Math 2306 Lecture 33

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Outline

Superposition Principle and Solution Matrix

Theorem (5.5)

Let $\boldsymbol{u}(t)$ be a solution of $\boldsymbol{y}' = P(t)\boldsymbol{y} + \boldsymbol{g}_1(t),\ a < t < b$, and let $\boldsymbol{v}(t)$ be a solution of $\boldsymbol{y}' = P(t)\boldsymbol{y} + \boldsymbol{g}_2(t),\ a < t < b$. Let a_1 and a_2 be any constants. Then the vector function $\boldsymbol{y}_p(t) = a_1\boldsymbol{u}(t) + a_2\boldsymbol{v}(t)$ is a particular solution of $\underline{\boldsymbol{y}' = P(t)\boldsymbol{y} + a_1\boldsymbol{g}_1(t) + a_2\boldsymbol{g}_2(t),\ a < t < b}$.

Definition

Let $\{\boldsymbol{y}_1(t), \boldsymbol{y}_2(t), \ldots, \boldsymbol{y}_n(t)\}$ be a set of solutions of a homogeneous first order linear system $\boldsymbol{y}' = P(t)\boldsymbol{y}$. The $n \times n$ matrix whose columns consist of solutions $\underline{\Psi(t) = [\boldsymbol{y}_1(t), \boldsymbol{y}_2(t), \ldots, \boldsymbol{y}_n(t)]\text{ is a solution matrix.}}$

Fundamental Matrix

Theorem (5.6)

Consider the homogeneous linear first order system

$$\boldsymbol{y}' = P(t)\boldsymbol{y}, \quad a < t < b. \tag{1}$$

1. *Let* $\Psi(t)$ *be any solution matrix of Eq. (1). Then* $\Psi(t)$ *satisfies the matrix differential equation*
2. *Let* Ψ_0 *represent any given constant* $n \times n$ *matrix, and let* t_0 *be any fixed point in the interval* (a, b)*. Then there is a unique solution* $n \times n$ *matrix* $\Psi(t)$ *that solves the initial value problem*
3. *If* $\Psi(t)$ *is any fundamental matrix and* $\hat{\Psi}(t)$ *is any solution matrix of Eq. (1), then there exists an* $n \times n$ *constant matrix* C *such that*

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3. *If* $\Psi(t)$ *is any fundamental matrix and* $\hat{\Psi}(t)$ *is any solution matrix of Eq. (1), then there exists an* $n \times n$ *constant matrix* C *such that* $\hat{\Psi}(t) = \Psi(t) \cdot C, \ a < t < b$. *Moreover, the matrix* $\hat{\Psi}(t)$ *is also a fundamental matrix if and only if* $\det(C) \neq 0$.

Example 5.24

Find the solution matrix that satisfies the following initial value problem $\boldsymbol{y}'(t) = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \boldsymbol{y}(t),\ \boldsymbol{y}(0) = \boldsymbol{y}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and the given eigenpairs $\left(3, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$ and $\left(-1, \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right)$.

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Solution: Given the eigenpairs, we obtain two solutions of the linear system $\boldsymbol{y}_1(t) = e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{y}_2(t) = e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$,

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<u>*Solution*</u>: Given the eigenpairs, we obtain two solutions of the linear system $\boldsymbol{y}_1(t) = e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{y}_2(t) = e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$, thus $\Psi(t) = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}$. This is also a fundamental matrix since eigenvalues are real and distinct (Theorem 5.4).

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Note that $\Psi(t)$ does not satisfy the initial value problem because

$$\Psi(0) = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \Psi_0$$

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$$\Rightarrow \quad C = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\frac{1}{4} \begin{pmatrix} -2 & -1 \\ -2 & 1 \end{pmatrix}$$

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Thus, the solution matrix of the initial value problem is

$$\begin{aligned} \hat{\Psi}(t) &= \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix} \begin{pmatrix} 1/2 & 1/4 \\ 1/2 & -1/4 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}\left(e^{3t} + e^{-t}\right) & \frac{1}{4}\left(e^{3t} - e^{-t}\right) \\ e^{3t} - e^{-t} & -\frac{1}{2}\left(e^{3t} + e^{-t}\right) \end{pmatrix} \end{aligned}$$

The Method of Variation of Parameters

Consider the non-homogeneous initial value problem

$$\boldsymbol{y}'(t) = P(t)\boldsymbol{y}(t) + \boldsymbol{g}(t), \;\; \boldsymbol{y}(t_0) = \boldsymbol{y}_0, \;\; a < t < b, \qquad (2)$$

where the $n \times n$ coefficient matrix $P(t)$ and the $n \times 1$ vector function $\boldsymbol{g}(t)$ are continuous on (a, b), and $t_0 \in (a, b)$.

