June 8 Math 3260 sec. 51 Summer 2023

Section 1.5: Solution Sets of Linear Systems

We said that a linear system $A\mathbf{x} = \mathbf{b}$ is **homogeneous** if $\mathbf{b} = \mathbf{0}$. That is, a homogeneous system is one of the form

$$A\mathbf{x} = \mathbf{0}$$

for some $m \times n$ matrix A and where $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$.

Two Theorems

(1) The homogeneous equation $A\mathbf{x} = \mathbf{0}$ is always consistent because the trivial solution $\mathbf{x} = \mathbf{0}$ is a solution.

(2) Moreover, it has nontrivial solutions if and only if the system has at least one free variable.

Homogeneous Linear Systems

We can determine whether a homogeneous system $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions using an augmented matrix $[A\,\mathbf{0}]$. We looked at some examples.

(a) $$\begin{array}{rcrcl} 2x_1 & + & x_2 & = & 0 \\ x_1 & - & 3x_2 & = & 0 \end{array}$$

The augmented matrix $\begin{bmatrix} 2 & 1 & 0 \\ 1 & -3 & 0 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.
So the solution set is $\{(0,0)\}$. There are no nontrivial solutions.

Homogeneous Linear Systems

$$\text{(b)}\quad \begin{array}{rcrcrcr} 3x_1 & + & 5x_2 & - & 4x_3 & = & 0 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ 6x_1 & + & x_2 & - & 8x_3 & = & 0 \end{array}$$

Using the augmented matrix and row operations gives

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can see that there are nontrivial solutions because there are three variables but only two pivot columns. x_3 is a free variable.

Solution of Homogeneous Linear System

$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The rref can be used to describe the solution set in various ways.

Parametric description: $\left\{ \begin{array}{lcl} x_1 & = & \frac{4}{3}x_3 \\ x_2 & = & 0 \\ x_3 & & \text{is free} \end{array} \right.$

Parametric Vector Form: $\mathbf{x} = t \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$

In terms of span*: $\mathbf{x} \in \text{Span}\left\{ \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} \right\}$

The symbol "$\in$" means *is an element of*.

* Later, we'll say that we're describing the set as a *subspace* of $\mathbb{R}^m$.

Geometry $\mathbf{x} = t\left(\frac{4}{3}, 0, 1\right)$ in $\mathbb{R}^3$

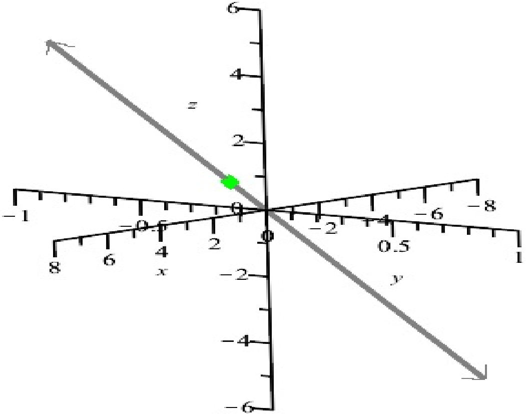

Figure: Plot of the line $\mathbf{x} = t\begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$. The point $\left(\frac{4}{3}, 0, 1\right)$ is shown in green.

(c) $x_1 - 2x_2 + 5x_3 = 0$

The augmented matrix $[\ 1 \quad -2 \quad 5 \quad 0\ ]$ is already an rref. There are nontrivial solutions because there are two free variables. We expressed the solution set

Parametric description: $\begin{cases} x_1 = 2x_2 - 5x_3 \\ x_2, x_3 \text{ are free} \end{cases}$

Parametric Vector Form: $\mathbf{x} = s\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}$

In terms of span: $\mathbf{x} \in \text{Span}\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}\right\}$

This is a plane in $\mathbb{R}^3$ that contains the points $(0,0,0)$, $(2,1,0)$, and $(-5,0,1)$.

Geometry

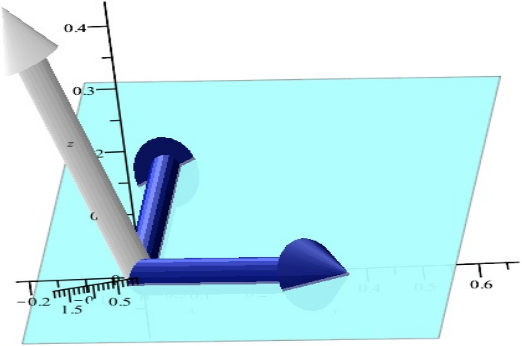

Figure: Plot of the plane $\mathbf{x} = s\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$. The blue vectors are in the directions of $(2, 1, 0)$ and $(-5, 0, 1)$. The white vector is *normal* (i.e., perpendicular) to the plane.

Nonhomogeneous Systems

Find all solutions of the nonhomogeneous system of equations

$$\begin{array}{rcrcrcr} 3x_1 & + & 5x_2 & - & 4x_3 & = & 7 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & -1 \\ 6x_1 & + & x_2 & - & 8x_3 & = & -4 \end{array}$$

We can use an augmented matrix

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \xrightarrow[\text{TI 92}]{\text{rref}} \begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

↑ = sign here

The solutions in parametric description are

$x_1 = -1 + \frac{4}{3}x_3$

$x_2 = 2$

x_3 is free

Let's convert to parametric vector form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

Geometry $\mathbf{x} = (-1, 2, 0) + t\left(\frac{4}{3}, 0, 1\right)$ in $\mathbb{R}^3$

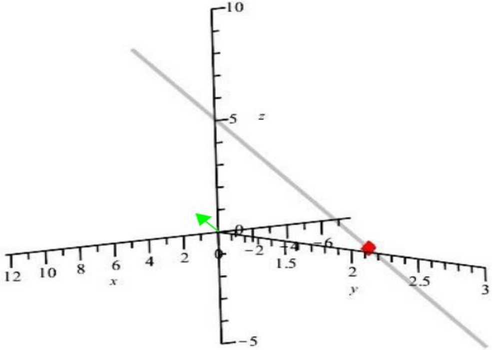

Figure: Plot of the line $\mathbf{x} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$. The point $(-1, 2, 0)$ is shown in red, and the vector $\left(\frac{4}{3}, 0, 1\right)$ is shown in green.

Solutions of Nonhomogeneous Systems

Note that the solution in this example has the form

Parametric vector form

$$\mathbf{x} = \mathbf{p} + t\mathbf{v}$$

with **p** and **v** fixed vectors and t a varying parameter. Also note that the $t\mathbf{v}$ part is the solution to the previous example with the right hand side all zeros. This is no coincidence!

Definition

The vector **p** satisfying the nonhomogeneous system $A\mathbf{p} = \mathbf{b}$ is called a **particular solution**.

The term $t\mathbf{v}$ is called a solution to the associated homogeneous equation.

General Solution Nonhomogeneous Equation

Theorem

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for a given $\mathbf{b}$. Let $\mathbf{p}$ be a particular solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form

$$\mathbf{x} = \mathbf{p} + \mathbf{v}_h,$$

where $\mathbf{v}_h$ is any solution of the associated homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Remark: We can use a row reduction technique to get all parts of the solution in one process.

Example

Find the solution set of the following system. Express the solution set in parametric vector form.

$$\begin{array}{rcrcrcrcl} x_1 & - & 2x_2 & & & + & x_4 & = & 2 \\ 3x_1 & - & 6x_2 & + & x_3 & - & x_4 & = & 7 \end{array}$$

We can use an augmented matrix

$$\begin{bmatrix} 1 & -2 & 0 & 1 & 2 \\ 3 & -6 & 1 & -1 & 7 \end{bmatrix} \xrightarrow[\text{TI-92}]{\text{rref}} \begin{bmatrix} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -4 & 1 \end{bmatrix}$$

Describing the solutions

$x_1 = 2 + 2x_2 - x_4$

x_2 - free

$x_3 = 1 + 4x_4$

x_4 - free

Converting to parametric vector form.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 + 2x_2 - x_4 \\ x_2 \\ 1 + 4x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -x_4 \\ 0 \\ 4x_4 \\ x_4 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\vec{p}} + \underbrace{x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 4 \\ 1 \end{bmatrix}}_{\vec{v}_h}$$

The solutions are

$$\vec{x} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 4 \\ 1 \end{bmatrix}$$

$s, t \in \mathbb{R}$.

Section 1.7: Linear Independence

We already know that a homogeneous equation $A\mathbf{x} = \mathbf{0}$ can be thought of as an equation in the column vectors of the matrix $A = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n]$ as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots x_n\mathbf{a}_n = \mathbf{0}.$$

And, we know that at least one solution (the trivial one $x_1 = x_2 = \cdots = x_n = 0$) always exists.

Remark: The existence, or not, of a nontrivial solution is a property of the set of vectors $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$.

Definition: Linear Independence

Definition:

An indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution.

If a set of vectors is not linearly independent, we say that it is **linearly dependent**.

Linear Dependence & Independence

We can restate this definition:

Definition:

The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exists a set of weights $c_1, c_2, \ldots, c_p$, *at least one of which is nonzero*, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_p\mathbf{v}_p = \mathbf{0}.$$

Remark: The phrase "*at least one of which is nonzero*" is a reference to a **nontrivial solution.**

Definition:

An equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots c_p\mathbf{v}_p = \mathbf{0}$, with at least one $c_i \neq 0$, is called a **linear dependence relation.**

Theorem on Linear Independence

Theorem:

The columns of a matrix A are linearly **independent** if and only if the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Remark: This follows from the definition of linear independence. This connects a homogeneous system $A\mathbf{x} = \mathbf{0}$ with a property of the columns of A as a set of vectors.

Example

(a) Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent or linearly independent.

An option is to create a matrix with $\vec{v}_1$ and $\vec{v}_2$ as columns. Say $A = [\vec{v}_1 \;\; \vec{v}_2]$.

Consider the homogeneous eqn. $A\vec{x} = \vec{0}$.

The augmented matrix is

$$\begin{bmatrix} 2 & 1 & 0 \\ 4 & -2 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$A\vec{x} = \vec{0}$ has no notrivial solutions

Hence the columns of A are linearly independent.

That is, $\{\vec{v}_1, \vec{v}_2\}$ is

linearly independent.

Example

(b) Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent or linearly independent.

Note $\vec{v}_1 + \vec{v}_2 = \vec{v}_3$. We can create a linear dependence relation by subtracting $\vec{v}_3$ to get

$$\vec{v}_1 + \vec{v}_2 - \vec{v}_3 = \vec{0}$$

The coefficients are 1, 1 and -1,

so at least one of them is non zero..

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly

dependent.

Example

(c) Determine if the set of vectors is linearly dependent or linearly independent. If dependent, find a linear dependence relation.

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$$

Call them $\vec{v}_i$ in the order given.

Let $A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{v}_4]$ and consider $A\vec{x} = \vec{0}$.

The augmented matrix

$$\begin{bmatrix} 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ 0 & 2 & 3 & 2 & 0 \\ 0 & 1 & 3 & 0 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 1/3 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The columns are linearly dependent since the system has free variable(s).

The solution to $A\vec{x} = \vec{0}$ satisfies

$$x_1 = -\frac{1}{3}x_4$$

$$x_2 = -2x_4$$

$$x_3 = \frac{2}{3}x_4$$

x_4 is free

We can write

$$-\frac{1}{3}x_4\vec{v}_1 - 2x_4\vec{v}_2 + \frac{2}{3}x_4\vec{v}_3 + x_4\vec{v}_4 = \vec{0}$$

Setting $x_4 = -3$ gives the

linear dependence relation

$$\vec{v}_1 + 6\vec{v}_2 - 2\vec{v}_3 - 3\vec{v}_4 = \vec{0}$$

Note: This isn't the only possible lin. dependence relation. We could choose a different x_4 value (e.g. $x_4 = 1$). The coefficients can be different, but they'll all have a common ratio relationship.

Theorem

Theorem

An indexed set of two or more vectors is linearly dependent if and only if at least one vector in the set is a linear combination of the others in the set.

Example: Let **u** and **v** be any nonzero vectors in $\mathbb{R}^3$. Show that if **w** is any vector in Span$\{\mathbf{u}, \mathbf{v}\}$, then the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly **dependent.**

Since $\vec{w}$ is in Span$\{\vec{u}, \vec{v}\}$,

$\vec{w} = c_1\vec{u} + c_2\vec{v}$ for some scalars

c_1 and c_2. We can rearrange this

to get

$$c_1\vec{u} + c_2\vec{v} - \vec{w} = \vec{0}.$$

This is a linear dependence relation because, the coefficient of $\vec{w}$ is -1, which is not zero.

Hence $\{\vec{u}, \vec{v}, \vec{w}\}$ is necessarily linearly dependent.

Caveat!

A set may be linearly dependent even if all proper subsets are linearly independent. For example, consider

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Each set $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$, and $\{\mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. (You can easily verify this.)

However,

$$\mathbf{v}_3 = \mathbf{v}_2 - \mathbf{v}_1 \quad \text{i.e.} \quad \mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0},$$

so the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

This means that you can't just consider two vectors at a time.

Two More Theorems

Theorem:

If a set contains more vectors than there are entries in each vector, then the set is linearly **dependent**. That is, if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a set of vector in $\mathbb{R}^n$, and $p > n$, then the set is linearly dependent.

For example, if you have 7 vectors, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7\}$, and each of these is a vector in $\mathbb{R}^5$, i.e., $\mathbf{v}_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \\ v_{41} \\ v_{51} \end{bmatrix}$ and so forth, then they must be **linearly dependent** because $7 > 5$.

Two More Theorems

Theorem:

Any set of vectors that contains the zero vector is linearly **dependent.**

Consider the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p, \mathbf{0}\}$ in $\mathbb{R}^n$. Note that

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_p + 1\mathbf{0} = \mathbf{0}$$

is a **linear dependence relation** because the last coefficient $c_{p+1} = 1$ is nonzero. It doesn't matter what the other vectors are or what the values of p and n are relative to one another!

Examples

Without doing any computations, determine, with justification, whether the given set is linearly dependent or linearly independent.

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \right\}$

This is 4 vectors in $\mathbb{R}^3$. Then are linearly dependent because $4 > 3$.

Examples

(b) $\left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -8 \\ 1 \end{bmatrix}, \right\}$

This set contains $\vec{0}$. It is linearly dependent.