June 6 Math 3260 sec. 51 Summer 2023

Section 1.3: Vector Equations

We defined vectors (specifically real vectors) as $n \times 1$ column matrices. The set of all vectors of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

constitutes the set $\mathbb{R}^n$ (read "R n").

For example $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a vector in $\mathbb{R}^2$, $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a vector in $\mathbb{R}^3$, and so forth.

Vectors in $\mathbb{R}^n$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

The numbers, $x_1, x_2, \ldots, x_n$ in the vector are called **components** or **entries**. For vectors in $\mathbb{R}^n$, these are elements of the real numbers (denoted $\mathbb{R}$).

Scalars: $\mathbb{R}^n$ is an example of something called a **vector space**. Such a structure is always associated with another set (known as a *field*) of objects called **scalars**.

$\boxed{\text{Our set of scalars will be } \mathbb{R}.}$

Some Matrix & Vector Notation

If $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ is a collection of vectors in $\mathbb{R}^m$, then we can express the $m \times n$ matrix having these vectors as its columns

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Note that the number a_{ij} is the i^{th} component of the vector $\mathbf{a}_j$.

Equivalence & Operations

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ be in $\mathbb{R}^n$ and c is a scalar.

$$\text{Equivalence:} \quad \mathbf{u} = \mathbf{v} \quad \Leftrightarrow \quad u_i = v_i \quad \text{for each} \quad i = 1, \dots, n$$

$$\text{Scalar Multiplication:} \quad c\mathbf{u} = \begin{bmatrix} cu_1 \\ \vdots \\ cu_n \end{bmatrix}$$

$$\text{Vector Addition:} \quad \mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix}$$

The Zero Vector (denoted $\mathbf{0}$ or $\vec{0}$) is the vector whose entries are all zeros.

Algebraic Properties on $\mathbb{R}^n$

For every $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^n$ and scalars c and d[1]

(i) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(ii) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$

(iii) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$

(iv) $\mathbf{u}+(-\mathbf{u})=-\mathbf{u}+\mathbf{u}=\mathbf{0}$

(v) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$

(vi) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$

(vii) $c(d\mathbf{u})=d(c\mathbf{u})=(cd)\mathbf{u}$

(viii) $1\mathbf{u}=\mathbf{u}$

Remark: These properties follow easily from the algebraic properties on $\mathbb{R}$. They provide a structure that we will later associate with **vector spaces.**

[1] The term $-\mathbf{u}$ denotes $(-1)\mathbf{u}$.

Linear Combination & Span

Definition

A **linear combination** of vectors $\mathbf{v}_1, \ldots \mathbf{v}_p$ in $\mathbb{R}^n$ is a vector $\mathbf{y}$ of the form

$$\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

where the scalars $c_1, \ldots, c_p$ are often called weights.

Definition

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set of vectors in $\mathbb{R}^n$. The set of all linear combinations of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is the **span** of the set S. It is denoted by

$$\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = \text{Span}(S).$$

Equivalent Statements

Suppose $\mathbf{b}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are vectors in $\mathbb{R}^m$. The following are equivalent:

- $\mathbf{b}$ is in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$,
- $\mathbf{b} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$ for some scalars $c_1, \ldots, c_p$,
- the vector equation $x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{b}$ has a solution,
- the linear system of equations whose augmented matrix is $[\mathbf{v}_1 \ \cdots \ \mathbf{v}_p \ \mathbf{b}]$ is consistent.

Example

Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 4 \\ -2 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 5 \\ -5 \\ k \end{bmatrix}$. For which value(s) of k, if any, is $\mathbf{b}$ in Span $\{\mathbf{v}_1, \mathbf{v}_2\}$?

For what k is $\vec{b} = c_1\vec{v}_1 + c_2\vec{v}_2$?

For what k is the system w/ augmented matrix $[\vec{v}_1 \ \vec{v}_2 \ \vec{b}]$ consistent?

$$\begin{bmatrix} 1 & -1 & 5 \\ 1 & 4 & -5 \\ 2 & -2 & k \end{bmatrix} \xrightarrow[-2R_1+R_3 \to R_3]{-R_1+R_2 \to R_2} \begin{bmatrix} 1 & -1 & 5 \\ 0 & 5 & -10 \\ 0 & 0 & k-10 \end{bmatrix}$$

The system is only consistent if

$k - 10 = 0$, i.e., $k = 10$

$\vec{b}$ is in Span$\{\vec{v}_1, \vec{v}_2\}$ if

$k = 10$.

Some Geometry

Recall: The vectors in $\mathbb{R}^2$ represent the points in the Cartesian plane.

Give a geometric description of the subset of $\mathbb{R}^2$ given by

$$\text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.$$

If $\vec{x}$ is in $\text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ then

$$\vec{x} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}.$$

Allowing x_1 to vary, this gives the x-axis.

The set of all points (x, y) with

$y = 0$.

Span$\{\mathbf{u}\}$ in $\mathbb{R}^3$

If $\mathbf{u}$ is any nonzero vector in $\mathbb{R}^3$, then Span$\{\mathbf{u}\}$ is a line through the origin parallel to $\mathbf{u}$.

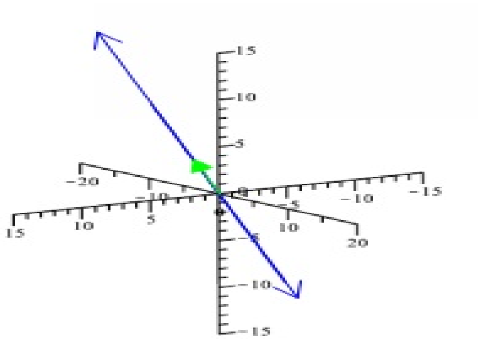

Figure: A nonzero vector $\mathbf{u}$ and the line Span$\{\mathbf{u}\}$ in $\mathbb{R}^3$.

Span$\{\mathbf{u}, \mathbf{v}\}$ in $\mathbb{R}^3$

If $\mathbf{u}$ and $\mathbf{v}$ are nonzero, and nonparallel vectors in $\mathbb{R}^3$, then Span$\{\mathbf{u}, \mathbf{v}\}$ is a plane containing the origin parallel to both vectors.

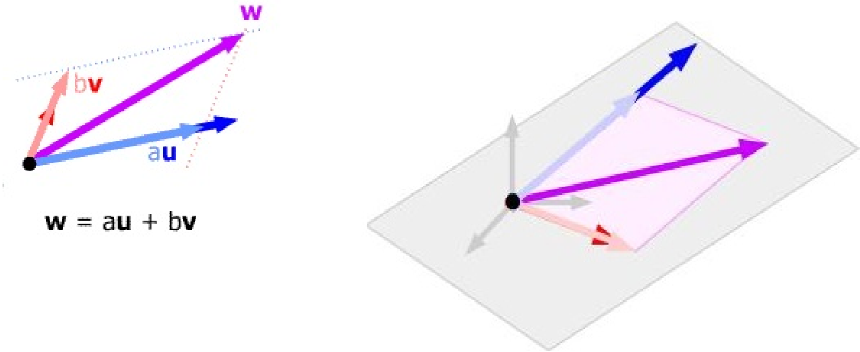

Figure: A vector $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$. If we let a and b vary, the collection of vectors Span$\{\mathbf{u}, \mathbf{v}\}$ is a plane.

Example

Let $\mathbf{u} = (1, 1)$ and $\mathbf{v} = (0, 2)$ in $\mathbb{R}^2$. Show that for every pair of real numbers a and b, that (a, b) is in Span$\{\mathbf{u}, \mathbf{v}\}$.

We want to show that

$$\begin{bmatrix} a \\ b \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

for some x_1, x_2 for all possible a, b.

This is true if the system w/ augmented matrix $\left[\vec{u} \;\; \vec{v} \;\; \begin{bmatrix} a \\ b \end{bmatrix} \right]$ is always consistent.

$$\begin{bmatrix} 1 & 0 & a \\ 1 & 2 & b \end{bmatrix} \xrightarrow{-R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 0 & a \\ 0 & 2 & b-a \end{bmatrix}$$

The third column will never be a pivot column, hence the system is consistent. That is, $\begin{bmatrix} a \\ b \end{bmatrix}$ is in Span $\{\vec{u}, \vec{v}\}$.

Note $\frac{1}{2}R_2 \to R_2$ $\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & \frac{b-a}{2} \end{bmatrix}$

$$\begin{bmatrix} a \\ b \end{bmatrix} = a\vec{u} + \left(\frac{b-a}{2}\right)\vec{v}.$$

Section 1.4: The Matrix Equation $A\mathbf{x} = \mathbf{b}$.

Definition

Let A be an $m \times n$ matrix whose columns are the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\cdots$, $\mathbf{a}_n$ (each in $\mathbb{R}^m$), and let $\mathbf{x}$ be a vector in $\mathbb{R}^n$. Then the product of A and $\mathbf{x}$, denoted by

$$A\mathbf{x}$$

is the linear combination of the columns of A whose weights are the corresponding entries in $\mathbf{x}$. That is

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n.$$

Remark: Note that based on the definition of scalar multiplication and vector addition, the product is a vector in $\mathbb{R}^m$.

