June 29 Math 3260 sec. 51 Summer 2023

Section 4.2: Null & Column Spaces, Row Space, Linear Transformations

Definition

Definition: Let A be an $m \times n$ matrix. The **null space** of A, denoted by Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. That is

$$\text{Nul}\, A = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}.$$

Theorem:

If A is an $m \times n$ matrix, then $\text{Nul}(A)$ is a subspace of $\mathbb{R}^n$.

Column Space

Definition:

The **column space** of an $m \times n$ matrix A, denoted $\operatorname{Col} A$, is the set of all linear combinations of the columns of A. If $A = [\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n]$, then

$$\operatorname{Col} A = \operatorname{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}.$$

Theorem:

The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

Row Space

Definition:

The **row space**, denoted $\text{Row}\, A$, of an $m \times n$ matrix A is the subspace of $\mathbb{R}^n$ spanned by the rows of A.

It's written into the definition of the row space that it is a subspace of $\mathbb{R}^n$

Theorem

If two matrices A and B are row equivalent, then their row spaces are the same.

Fundamental Subspaces

People often refer to four fundamental subspaces associated with an $m \times n$ matrix. The fourth one is the null space of A^T.

Remark: Since the rows of A are the columns of A^T and vice versa, it's not surprising that

$$\text{Col}(A) = \text{Row}(A^T) \quad \text{and} \quad \text{Row}(A) = \text{Col}(A^T).$$

Remark: We can summarize that for $m \times n$ matrix A

$$\text{Col}(A) \quad \text{and} \quad \text{Nul}(A^T) \text{ are subspaces of } \mathbb{R}^m,$$

and

$$\text{Row}(A) \quad \text{and} \quad \text{Nul}(A) \text{ are subspaces of } \mathbb{R}^n.$$

Example

The following matrices are row equivalent. Use them to find an explicit description (i.e., a spanning set) for $\text{Row}(A)$ and $\text{Nul}(A)$.

$$A = \begin{bmatrix} -2 & 2 & -3 & -2 \\ 3 & -3 & 3 & 1 \\ 2 & -2 & 2 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For Row(A), we can use the rows of A or B.

$$\text{Row}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Nul(A) contains all solutions to $A\vec{x} = \vec{0}$

$$\begin{bmatrix} A & \vec{0} \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} B & \vec{0} \end{bmatrix},$$

From B, $\vec{x}$ satisfies

$x_1 = x_2$

$x_3 = 0$

$x_4 = 0$

x_2 is free

$$\vec{x} = \begin{bmatrix} x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Nul}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Linear Transformation

Definition:

Let V and W be vector spaces. A **linear transformation** $T : V \to W$ is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W such that

(i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for every $\mathbf{u}, \mathbf{v}$ in V, and

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for every $\mathbf{u}$ in V and scalar c.

Remark: The only difference between this definition and our previous one is that the domain and codomain spaces can be any vector spaces.

Example

Let $C^1(\mathbb{R})$ denote[1] the set of all real valued functions that are differentiable and $C^0(\mathbb{R})$ the set of all continuous real valued functions. Note that differentiation is a linear transformation. That is

$$D: C^1(\mathbb{R}) \longrightarrow C^0(\mathbb{R}), \quad D(f) = f'$$

satisfies the two conditions in the previous definition.

We know from calculus that if f and g are differentiable and c is a scalar, then

$$\frac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x) \quad \text{and} \quad \frac{d}{dx}(cf(x)) = cf'(x).$$

Using the current notation, we can write these statements like

$$D(f+g) = D(f)+D(g) \quad \text{and} \quad D(cf) = cD(f).$$

[1]This could also be written as $C^1(-\infty, \infty)$.

Example

Consider the derivative transformation on $C^1(\mathbb{R})$

$$D : C^1(\mathbb{R}) \longrightarrow C^0(\mathbb{R})$$

$$f \mapsto f'$$

Characterize the subset of $C^1(\mathbb{R})$ such that $D(f) = 0$. ← this is the zero function: $y = 0$

These are the constant functions $f(x) = C$ for some real number C.

