2.5 Nonhomogeneous Equations and Undetermined Coefficients

2.5.1 The General Solution of a Linear Nonhomogeneous Equation

The linear non-homogeneous equation has the form

$$y'' + p(t)y' + q(t)y = g(t), \quad a < t < b.$$

The general solution of the nonhomogeneous equation = The general solution of the homogeneous equation + A particular solution of the nonhomogeneous equation.

Example 2.14. Verify that $y_P(t) = 2e^{t-\pi}$ is a solution of $y'' + 4y = 10e^{t-\pi}$. Determine the complementary solution and find the solution using the initial conditions $y(\pi) = 2$ and $y'(\pi) = 0$.

Theorem 2.6. *Let $u(t)$ be a solution of $y'' + p(t)y' + q(t)y = g_1(t)$, $a < t < b$. Let $v(t)$ be a solution of $y'' + p(t)y' + q(t)y = g_2(t)$, $a < t < b$. Let a_1 and a_2 be any constants. Then the function* ____________________ *is a particular solution of* ____________________ .

This theorem allows us to break the process of solving the equations with a complicated right hand side into several steps.

Example 2.15. Given that $u(t) = 2e^{4t}$ is a particular solution of $y'' - y' - 2y = 20e^{4t}$ and $v(t) = 3t - 4$ is a particular solution of $y'' - y' - 2y = 5 - 6t$, find a particular solution of $y'' - y' - 2y = -5e^{4t} + 20 - 24t$.

2.5.2 Method of Undetermined Coefficients

The idea here is to guess what particular solution solves the nonhomogeneous equation. We will try a y_p that is similar to $g(t)$.

Example 2.16. Solve the differential equation

$$y'' + 3y' + 2y = 5e^{2t}.$$

Solution:

Example 2.17. Solve the differential equation

$$3y'' + y' - 2y = 2\cos t$$

Solution:

Check Figure 2.5.1 on page 156 of the textbook to see the remaining particular solutions you should try for different forms of $g(t)$ ($f(x)$ in your textbook).

Caution: This method has problems when proposed y_p contains elements of y_c.

Example 2.18. Find the general solution for

$$y'' + 3y' + 2y = 5e^{-2t}.$$

Solution:

Example 2.19. Find the correct form for $y_p(t)$ when

$$y'' + 4y = 2t^2 + 5\sin(2t) + e^{3t}$$

Solution:

Example 2.20. Find the general solution of

$$y'' - y' - 2y = 3t^3.$$

Solution:

2.5.3 Method of Variation of Parameters

This approach is more general in the sense that it helps deal with more complicated functions for $g(t)$.

Consider the general linear second order nonhomogeneous differential equation $y'' + p(t)y' + q(t)y = g(t)$, where $p(t)$, $q(t)$, and $g(t)$ are continuous on the t-interval of interest.

Assume

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

where $\{y_1(t), y_2(t)\}$ is a fundamental set of solutions of $y'' + p(t)y' + q(t)y = 0$.

Example 2.21. Determine the complementary solution for

$$y'' - 4y = 10.$$

After that, use the method of variation of parameters to find the particular solution.
Solution:

Example 2.22. Find the general solution for

$$y'' - 3y' + 2y = \frac{1}{1 + e^{-t}}$$

<u>*Solution*</u>: