Math 2306 Lecture 33

Dr. Lihong Zhao

lzhao12@kennesaw.edu

Friday, November 7, 2025

Outline

Superposition Principle and Solution Matrix

Theorem (5.5)

Let $\boldsymbol{u}(t)$ be a solution of $\boldsymbol{y}' = P(t)\boldsymbol{y} + \boldsymbol{g}_1(t),\ a < t < b$, and let $\boldsymbol{v}(t)$ be a solution of $\boldsymbol{y}' = P(t)\boldsymbol{y} + \boldsymbol{g}_2(t),\ a < t < b$. Let a_1 and a_2 be any constants. Then the vector function $\boldsymbol{y}_p(t) = a_1\boldsymbol{u}(t) + a_2\boldsymbol{v}(t)$ is a particular solution of $\underline{\boldsymbol{y}' = P(t)\boldsymbol{y} + a_1\boldsymbol{g}_1(t) + a_2\boldsymbol{g}_2(t),\ a < t < b}$.

Definition

Let $\{\boldsymbol{y}_1(t), \boldsymbol{y}_2(t), \dots, \boldsymbol{y}_n(t)\}$ be a set of solutions of a homogeneous first order linear system $\boldsymbol{y}' = P(t)\boldsymbol{y}$. The $n \times n$ matrix whose columns consist of solutions $\underline{\Psi(t) = [\boldsymbol{y}_1(t), \boldsymbol{y}_2(t), \dots, \boldsymbol{y}_n(t)] \text{ is a solution matrix.}}$

Fundamental Matrix

Theorem (5.6)

Consider the homogeneous linear first order system

$$\mathbf{y}' = P(t)\mathbf{y}, \quad a < t < b. \tag{1}$$

1. *Let* $\Psi(t)$ *be any solution matrix of Eq. (1). Then* $\Psi(t)$ *satisfies the matrix differential equation*
2. *Let* Ψ_0 *represent any given constant* $n \times n$ *matrix, and let* t_0 *be any fixed point in the interval* (a, b). *Then there is a unique solution* $n \times n$ *matrix* $\Psi(t)$ *that solves the initial value problem*
3. *If* $\Psi(t)$ *is any fundamental matrix and* $\hat{\Psi}(t)$ *is any solution matrix of Eq. (1), then there exists an* $n \times n$ *constant matrix* C *such that*

Fundamental Matrix

Theorem (5.6)

Consider the homogeneous linear first order system

$$\mathbf{y}' = P(t)\mathbf{y}, \quad a < t < b. \tag{1}$$

1. *Let* $\Psi(t)$ *be any solution matrix of Eq. (1). Then* $\Psi(t)$ *satisfies the matrix differential equation* $\Psi'(t) = P(t)\Psi(t),\ a < t < b$.
2. *Let* Ψ_0 *represent any given constant* $n \times n$ *matrix, and let* t_0 *be any fixed point in the interval* (a, b). *Then there is a unique solution* $n \times n$ *matrix* $\Psi(t)$ *that solves the initial value problem*
3. *If* $\Psi(t)$ *is any fundamental matrix and* $\hat{\Psi}(t)$ *is any solution matrix of Eq. (1), then there exists an* $n \times n$ *constant matrix* C *such that*

Fundamental Matrix

Theorem (5.6)

Consider the homogeneous linear first order system

$$\mathbf{y}' = P(t)\mathbf{y}, \quad a < t < b. \tag{1}$$

1. *Let* $\Psi(t)$ *be any solution matrix of Eq. (1). Then* $\Psi(t)$ *satisfies the matrix differential equation* $\Psi'(t) = P(t)\Psi(t),\ a < t < b.$
2. *Let* Ψ_0 *represent any given constant* $n \times n$ *matrix, and let* t_0 *be any fixed point in the interval* (a, b)*. Then there is a unique solution* $n \times n$ *matrix* $\Psi(t)$ *that solves the initial value problem* $\Psi'(t) = P(t)\Psi(t),\ \Psi(t_0) = \Psi_0,\ a < t < b.$
3. *If* $\Psi(t)$ *is any fundamental matrix and* $\hat{\Psi}(t)$ *is any solution matrix of Eq. (1), then there exists an* $n \times n$ *constant matrix* C *such that*

Fundamental Matrix

Theorem (5.6)

Consider the homogeneous linear first order system

$$\boldsymbol{y}' = P(t)\boldsymbol{y}, \quad a < t < b. \tag{1}$$

1. *Let* $\Psi(t)$ *be any solution matrix of Eq. (1). Then* $\Psi(t)$ *satisfies the matrix differential equation* $\Psi'(t) = P(t)\Psi(t)$, $a < t < b$.
2. *Let* Ψ_0 *represent any given constant* $n \times n$ *matrix, and let* t_0 *be any fixed point in the interval* (a, b). *Then there is a unique solution* $n \times n$ *matrix* $\Psi(t)$ *that solves the initial value problem* $\Psi'(t) = P(t)\Psi(t)$, $\Psi(t_0) = \Psi_0$, $a < t < b$. *Moreover, if the constant matrix* Ψ_0 *is invertible (i.e., has nonzero determinant), then the matrix* $\Psi(t)$ *is a fundamental matrix of Eq. (1).*
3. *If* $\Psi(t)$ *is any fundamental matrix and* $\hat{\Psi}(t)$ *is any solution matrix of Eq. (1), then there exists an* $n \times n$ *constant matrix* C *such that*

Fundamental Matrix

Theorem (5.6)

Consider the homogeneous linear first order system

$$\boldsymbol{y}' = P(t)\boldsymbol{y}, \quad a < t < b. \tag{1}$$

1. *Let* $\Psi(t)$ *be any solution matrix of Eq. (1). Then* $\Psi(t)$ *satisfies the matrix differential equation* $\Psi'(t) = P(t)\Psi(t),\ a < t < b$.
2. *Let* Ψ_0 *represent any given constant* $n \times n$ *matrix, and let* t_0 *be any fixed point in the interval* (a, b). *Then there is a unique solution* $n \times n$ *matrix* $\Psi(t)$ *that solves the initial value problem* $\Psi'(t) = P(t)\Psi(t),\ \Psi(t_0) = \Psi_0,\ a < t < b$. *Moreover, if the constant matrix* Ψ_0 *is invertible (i.e., has nonzero determinant), then the matrix* $\Psi(t)$ *is a fundamental matrix of Eq. (1).*
3. *If* $\Psi(t)$ *is any fundamental matrix and* $\hat{\Psi}(t)$ *is any solution matrix of Eq. (1), then there exists an* $n \times n$ *constant matrix* C *such that* $\hat{\Psi}(t) = \Psi(t) \cdot C,\ a < t < b$.

Fundamental Matrix

Theorem (5.6)

Consider the homogeneous linear first order system

$$\mathbf{y}' = P(t)\mathbf{y}, \quad a < t < b. \tag{1}$$

1. *Let* $\Psi(t)$ *be any solution matrix of Eq. (1). Then* $\Psi(t)$ *satisfies the matrix differential equation* $\Psi'(t) = P(t)\Psi(t),\ a < t < b$.
2. *Let* Ψ_0 *represent any given constant* $n \times n$ *matrix, and let* t_0 *be any fixed point in the interval* (a, b). *Then there is a unique solution* $n \times n$ *matrix* $\Psi(t)$ *that solves the initial value problem* $\Psi'(t) = P(t)\Psi(t),\ \Psi(t_0) = \Psi_0,\ a < t < b$. *Moreover, if the constant matrix* Ψ_0 *is invertible (i.e., has nonzero determinant), then the matrix* $\Psi(t)$ *is a fundamental matrix of Eq. (1).*
3. *If* $\Psi(t)$ *is any fundamental matrix and* $\hat{\Psi}(t)$ *is any solution matrix of Eq. (1), then there exists an* $n \times n$ *constant matrix* C *such that* $\hat{\Psi}(t) = \Psi(t) \cdot C,\ a < t < b$. *Moreover, the matrix* $\hat{\Psi}(t)$ *is also a fundamental matrix if and only if* $\det(C) \neq 0$.