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We "vary the parameter" and look for particular solution of the form $\boldsymbol{y}_p(t) = \Psi(t)\boldsymbol{u}(t)$ where $\boldsymbol{u}(t)$ is an unknown $n \times 1$ matrix function to be determined.

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$$\begin{aligned} & [\Psi(t)\boldsymbol{u}(t)]' = P(t)[\Psi(t)\boldsymbol{u}(t)] + \boldsymbol{g}(t) \\ \Rightarrow \quad & \Psi'(t)\boldsymbol{u}(t) + \Psi(t)\boldsymbol{u}'(t) = P(t)\Psi(t)\boldsymbol{u}(t) + \boldsymbol{g}(t) \end{aligned}$$

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\Rightarrow \quad & \boldsymbol{u}'(t) = \Psi^{-1}(t)\boldsymbol{g}(t) \text{ as the fundamental matrix } \Psi(t) \text{ is invertible}
\end{aligned}$$

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Solve the IVP $\boldsymbol{y}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \boldsymbol{y} + \begin{pmatrix} e^{2t} \\ -2t \end{pmatrix}, \quad \boldsymbol{y}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$

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Solution: First find the complimentary solution.

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$$\begin{pmatrix} 1-(-1) & 2 \\ 2 & 1-(-1) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_1 = -v_2$$

For $\lambda_2 = 3$,

$$\begin{pmatrix} 1-3 & 2 \\ 2 & 1-3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Example 5.25

Solve the IVP $\boldsymbol{y}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \boldsymbol{y} + \begin{pmatrix} e^{2t} \\ -2t \end{pmatrix}, \quad \boldsymbol{y}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$

Solution: First find the complimentary solution.

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (\lambda - 3)(\lambda + 1) = 0 \Rightarrow \lambda_1 = -1, \ \lambda_2 = 3$$

For $\lambda_1 = -1$,

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$$\begin{pmatrix} 1-3 & 2 \\ 2 & 1-3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_1 = v_2$$

Pick $v_1 = 1$ for both eigenvalues, we obtain the eigenpairs $\left(-1, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$ and $\left(3, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$.

Example 5.25 (cont'd)

Example 5.25 (cont'd)

The complimentary solution is $\boldsymbol{y}_c(t) = \begin{pmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$.

Example 5.25 (cont'd)

The complimentary solution is $\boldsymbol{y}_c(t) = \begin{pmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$.

To find the particular solution, we need to solve $\Psi(t)\boldsymbol{u}'(t) = \boldsymbol{g}(t)$.

Example 5.25 (cont'd)

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To find the particular solution, we need to solve $\Psi(t)\boldsymbol{u}'(t) = \boldsymbol{g}(t)$. Take the inverse of Ψ we have

$$\Psi^{-1}(t) = \frac{1}{2e^{2t}} \begin{pmatrix} e^{3t} & -e^{3t} \\ e^{-t} & e^{-t} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{t} & -e^{t} \\ e^{-3t} & e^{-3t} \end{pmatrix}.$$

Example 5.25 (cont'd)

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Then

$$\boldsymbol{u}'(t) = \Psi^{-1}(t)\boldsymbol{g}(t) = \frac{1}{2} \begin{pmatrix} e^{t} & -e^{t} \\ e^{-3t} & e^{-3t} \end{pmatrix} \begin{pmatrix} e^{2t} \\ -2t \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e^{3t} + te^{t} \\ \frac{1}{2}e^{-t} - te^{-3t} \end{pmatrix}.$$

Example 5.25 (cont'd)

The complimentary solution is $\boldsymbol{y}_c(t) = \begin{pmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$.

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$$\Psi^{-1}(t) = \frac{1}{2e^{2t}} \begin{pmatrix} e^{3t} & -e^{3t} \\ e^{-t} & e^{-t} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{t} & -e^{t} \\ e^{-3t} & e^{-3t} \end{pmatrix}.$$

Then

$$\boldsymbol{u}'(t) = \Psi^{-1}(t)\boldsymbol{g}(t) = \frac{1}{2} \begin{pmatrix} e^{t} & -e^{t} \\ e^{-3t} & e^{-3t} \end{pmatrix} \begin{pmatrix} e^{2t} \\ -2t \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e^{3t} + te^{t} \\ \frac{1}{2}e^{-t} - te^{-3t} \end{pmatrix}.$$

Integrating

$$\int_{t_0}^{t} \boldsymbol{u}'(s)\,\mathrm{d}s = \begin{pmatrix} \int_{t_0}^{t} \frac{1}{2}e^{3s} + se^{s}\,\mathrm{d}s \\ \int_{t_0}^{t} \frac{1}{2}e^{-s} - se^{-3s}\,\mathrm{d}s \end{pmatrix} = \begin{pmatrix} \frac{e^{3t}}{6} - e^{t} + te^{t} \\ -\frac{e^{-t}}{2} + \frac{e^{-3t}}{9} + \frac{te^{-3t}}{3} \end{pmatrix}.$$