Kernel and Range

Definition:

The **kernel** of a linear transformation $T : V \longrightarrow W$ is the set of all vectors $\mathbf{x}$ in V such that $T(\mathbf{x}) = \mathbf{0}$. (All solutions to a homogeneous equation.)

A null space is a **kernel.**

Definition:

The **range** of a linear transformation $T : V \longrightarrow W$ is the set of all vectors in W of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in V. (All the images of the transformation.)

A column space is a **range.**

Range & Kernel as Subspaces

Theorem:

Given a linear transformation $T : V \longrightarrow W$,

- the range of T is a subspace of W,
- and the kernel of T is a subspace of V.

Remark: This generalizes the result for column and null spaces. If $T : \mathbb{R}^n \to \mathbb{R}^m$, $T(\mathbf{x}) = A\mathbf{x}$. Then $\mathrm{Col}(A)$ is the range of T and is a subspace of $\mathbb{R}^m$. And $\mathrm{Nul}(A)$ is the kernel of T and is a subspace of $\mathbb{R}^n$.

Example

Consider $T : C^1(\mathbb{R}) \longrightarrow C^0(\mathbb{R})$ defined by

$$T(f) = \frac{df}{dx} + \alpha f(x), \quad \alpha \text{ a fixed constant.}$$

(a) Express the equation that a function y must satisfy if y is in the kernel of T.

If y is in the kernel, then

$T(y) = 0$; $T(y) = \frac{dy}{dx} + \alpha y$. The

equation is $\frac{dy}{dx} + \alpha y = 0$

$$T : C^1(\mathbb{R}) \longrightarrow C^0(\mathbb{R}), \quad T(f) = \frac{df}{dx} + \alpha f(x)$$

(b) Show that for any scalar c, $y = ce^{-\alpha x}$ is in the kernel of T.

If y is in the kernel of T, then

$\frac{dy}{dx} + \alpha y = 0$. Let $y = ce^{-\alpha x}$.

$$\frac{dy}{dx} = ce^{-\alpha x}(-\alpha) = -\alpha c e^{-\alpha x}$$

$$\frac{dy}{dx} + \alpha y \stackrel{?}{=} 0$$

$$-\alpha c e^{-\alpha x} + \alpha c e^{-\alpha x} \stackrel{?}{=} 0$$

$$0 = 0$$

So yes $y = ce^{-\alpha x}$ is in the kernel.

Section 4.3: Linearly Independent Sets and Bases

Definition:

A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in a vector space V is said to be **linearly independent** if the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \tag{1}$$

has only the trivial solutions $c_1 = c_2 = \cdots = c_p = 0$.

The set is **linearly dependent** if there exist a nontrivial solution (at least one of the weights c_i is nonzero).

If there is a nontrivial solution $c_1, \ldots, c_p$, then equation (1) is called a **linear dependence relation**.

Linearly Dependent Sets

Theorem:

Consider the ordered set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in a vector space V, where $p \geq 2$ and $\mathbf{v}_1 \neq \mathbf{0}$. This set is **linearly dependent** if and only if there is some $j > 1$ such that $\mathbf{v}_j$ is a linear combination of the preceding vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

This says that

1. If one of the vectors, say $\mathbf{v}_j$ can be written as a linear combo of the ones that come before it, the set is linearly dependent, and
2. if the set is linearly dependent, it must be possible to write one of the vectors as a linear combo of the others.

Example

Determine if the set $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is linearly dependent or independent in $\mathbb{P}_2$, where

$$\mathbf{p}_1 = 1, \quad \mathbf{p}_2 = 2t, \quad \text{and} \quad \mathbf{p}_3 = t - 3.$$

Note that $\vec{p}_3 = \frac{1}{2}\vec{p}_2 - 3\vec{p}_1$

Since $\vec{p}_3$ is a linear combo of $\vec{p}_1$ and $\vec{p}_2$, the set is linearly dependent.