Example 5.24

Find the solution matrix that satisfies the following initial value problem $\boldsymbol{y}'(t) = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \boldsymbol{y}(t)$, $\boldsymbol{y}(0) = \boldsymbol{y}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and the given eigenpairs $\left(3, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$ and $\left(-1, \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right)$.

Example 5.24 (cont'd)

The Method of Variation of Parameters

Consider the non-homogeneous initial value problem

$$\boldsymbol{y}'(t) = P(t)\boldsymbol{y}(t) + \boldsymbol{g}(t), \ \ \boldsymbol{y}(t_0) = \boldsymbol{y}_0, \ \ a < t < b, \tag{2}$$

where the $n \times n$ coefficient matrix $P(t)$ and the $n \times 1$ vector function $\boldsymbol{g}(t)$ are continuous on (a, b), and $t_0 \in (a, b)$.

The Method of Variation of Parameters

Consider the non-homogeneous initial value problem

$$\boldsymbol{y}'(t) = P(t)\boldsymbol{y}(t) + \boldsymbol{g}(t),\ \ \boldsymbol{y}(t_0) = \boldsymbol{y}_0,\ \ a < t < b, \qquad (2)$$

where the $n \times n$ coefficient matrix $P(t)$ and the $n \times 1$ vector function $\boldsymbol{g}(t)$ are continuous on (a, b), and $t_0 \in (a, b)$.
Assume that we know a fundamental matrix $\Psi(t)$ such that $\Psi'(t) = P(t)\Psi(t),\ \ a < t < b$. The complementary solution is $\boldsymbol{y}_c(t) = \Psi(t)\boldsymbol{c}$ where $\boldsymbol{c}$ is an arbitrary $n \times 1$ vector.

The Method of Variation of Parameters

Consider the non-homogeneous initial value problem

$$\boldsymbol{y}'(t) = P(t)\boldsymbol{y}(t) + \boldsymbol{g}(t),\ \ \boldsymbol{y}(t_0) = \boldsymbol{y}_0,\ \ a < t < b, \tag{2}$$

where the $n \times n$ coefficient matrix $P(t)$ and the $n \times 1$ vector function $\boldsymbol{g}(t)$ are continuous on (a, b), and $t_0 \in (a, b)$.
Assume that we know a fundamental matrix $\Psi(t)$ such that $\Psi'(t) = P(t)\Psi(t),\ \ a < t < b$. The complementary solution is $\boldsymbol{y}_c(t) = \Psi(t)\boldsymbol{c}$ where $\boldsymbol{c}$ is an arbitrary $n \times 1$ vector.
We "vary the parameter" and look for particular solution of the form $\boldsymbol{y}_p(t) = \Psi(t)\boldsymbol{u}(t)$ where $\boldsymbol{u}(t)$ is an unknown $n \times 1$ matrix function to be determined.

The Method of Variation of Parameters

Consider the non-homogeneous initial value problem

$$\boldsymbol{y}'(t) = P(t)\boldsymbol{y}(t) + \boldsymbol{g}(t), \ \ \boldsymbol{y}(t_0) = \boldsymbol{y}_0, \ \ a < t < b, \qquad (2)$$

where the $n \times n$ coefficient matrix $P(t)$ and the $n \times 1$ vector function $\boldsymbol{g}(t)$ are continuous on (a, b), and $t_0 \in (a, b)$.
Assume that we know a fundamental matrix $\Psi(t)$ such that $\Psi'(t) = P(t)\Psi(t), \ a < t < b$. The complementary solution is $\boldsymbol{y}_c(t) = \Psi(t)\boldsymbol{c}$ where $\boldsymbol{c}$ is an arbitrary $n \times 1$ vector.
We "vary the parameter" and look for particular solution of the form $\boldsymbol{y}_p(t) = \Psi(t)\boldsymbol{u}(t)$ where $\boldsymbol{u}(t)$ is an unknown $n \times 1$ matrix function to be determined. Substituting it into Eq. (2) leads to

$$[\Psi(t)\boldsymbol{u}(t)]' = P(t)[\Psi(t)\boldsymbol{u}(t)] + \boldsymbol{g}(t)$$

The Method of Variation of Parameters

Consider the non-homogeneous initial value problem

$$\boldsymbol{y}'(t) = P(t)\boldsymbol{y}(t) + \boldsymbol{g}(t), \ \ \boldsymbol{y}(t_0) = \boldsymbol{y}_0, \ \ a < t < b, \qquad (2)$$

where the $n \times n$ coefficient matrix $P(t)$ and the $n \times 1$ vector function $\boldsymbol{g}(t)$ are continuous on (a, b), and $t_0 \in (a, b)$.
Assume that we know a fundamental matrix $\Psi(t)$ such that $\Psi'(t) = P(t)\Psi(t), \ \ a < t < b$. The complementary solution is $\boldsymbol{y}_c(t) = \Psi(t)\boldsymbol{c}$ where $\boldsymbol{c}$ is an arbitrary $n \times 1$ vector.
We "vary the parameter" and look for particular solution of the form $\boldsymbol{y}_p(t) = \Psi(t)\boldsymbol{u}(t)$ where $\boldsymbol{u}(t)$ is an unknown $n \times 1$ matrix function to be determined. Substituting it into Eq. (2) leads to

$$\begin{aligned} & [\Psi(t)\boldsymbol{u}(t)]' = P(t)[\Psi(t)\boldsymbol{u}(t)] + \boldsymbol{g}(t) \\ \Rightarrow \ \ & \Psi'(t)\boldsymbol{u}(t) + \Psi(t)\boldsymbol{u}'(t) = P(t)\Psi(t)\boldsymbol{u}(t) + \boldsymbol{g}(t) \end{aligned}$$

The Method of Variation of Parameters

Consider the non-homogeneous initial value problem

$$\boldsymbol{y}'(t) = P(t)\boldsymbol{y}(t) + \boldsymbol{g}(t), \; \boldsymbol{y}(t_0) = \boldsymbol{y}_0, \; a < t < b, \tag{2}$$

where the $n \times n$ coefficient matrix $P(t)$ and the $n \times 1$ vector function $\boldsymbol{g}(t)$ are continuous on (a, b), and $t_0 \in (a, b)$.
Assume that we know a fundamental matrix $\Psi(t)$ such that $\Psi'(t) = P(t)\Psi(t), \; a < t < b$. The complementary solution is $\boldsymbol{y}_c(t) = \Psi(t)\boldsymbol{c}$ where $\boldsymbol{c}$ is an arbitrary $n \times 1$ vector.
We "vary the parameter" and look for particular solution of the form $\boldsymbol{y}_p(t) = \Psi(t)\boldsymbol{u}(t)$ where $\boldsymbol{u}(t)$ is an unknown $n \times 1$ matrix function to be determined. Substituting it into Eq. (2) leads to

$$\begin{aligned}
& [\Psi(t)\boldsymbol{u}(t)]' = P(t)[\Psi(t)\boldsymbol{u}(t)] + \boldsymbol{g}(t) \\
\Rightarrow \quad & \Psi'(t)\boldsymbol{u}(t) + \Psi(t)\boldsymbol{u}'(t) = P(t)\Psi(t)\boldsymbol{u}(t) + \boldsymbol{g}(t) \\
\Rightarrow \quad & \Psi(t)\boldsymbol{u}'(t) = \boldsymbol{g}(t)
\end{aligned}$$

The Method of Variation of Parameters

Consider the non-homogeneous initial value problem

$$\boldsymbol{y}'(t) = P(t)\boldsymbol{y}(t) + \boldsymbol{g}(t), \;\; \boldsymbol{y}(t_0) = \boldsymbol{y}_0, \;\; a < t < b, \qquad (2)$$

where the $n \times n$ coefficient matrix $P(t)$ and the $n \times 1$ vector function $\boldsymbol{g}(t)$ are continuous on (a, b), and $t_0 \in (a, b)$.
Assume that we know a fundamental matrix $\Psi(t)$ such that $\Psi'(t) = P(t)\Psi(t), \;\; a < t < b$. The complementary solution is $\boldsymbol{y}_c(t) = \Psi(t)\boldsymbol{c}$ where $\boldsymbol{c}$ is an arbitrary $n \times 1$ vector.
We "vary the parameter" and look for particular solution of the form $\boldsymbol{y}_p(t) = \Psi(t)\boldsymbol{u}(t)$ where $\boldsymbol{u}(t)$ is an unknown $n \times 1$ matrix function to be determined. Substituting it into Eq. (2) leads to