Example: Find the product $A\mathbf{x}$.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ -2 & -1 & 4 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{a}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$A\vec{x} = 2\begin{bmatrix} 1 \\ -2 \end{bmatrix} + 1\begin{bmatrix} 0 \\ -1 \end{bmatrix} + (-1)\begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -4 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ -9 \end{bmatrix},$$

Example: Find the product $A\mathbf{x}$.

$$A = \begin{bmatrix} 2 & 4 \\ -1 & 1 \\ 0 & 3 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$\vec{a}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \qquad \vec{a}_2 = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

$$A\vec{x} = -3\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -6 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 8 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$$

Example

Is the product $A\mathbf{x}$ defined if $A = \begin{bmatrix} 2 & 4 \\ -1 & 1 \\ 0 & 3 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$?

This is not defined.

The number of entries in the vector has to match the number of columns in the matrix. This is not true for this matrix A and vector **x**.

Linear Systems, Vector Equations, & Matrix Equations

Write the linear system as a vector equation and then as a matrix equation of the form $A\mathbf{x} = \mathbf{b}$.

$$\begin{array}{rcrcrcr} 2x_1 & - & 3x_2 & + & x_3 & = & 2 \\ x_1 & + & x_2 & + & & = & -1 \end{array}$$

As a vector equation.

$$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

As a matrix equation

$$\begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$A \qquad \vec{x} \qquad \vec{b}$$

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Theorem

Theorem

If A is the $m \times n$ matrix whose columns are the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\cdots, \mathbf{a}_n$, and $\mathbf{b}$ is in $\mathbb{R}^m$, then the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

which, in turn, has the same solution set as the linear system of equations whose augmented matrix is

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n \quad \mathbf{b}].$$

Corollary

Corollary

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.

Remark

In other words, if $A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n]$, then the corresponding linear system, $A\mathbf{x} = \mathbf{b}$, is consistent if and only if $\mathbf{b}$ is in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$.

Example

Characterize the set of all vectors $\mathbf{b} = (b_1, b_2, b_3)$ such that $A\mathbf{x} = \mathbf{b}$ has a solution where

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}.$$

We want

$A\vec{x} = \vec{b}$ solvable

This will hold if the system w/ augmented matrix $[A \ \vec{b}]$ is consistent.

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \quad \begin{matrix} 4R_1 + R_2 \to R_2 \\ 3R_1 + R_3 \to R_3 \end{matrix} \quad \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\left[\begin{array}{cccc} 1 & 3 & 4 & b_1 \\ 0 & 7 & 5 & b_3+3b_1 \\ 0 & 14 & 10 & b_2+4b_1 \end{array}\right]$$

$-2R_2+R_3 \to R_3$

$$\left[\begin{array}{cccc} 1 & 3 & 4 & b_1 \\ 0 & 7 & 5 & b_3+3b_1 \\ 0 & 0 & 0 & b_2-2b_1-2b_3 \end{array}\right]$$

The system is only consistent if

$$-2b_1+b_2-2b_3=0$$

So $A\vec{x}=\vec{b}$ will be solvable if $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ satisfies: $-2b_1+b_2-2b_3=0$

We can state this as $\vec{b}$ belonging to a given Span.

$$-2b_1 + b_2 - 2b_3 = 0$$

Think of b_1 as basic and b_2, b_3 as free vaiebles.

$$b_1 = \frac{1}{2} b_2 - b_3$$

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} b_2 - b_3 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} b_2 \\ b_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -b_3 \\ 0 \\ b_3 \end{bmatrix}$$

$$= b_2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

We can say that $A\vec{x} = \vec{b}$ is solvable if $\vec{b}$ is in

$$\operatorname{Span}\left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Theorem (first in a string of equivalency theorems)

Theorem

Let A be an $m \times n$ matrix. Then the following are logically equivalent (i.e. they are either all true or are all false).

(a) For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

(b) Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

(c) The columns of A span $\mathbb{R}^m$.

(d) A has a pivot position in every row.

Remark: That last statement, (d), is about *coefficient* matrix A. It's not about an augmented matrix $[A \quad \mathbf{b}]$.

A Special Product

Definition

Consider two vectors $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ in $\mathbb{R}^n$. The **dot product** of $\mathbf{x}$ and $\mathbf{y}$, denoted

$$\mathbf{x} \cdot \mathbf{y},$$

is defined by

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Remark: Note that the dot product of two vectors is a scalar. This is an example of an *inner product*. It's sometimes called a *scalar product*.

Computing $A\mathbf{x}$

The dot product can be used as an alternative way of computing a product $A\mathbf{x}$. If A is an $m \times n$ matrix and $\mathbf{x}$ is a vector in $\mathbb{R}^n$, then the i^{th} component of the product $A\mathbf{x}$ is the dot product of the i^{th} **row** of A and the vector $\mathbf{x}$.

