5.8 Nonhomogeneous Linear Systems

We now address the problem of finding the general solution of a nonhomogeneous first order linear system ______________________________.

Theorem 5.5. *Let $\boldsymbol{u}(t)$ be a solution of $\boldsymbol{y}' = P(t)\boldsymbol{y} + \boldsymbol{g}_1(t)$, $a < t < b$, and let $\boldsymbol{v}(t)$ be a solution of $\boldsymbol{y}' = P(t)\boldsymbol{y} + \boldsymbol{g}_2(t)$, $a < t < b$. Let a_1 and a_2 be any constants. Then the vector function $\boldsymbol{y}_p(t) = a_1\boldsymbol{u}(t) + a_2\boldsymbol{v}(t)$ is a particular solution of* ________________________________.

Definition 5.5. *Let $\{\boldsymbol{y}_1(t), \boldsymbol{y}_2(t), \ldots, \boldsymbol{y}_n(t)\}$ be a set of solutions of a homogeneous first order linear system $\boldsymbol{y}' = P(t)\boldsymbol{y}$. The $n \times n$ matrix whose columns consist of solutions* ____.

Theorem 5.6. *Consider the homogeneous linear first order system*

$$\boldsymbol{y}' = P(t)\boldsymbol{y}, \quad a < t < b. \tag{5.2}$$

1. *Let $\Psi(t)$ be any solution matrix of Eq. (5.2). Then $\Psi(t)$ satisfies the matrix differential equation*

2. *Let Ψ_0 represent any given constant $n \times n$ matrix, and let t_0 be any fixed point in the interval (a, b). Then there is a unique solution $n \times n$ matrix $\Psi(t)$ that solves the initial value problem*

3. *If $\Psi(t)$ is any fundamental matrix and $\hat{\Psi}(t)$ is any solution matrix of Eq. (5.2), then there exists an $n \times n$ constant matrix C such that*

Example 5.24. Find the solution matrix that satisfies the following initial value problem

$$\boldsymbol{y}'(t) = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \boldsymbol{y}(t),\ \boldsymbol{y}(0) = \boldsymbol{y}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and the given eigenpairs $\left(3, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ and $\left(-1, \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)$.

<u>*Solution*</u>:

The Method of Variation of Parameters

Consider the non-homogeneous initial value problem

$$\boldsymbol{y}'(t) = P(t)\boldsymbol{y}(t) + \boldsymbol{g}(t),\ \ \boldsymbol{y}(t_0) = \boldsymbol{y}_0,\ \ a < t < b, \tag{5.3}$$

where the $n \times n$ coefficient matrix $P(t)$ and the $n \times 1$ vector function $\boldsymbol{g}(t)$ are continuous on (a, b), and $t_0 \in (a, b)$.

Assume that we know a fundamental matrix $\Psi(t)$ such that $\Psi'(t) = P(t)\Psi(t),\ \ a < t < b$. The complementary solution is $\boldsymbol{y}_c(t) = \Psi(t)\boldsymbol{c}$ where $\boldsymbol{c}$ is an arbitrary $n \times 1$ vector.

We "vary the parameter" and look for particular solution of the form ____________________________.

Example 5.25. Solve the initial value problem

$$\boldsymbol{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \boldsymbol{y} + \begin{bmatrix} e^{2t} \\ -2t \end{bmatrix}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solution:

Example 5.25 continued.

Example 5.26. Consider the system

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{y} + \begin{bmatrix} t \\ -1 \end{bmatrix}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

with the particular solution $\boldsymbol{y}_p(t) = t\boldsymbol{a} + \boldsymbol{b}$, where $\boldsymbol{a},\ \boldsymbol{b}$ are column vectors.

Solve the initial value problem given the eigenpairs $\left(1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ and $\left(-1, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$.

<u>*Solution*</u>:

Example 5.26 continued.