$$\begin{aligned}
& [\Psi(t)\boldsymbol{u}(t)]' = P(t)[\Psi(t)\boldsymbol{u}(t)] + \boldsymbol{g}(t) \\
\Rightarrow \;\; & \Psi'(t)\boldsymbol{u}(t) + \Psi(t)\boldsymbol{u}'(t) = P(t)\Psi(t)\boldsymbol{u}(t) + \boldsymbol{g}(t) \\
\Rightarrow \;\; & \Psi(t)\boldsymbol{u}'(t) = \boldsymbol{g}(t) \\
\Rightarrow \;\; & \boldsymbol{u}'(t) = \Psi^{-1}(t)\boldsymbol{g}(t)
\end{aligned}$$

The Method of Variation of Parameters

Consider the non-homogeneous initial value problem

$$\boldsymbol{y}'(t) = P(t)\boldsymbol{y}(t) + \boldsymbol{g}(t), \;\; \boldsymbol{y}(t_0) = \boldsymbol{y}_0, \;\; a < t < b, \qquad (2)$$

where the $n \times n$ coefficient matrix $P(t)$ and the $n \times 1$ vector function $\boldsymbol{g}(t)$ are continuous on (a, b), and $t_0 \in (a, b)$.
Assume that we know a fundamental matrix $\Psi(t)$ such that $\Psi'(t) = P(t)\Psi(t), \;\; a < t < b.$ The complementary solution is $\boldsymbol{y}_c(t) = \Psi(t)\boldsymbol{c}$ where $\boldsymbol{c}$ is an arbitrary $n \times 1$ vector.
We "vary the parameter" and look for particular solution of the form $\boldsymbol{y}_p(t) = \Psi(t)\boldsymbol{u}(t)$ where $\boldsymbol{u}(t)$ is an unknown $n \times 1$ matrix function to be determined. Substituting it into Eq. (2) leads to

$$\begin{aligned}
& [\Psi(t)\boldsymbol{u}(t)]' = P(t)[\Psi(t)\boldsymbol{u}(t)] + \boldsymbol{g}(t) \\
\Rightarrow \;\; & \Psi'(t)\boldsymbol{u}(t) + \Psi(t)\boldsymbol{u}'(t) = P(t)\Psi(t)\boldsymbol{u}(t) + \boldsymbol{g}(t) \\
\Rightarrow \;\; & \Psi(t)\boldsymbol{u}'(t) = \boldsymbol{g}(t) \\
\Rightarrow \;\; & \boldsymbol{u}'(t) = \Psi^{-1}(t)\boldsymbol{g}(t) \text{ as the fundamental matrix } \Psi(t) \text{ is invertible}
\end{aligned}$$

Example 5.25

Solve the IVP $\boldsymbol{y}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \boldsymbol{y} + \begin{pmatrix} e^{2t} \\ -2t \end{pmatrix}, \quad \boldsymbol{y}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$

Example 5.25 (cont'd)

Example 5.26

Consider the system $\boldsymbol{y}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \boldsymbol{y} + \begin{pmatrix} t \\ -1 \end{pmatrix}, \ \boldsymbol{y}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ with the particular solution $\boldsymbol{y}_p(t) = t\boldsymbol{a} + \boldsymbol{b}$, where $\boldsymbol{a}$, $\boldsymbol{b}$ are column vectors. Solve the initial value problem given the eigenpairs $\left(1, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$ and $\left(-1, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$.

Example 5.26 (cont'd)

Summary

Today we learned/reviewed

- more on nonhomogeneous linear systems
  - superposition principle and solution matrix
  - fundamental matrix
  - the method of variation of parameters
  - the method of undetermined coefficients

next time, we will learn

- Euler's method