Linear Dependence

Definition 1. A set of functions $f_1, f_2, \ldots, f_n$ are said to be linearly dependent on an interval I if there exists a set of constants $c_1, c_2, \ldots, c_n$ not all zero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all } x \text{ in } I.$$

Let's look at some actual examples of functions that are or are not linearly dependent. The simplest case is when you have two functions, say $f(x)$ and $g(x)$. We can say the following:

Linear Dependence of 2 Functions: Two functions f and g are linearly dependent if one is a constant multiple of the other. Below is a table of examples of pairs of functions that are linearly dependent or linearly independent with some explanations.

$f(x)$	$g(x)$	Dependent?	Why	Lin. Dep. Relation
x	$-3x$	yes dependent	$g(x) = -3f(x)$	$3f(x) + g(x) = 0$
e^{4x+1}	e^{4x}	yes dependent	$f(x) = eg(x)$	$f(x) - eg(x) = 0$
x^3	$2x$	no independent	$f(x) \neq cg(x)$ for any c	There isn't one.
$\cos^2 x$	$1 - \sin^2 x$	yes dependent	$f(x) = g(x)$ (trig ID)	$f(x) - g(x) = 0$
0	$\cos(x)$	yes dependent	$f(x) = 0g(x)$	$1f(x) + 0g(x) = 0$ (note one coefficient is not zero)
$\cos x$	$\sin x$	no independent	$f(x) \neq cg(x)$ for any c	There isn't one.

So when you only have two functions, the only way they can be linearly dependent is if one is just the other multiplied by a constant.

With three functions, it's a little more complicated, but we can extend the idea. We can say that three functions, say f, g, and h are linearly dependent if one can be written by additing multiples of the others together.

Linear Dependence of 3 Functions: Three functions f, g, and h are linearly dependent if one of them is a linear combination of the other two. Here's a similar table with examples involving three functions.

$f(x)$	$g(x)$	$h(x)$	Dependent?	Why	Lin. Dep. Relation
x	1	$x-1$	yes dependent	$h(x)=f(x)-g(x)$	$f(x)-g(x)-h(x)=0$
x	1	x^2	no independent	$h(x)$ is not equal to $c_1f(x)+c_2g(x)$ for any numbers c_1, c_2	There isn't one.
$\cos(2x)$	1	$\sin^2 x$	yes dependent	$h(x)=\frac{1}{2}g(x)-\frac{1}{2}f(x)$ (Trig ID)	$f(x)-g(x)+2h(x)=0$
e^x	$2e^{2x}$	e^x+e^{2x}	yes dependent	$h(x)=f(x)+\frac{1}{2}g(x)$	$f(x)+\frac{1}{2}g(x)-h(x)=0$
0	$\cos(x)$	$\sin(x)$	yes dependent	$f(x)=0g(x)+0h(x)$	$1f(x)+0g(x)+0h(x)=0$ (note one coefficient is not zero)
$\cos x$	$\sin x$	x	no independent	$h(x)$ is not equal to $c_1f(x)+c_2g(x)$ for any numbers c_1, c_2	There isn't one.

Four or More Functions This generalizes to any number of functions. If I have four functions, f, g, h, and u, they are linearly dependent if I can write one of them as a linear combination of the others (i.e. one of them is formed by adding constant multiples of the others together). So, for example

$$f(x)=1, \quad g(x)=x, \quad h(x)=x^2, \quad \text{and} \quad u(x)=3x^2+4x-5$$

would be linearly dependent because

$$u(x)=3h(x)+4g(x)-5f(x).$$

Notice that this is the same as saying that

$$3h(x)+4g(x)-5f(x)-u(x)=0.$$

But

$$f(x)=1, \quad g(x)=x, \quad h(x)=x^2, \quad \text{and} \quad u(x)=x^3$$

are linearly independent because there's no way to create x^3 just by multiplying 1, x, and x^2 by numbers and adding them together (for all possible values of x). That is the equation

$$c_1(1)+c_2x+c_3x^2+c_4x^3=0 \quad \text{for all } x$$

is only true if all of the c's are set equal to zero.

Why do we care? The point is that we want to know what all the solutions to a linear differential equation can be. Take the simple example

$$y''-3y'+2y=0.$$

It has characteristic equation $m^2-3m+2=0$ with roots $m=1$ and $m=2$. So we can say that all the solutions are linear combinations of the two functions e^x and e^{2x}.

Someone could say that all the solutions are linear combinations of e^x, e^{2x} and $3e^x$. But we don't say that because it's misleading. Why would you add $3e^x$ to the list when it doesn't really add anything? We already have e^x, so $3e^x$ is not really new information. It makes it look like we might be able to have three initial conditions, but that's not true.

What's the difference between these sets (aside from the number of functions being 3 instead of 2)?

$$e^x, \quad e^{2x} \quad \text{is linearly independent}$$

but

$$e^x, \quad e^{2x}, \quad 3e^x \quad \text{is linearly Dependent.}$$

In fact

$$c_1e^x + c_2e^{2x} = 0 \quad \text{for all } x \text{ is } \textbf{only true} \text{ if both } c\text{'s are zero.}$$

While

$$-3(e^x) + 0(e^{2x}) + 1(3e^x) = 0 \quad \text{for all x.}$$