Example 5.25 (cont'd)

Thus we obtain the particular solution

$$\begin{aligned} \boldsymbol{y}_p(t) &= \Psi(t)\boldsymbol{u}(t) = \begin{pmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{pmatrix} \begin{pmatrix} \frac{e^{3t}}{6} - e^t + te^t \\ -\frac{e^{-t}}{2} + \frac{e^{-3t}}{9} + \frac{te^{-3t}}{3} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{e^{2t}}{3} + \frac{4t}{3} - \frac{8}{9} \\ -\frac{2e^{2t}}{3} - \frac{2t}{3} + \frac{10}{9} \end{pmatrix}. \end{aligned}$$

Example 5.25 (cont'd)

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The general solution is

$$\boldsymbol{y}(t) = \begin{pmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} -\frac{e^{2t}}{3} + \frac{4t}{3} - \frac{8}{9} \\ -\frac{2e^{2t}}{3} - \frac{2t}{3} + \frac{10}{9} \end{pmatrix}.$$

Example 5.25 (cont'd)

Thus we obtain the particular solution

$$\begin{aligned}
\boldsymbol{y}_p(t) &= \Psi(t)\boldsymbol{u}(t) = \begin{pmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{pmatrix} \begin{pmatrix} \frac{e^{3t}}{6} - e^t + te^t \\ -\frac{e^{-t}}{2} + \frac{e^{-3t}}{9} + \frac{te^{-3t}}{3} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{e^{2t}}{3} + \frac{4t}{3} - \frac{8}{9} \\ -\frac{2e^{2t}}{3} - \frac{2t}{3} + \frac{10}{9} \end{pmatrix}.
\end{aligned}$$

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Imposing the initial condition to find c_1 and c_2,

$$\boldsymbol{y}(0) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} -\frac{11}{9} \\ \frac{4}{9} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Example 5.25 (cont'd)

Thus we obtain the particular solution

$$\begin{aligned}\boldsymbol{y}_p(t) &= \Psi(t)\boldsymbol{u}(t) = \begin{pmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{pmatrix} \begin{pmatrix} \frac{e^{3t}}{6} - e^t + te^t \\ -\frac{e^{-t}}{2} + \frac{e^{-3t}}{9} + \frac{te^{-3t}}{3} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{e^{2t}}{3} + \frac{4t}{3} - \frac{8}{9} \\ -\frac{2e^{2t}}{3} - \frac{2t}{3} + \frac{10}{9} \end{pmatrix}.\end{aligned}$$

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$$\boldsymbol{y}(t) = \begin{pmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} -\frac{e^{2t}}{3} + \frac{4t}{3} - \frac{8}{9} \\ -\frac{2e^{2t}}{3} - \frac{2t}{3} + \frac{10}{9} \end{pmatrix}.$$

Imposing the initial condition to find c_1 and c_2,

$$\boldsymbol{y}(0) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} -\frac{11}{9} \\ \frac{4}{9} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} c_1 = \frac{5}{6} \\ c_2 = \frac{7}{18} \end{cases}$$

Example 5.25 (cont'd)

The solution to the initial value problem is

$$\boldsymbol{y}(t) = \begin{pmatrix} e^{-t} & e^{3t} \\ -e^{-t} & e^{3t} \end{pmatrix} \begin{pmatrix} 5/6 \\ 7/18 \end{pmatrix} + \begin{pmatrix} -\frac{e^{2t}}{3} + \frac{4t}{3} - \frac{8}{9} \\ -\frac{2e^{2t}}{3} - \frac{2t}{3} + \frac{10}{9} \end{pmatrix}.$$

Example 5.26

Consider the system $\boldsymbol{y}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \boldsymbol{y} + \begin{pmatrix} t \\ -1 \end{pmatrix}$, $\boldsymbol{y}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ with the particular solution $\boldsymbol{y}_p(t) = t\boldsymbol{a} + \boldsymbol{b}$, where $\boldsymbol{a}$, $\boldsymbol{b}$ are column vectors. Solve the initial value problem given the eigenpairs $\left(1, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$ and $\left(-1, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$.

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Solution: Observe that the complementary solution is

$$\boldsymbol{y}_c(t) = c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

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We're given $\boldsymbol{y}_p(t) = t\boldsymbol{a} + \boldsymbol{b}$, so $\boldsymbol{y}_p'(t) = \boldsymbol{a}$.

