf{p})(t) = c\mathbf{p}(t) = cp_0 + cp_1 t + \cdots + cp_n t^n.$$

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[2] $\mathbf{q}(t) = q_0 + q_1 t + \cdots + q_n t^n$

Example

$\vec{p}(t) = p_0 + p_1 t + p_2 t^2 + \dots + p_n t^n$

What is the zero vector **0** in $\mathbb{P}_n$?

Let $\mathbf{0}(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$. Find the values of $a_0, \ldots, a_n$.

For $\vec{p}$ in $\mathbb{P}_n$, $\vec{p} + \vec{0} = \vec{p}$.

$$(\vec{p} + \vec{0})(t) = \vec{p}(t) + \vec{0}(t)$$

$$= (p_0 + a_0) + (p_1 + a_1)t + \dots + (p_n + a_n)t^n$$

$$= p_0 + p_1 t + \dots + p_n t^n$$

$$p_0 + a_0 = p_0 \Rightarrow a_0 = 0$$

$$p_1 + a_1 = p_1 \Rightarrow a_1 = 0$$

$$\vdots$$

$$p_n + a_n = p_n \Rightarrow a_n = 0$$

That is

$$\vec{0}(t) = 0 + 0t + 0t^2 + \dots + 0t^n$$

Example

If $\mathbf{p}(t) = p_0 + p_1 t + \cdots + p_n t^n$, what is the vector $-\mathbf{p}$?

Let $-\mathbf{p}(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n$. Find the values of $c_0, \ldots, c_n$.

We know that $\vec{p} + (-\vec{p}) = \vec{0}$

$$\left(\vec{p} + (-\vec{p})\right)(t) = \vec{p}(t) + (-\vec{p}(t))$$

$$= (p_0 + c_0) + (p_1 + c_1)t + \ldots + (p_n + c_n)t^n$$

$$= 0 + 0t + \ldots + 0t^n$$

Equating these

$$p_0 + c_0 = 0 \implies c_0 = -p_0$$

$$p_1 + c_1 = 0 \implies c_1 = -p_1$$

$$\vdots$$

$$p_n + c_n = 0 \implies c_n = -p_n$$

Hence

$$-\vec{p}(t) = -p_0 - p_1 t - \dots - p_n t^n$$

A set that is not a Vector Space

Let $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; x \leq 0, \; y \leq 0 \right\}$ with regular vector addition and scalar multiplication in $\mathbb{R}^2$. Note V is the third quadrant in the xy-plane.

(1) Does property 1. hold for V?

Let $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} z \\ w \end{bmatrix}$

be in V. So $x \leq 0$, $y \leq 0$, $z \leq 0$ and

$w \leq 0$.

$$\vec{u} + \vec{v} = \begin{bmatrix} x + z \\ y + w \end{bmatrix}$$

$x + z \leq 0$ and $y + w \leq 0$

Hence $\vec{u} + \vec{v}$ is in V.

A set that is not a Vector Space

Let $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \; x \leq 0,\; y \leq 0 \right\}$ with regular vector addition and scalar multiplication in $\mathbb{R}^2$. Note V is the third quadrant in the xy-plane.

(2) Does property 6. hold for V?

Note that $\vec{u} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ is in V.

Let $c = -1$ $\quad c\vec{u} = -1 \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$c\vec{u}$ is not in V. Property six fails.

V is not a vector space.