For example

$$\begin{bmatrix} 1 & 0 & -3 \\ -2 & -1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} (1)(2) + (0)(1) + (-3)(-1) \\ (-2)(2) + (-1)(1) + (4)(-1) \end{bmatrix} = \begin{bmatrix} 5 \\ -9 \end{bmatrix}$$

Evaluate

Use the dot product approach to compute each product.

$$\begin{bmatrix} 2 & 4 \\ -1 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2(-3) + 4(2) \\ -1(-3) + 1(2) \\ 0(-3) + 3(2) \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1x_1 + 0x_2 + 0x_3 \\ 0x_1 + 1x_2 + 0x_3 \\ 0x_1 + 0x_2 + 1x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Identity Matrix

Definition: Identity Matrix

We'll call an $n \times n$ matrix with 1's on the diagonal and 0's everywhere else—i.e. one that looks like

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

the $n \times n$ **identity** matrix and denote it by I_n. (We'll drop the subscript if it's obvious from the context.)

This matrix has the property that for each $\mathbf{x}$ in $\mathbb{R}^n$

$$I_n\mathbf{x} = \mathbf{x}.$$

Properties of the Matrix Product

Theorem

If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and c is any scalar, then

(a) $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$, and

(b) $A(c\mathbf{u}) = cA\mathbf{u}$.

Remark: These two properties are a pretty **big deal**! These are the two properties that constitute being **Linear** (as in *Linear Algebra*).

Section 1.5: Solution Sets of Linear Systems

Definition

A linear system is said to be **homogeneous** if it can be written in the form

$$A\mathbf{x} = \mathbf{0}$$

for some $m \times n$ matrix A and where $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$.

Theorem

Theorem: A homogeneous system $A\mathbf{x} = \mathbf{0}$ always has at least one solution, $\mathbf{x} = \mathbf{0}$, called the **trivial solution**.

Homogeneous Linear Systems

We know that the homogeneous system

$$A\mathbf{x} = \mathbf{0}$$

is always consistent because $\mathbf{x} = \mathbf{0}$ (in $\mathbb{R}^n$) is necessarily a solution.

The interesting question is:

Does $A\mathbf{x} = \mathbf{0}$ have any **nontrivial** *solutions?*

Theorem

The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the system has at least one free variable.

Homogeneous Linear Systems

Example:

Determine if the homogeneous system has a nontrivial solution. Describe the solution set.

(a) $$\begin{array}{rcrcl} 2x_1 & + & x_2 & = & 0 \\ x_1 & - & 3x_2 & = & 0 \end{array}$$

Consider the augmented matrix,

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & -3 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

There are no nontrivial solutions

The solution set is $\{(0,0)\}$.

(b)
$$\begin{array}{rcrcrcr} 3x_1 & + & 5x_2 & - & 4x_3 & = & 0 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ 6x_1 & + & x_2 & - & 8x_3 & = & 0 \end{array}$$

Using an augmented matrix

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3 will be free, so there are non trivial solutions. Solving for x_1, x_2, x_3

$$x_1 = \frac{4}{3}x_3$$

$$x_2 = 0$$

x_3 is free

We can write $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$

The solution set contains all vectors of the form

$$\vec{x} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}, \quad x_3 \text{ in } \mathbb{R}.$$

(c) $x_1 - 2x_2 + 5x_3 = 0$

$$\begin{bmatrix} 1 & -2 & 5 & 0 \end{bmatrix}$$

x_1 is basic, x_2, x_3 are free

so there are nontrivial solutions

$x_1 = 2x_2 - 5x_3$

x_2, x_3 - free

solutions satisfy

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 - 5x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -5x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$\vec{x} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$$

x_2, x_3 in $\mathbb{R}$

Parametric Vector Form of a Solution Set

Example (b) had a solution set consisting of vectors of the form $\mathbf{x} = x_3\mathbf{v}$. Example (c)'s solution set consisted of vectors that look like $\mathbf{x} = x_2\mathbf{u} + x_3\mathbf{v}$. Instead of using the variables x_2 and/or x_3 we often substitute **parameters** such as s or t.

Parametric Vector Form of a Solution Set

The forms

$$\mathbf{x} = s\mathbf{u}, \quad \text{or} \quad \mathbf{x} = s\mathbf{u} + t\mathbf{v}$$

are called **parametric vector forms.**

Remark: Since these are **linear combinations**, an alternative way to express the solution sets would be

$$\text{Span}\{\mathbf{u}\} \quad \text{or} \quad \text{Span}\{\mathbf{u}, \mathbf{v}\}.$$