A linear dependence relation is

$$3\vec{p}_1 - \frac{1}{2}\vec{p}_2 + \vec{p}_3 = \vec{0}$$

Basis

Definition:

Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a **basis** of H provided

(i) $\mathcal{B}$ is linearly independent, and

(ii) $H = \text{Span}(\mathcal{B})$.

Remark: We can think of a basis as a *minimal spanning set*. All of the *information* needed to construct vectors in H is contained in the basis, and none of this information is repeated.

A Basis for $\mathbb{R}^2$

Recall that $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and any vector $\mathbf{x}$ in $\mathbb{R}^2$ can be written as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Moreover, the equation $c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is only satisfied when $c_1 = 0$ and $c_2 = 0$.

So the set of vectors $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a **basis** for $\mathbb{R}^2$.

This set of two vectors $\{\mathbf{e}_1, \mathbf{e}_2\}$, along with the two operations **vector addition** and **scalar multiplication**, is all that is needed to build all of $\mathbb{R}^2$!

Standard or Elementary Basis in $\mathbb{R}^n$

The columns of the $n \times n$ identity matrix provide an obvious basis for $\mathbb{R}^n$. This is called the **standard basis** or the **elementary basis** for $\mathbb{R}^n$.

The examples in $\mathbb{R}^3$ and $\mathbb{R}^4$ are

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \text{and} \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\},$$

respectively.

When we want an ordered basis, we order these in the obvious way, $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$.

Other Vector Spaces

The set $\{1, t, t^2, t^3\}$ is a basis[2] for $\mathbb{P}_3$.

Notice that for any vector $\mathbf{p}$ in $\mathbb{P}_3$,

$$\mathbf{p}(t) = p_0 1 + p_1 t + p_2 t^2 + p_3 t^3.$$

This is a linear combination of 1, t, t^2, and t^3. We already know that the zero polynomial

$$\mathbf{0}(t) = 01 + 0t + 0t^2 + 0t^3.$$

That is, the equation

$$c_0 + c_1 t + c_2 t^2 + c_3 t^3 = 0 \quad \Leftrightarrow \quad c_0 = c_1 = c_2 = c_3 = 0$$

[2]The set $\{1, t, \dots, t^n\}$ is called the **standard basis** for $\mathbb{P}_n$

Other Vector Spaces

The set $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for $M_{2\times 2}$.

The exercise is left to the reader. It must be shown that

- every matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be written as a linear combination of these vectors and
- this is a linearly independent set.

Prelude to a Spanning Set Theorem

Example: Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ be vectors in a vector space V, and suppose that

(1) $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and
(2) $\mathbf{v}_3 = \mathbf{v}_1 - 2\mathbf{v}_2$.

Show that $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

We have to show that for any vector $\vec{u}$ in H, $\vec{u}$ is in Span $\{\vec{v}_1, \vec{v}_2\}$. This means that $\vec{u}$ can be written as a linear combo of $\vec{v}_1$ and $\vec{v}_2$. Since $\vec{u}$ is in H, $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$

Since $\vec{v}_3 = \vec{v}_1 - 2\vec{v}_2$, we can substitute

$$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3(\vec{v}_1 - 2\vec{v}_2)$$

$$= (c_1 + c_3)\vec{v}_1 + (c_2 - 2c_3)\vec{v}_2$$

$$= k_1\vec{v}_1 + k_2\vec{v}_2 \quad \text{where}$$

$k_1 = c_1 + c_3$ and $k_2 = c_2 - 2c_3$.

So $\vec{u}$ is in $\text{Span}\{\vec{v}_1, \vec{v}_2\}$.

i.e. $H = \text{Span}\{\vec{v}_1, \vec{v}_2\}$.

Spanning Set Theorem

Theorem

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ be a set in a vector space V and $H =$Span(S).

a. If one of the vectors in S, say $\mathbf{v}_k$ is a linear combination of the other vectors in S, then the subset of S obtained by eliminating $\mathbf{v}_k$ still spans H.

b. If $H \neq \{\mathbf{0}\}$, then some subset of S is a basis for H.

If we start with a spanning set, we can eliminate *duplication* to construct a **basis**.

Column Space

Find a basis for the column space matrix B that is in reduced row echelon form

$$B = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Label the columns $\vec{b}_1, \dots, \vec{b}_5$ so

$$B = [\vec{b}_1\ \vec{b}_2\ \vec{b}_3\ \vec{b}_4\ b_5]$$

$$Col(B) = Span\{\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4, \vec{b}_5\}.$$

$\vec{b}_2 = 4\vec{b}_1$ so we can remove $\vec{b}_2$

$\vec{b}_4 = 2\vec{b}_1 - \vec{b}_3$ so we can remove $\vec{b}_4$

$\{\vec{b}_1, \vec{b}_3, \vec{b}_5\}$ is linearly independent.

So $\{\vec{b}_1, \vec{b}_3, \vec{b}_5\}$ is a basis for $\mathrm{Col}(B)$.

Using the rref

Theorem:

If $A = [\mathbf{a}_1 \cdots \mathbf{a}_n]$ and $B = [\mathbf{b}_1 \cdots \mathbf{b}_n]$ are row equivalent matrices, then Nul A = Nul B. That is, the equations

$$A\mathbf{x} = \mathbf{0} \quad \text{and} \quad B\mathbf{x} = \mathbf{0}$$

have the same solution set.

Remark: This means that $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ and $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ have **exactly the same linear dependence relationships**!

Remark: We've actually already used this when we used an rref to characterize a null space.

A Basis for a Column Space

Theorem

Let A be an $m \times n$ matrix. The pivot columns of a matrix A form a basis of $\mathrm{Col}(A)$.

Caveat: This means we can use row reduction to identify a basis, but the vectors in the basis will be from the original matrix A.

Example

Consider the matrix A shown with a row equivalent rref. Find a basis for $\mathrm{Col}(A)$.

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We just need the pivot columns.

These are 1, 3, and 5. A basis for Col(A) is

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 2 \\ 8 \end{bmatrix} \right\}.$$

Basis for a Row Space

Theorem:

If two matrices A and B are row equivalent, then their row spaces are the same.

Remark: This tells us that a basis for the row space of an $m \times n$ matrix A is the set of nonzero rows of its rref.

Remark: Note how this is different from the column space. For $\mathrm{Col}(A)$, take the vectors from A, but for $\mathrm{Row}(A)$ take the vectors from the rref.

Find a basis for Row(A)

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis is the nonzero rows of the rref. A basis for Row(A) is

$$\left\{ \begin{bmatrix} 1 \\ 4 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Bases for $\text{Col}(A)$, $\text{Row}(A)$, and $\text{Nul}(A)$

Given a matrix A, find the rref. Then

- The pivot columns of the original matrix A give a basis for $\text{Col}(A)$.
- The nonzero rows of rref(A) give a basis for $\text{Row}(A)$.
- Use the rref to solve $A\mathbf{x} = \mathbf{0}$ to identify a basis for $\text{Nul}(A)$.

Find a basis for Nul(A)

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We need solutions to $A\vec{x} = \vec{0}$

From the rref

$x_1 = -4x_2 - 2x_4$

x_2 - free

$x_3 = x_4$

x_4 - free

$x_5 = 0$

For $\vec{x}$ in Nul(A)

$$\vec{x} = \begin{bmatrix} -4x_2 - 2x_4 \\ x_2 \\ x_4 \\ x_4 \\ 0 \end{bmatrix} = \begin{bmatrix} -4x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2x_4 \\ 0 \\ x_4 \\ x_4 \\ 0 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

A basis for Nul(A) is $\left\{ \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.