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<u>*Solution*</u>: Observe that the complementary solution is
$\boldsymbol{y}_c(t) = c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

We're given $\boldsymbol{y}_p(t) = t\boldsymbol{a} + \boldsymbol{b}$, so $\boldsymbol{y}_p'(t) = \boldsymbol{a}$. Plugging into the system:

$$\boldsymbol{y}_p'(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \boldsymbol{y}_p + \begin{pmatrix} t \\ -1 \end{pmatrix}$$

$$\Rightarrow \quad \boldsymbol{a} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (t\boldsymbol{a} + \boldsymbol{b}) + \begin{pmatrix} t \\ -1 \end{pmatrix}$$

Example 5.26 (cont'd)

Example 5.26 (cont'd)

$$t\begin{pmatrix}0\\0\end{pmatrix}+\boldsymbol{a} \quad = t\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\boldsymbol{a}+\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\boldsymbol{b}+t\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\-1\end{pmatrix}$$

Example 5.26 (cont'd)

$$\begin{aligned} t\begin{pmatrix}0\\0\end{pmatrix}+\boldsymbol{a} &= t\begin{pmatrix}0&1\\1&0\end{pmatrix}\boldsymbol{a}+\begin{pmatrix}0&1\\1&0\end{pmatrix}\boldsymbol{b}+t\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\-1\end{pmatrix} \\ &= t\left[\begin{pmatrix}0&1\\1&0\end{pmatrix}\boldsymbol{a}+\begin{pmatrix}1\\0\end{pmatrix}\right]+\left[\begin{pmatrix}0&1\\1&0\end{pmatrix}\boldsymbol{b}+\begin{pmatrix}0\\-1\end{pmatrix}\right] \end{aligned}$$

Example 5.26 (cont'd)

$$
\begin{aligned}
t\begin{pmatrix}0\\0\end{pmatrix}+\boldsymbol{a} &= t\begin{pmatrix}0&1\\1&0\end{pmatrix}\boldsymbol{a}+\begin{pmatrix}0&1\\1&0\end{pmatrix}\boldsymbol{b}+t\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\-1\end{pmatrix}\\
&= t\left[\begin{pmatrix}0&1\\1&0\end{pmatrix}\boldsymbol{a}+\begin{pmatrix}1\\0\end{pmatrix}\right]+\left[\begin{pmatrix}0&1\\1&0\end{pmatrix}\boldsymbol{b}+\begin{pmatrix}0\\-1\end{pmatrix}\right]
\end{aligned}
$$

which leads to

$$
\begin{cases}
\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}0&1\\1&0\end{pmatrix}\boldsymbol{a}+\begin{pmatrix}1\\0\end{pmatrix}\\
\boldsymbol{a}=\begin{pmatrix}0&1\\1&0\end{pmatrix}\boldsymbol{b}+\begin{pmatrix}0\\-1\end{pmatrix}
\end{cases}
$$

Example 5.26 (cont'd)

$$
\begin{aligned}
t\begin{pmatrix}0\\0\end{pmatrix} + \boldsymbol{a} &= t\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\boldsymbol{a} + \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\boldsymbol{b} + t\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\-1\end{pmatrix}\\
&= t\left[\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\boldsymbol{a} + \begin{pmatrix}1\\0\end{pmatrix}\right] + \left[\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\boldsymbol{b} + \begin{pmatrix}0\\-1\end{pmatrix}\right]
\end{aligned}
$$

which leads to

$$
\begin{cases}
\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\boldsymbol{a} + \begin{pmatrix}1\\0\end{pmatrix}\\
\boldsymbol{a} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\boldsymbol{b} + \begin{pmatrix}0\\-1\end{pmatrix}
\end{cases}
\Rightarrow
\begin{cases}
\boldsymbol{a} = \begin{pmatrix}0\\-1\end{pmatrix}\\
\boldsymbol{b} = \begin{pmatrix}0\\0\end{pmatrix}
\end{cases}
$$

Example 5.26 (cont'd)

$$\begin{aligned} t\begin{pmatrix}0\\0\end{pmatrix}+\boldsymbol{a} &= t\begin{pmatrix}0&1\\1&0\end{pmatrix}\boldsymbol{a}+\begin{pmatrix}0&1\\1&0\end{pmatrix}\boldsymbol{b}+t\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\-1\end{pmatrix}\\ &= t\left[\begin{pmatrix}0&1\\1&0\end{pmatrix}\boldsymbol{a}+\begin{pmatrix}1\\0\end{pmatrix}\right]+\left[\begin{pmatrix}0&1\\1&0\end{pmatrix}\boldsymbol{b}+\begin{pmatrix}0\\-1\end{pmatrix}\right] \end{aligned}$$

which leads to

$$\begin{cases}\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}0&1\\1&0\end{pmatrix}\boldsymbol{a}+\begin{pmatrix}1\\0\end{pmatrix}\\ \boldsymbol{a}=\begin{pmatrix}0&1\\1&0\end{pmatrix}\boldsymbol{b}+\begin{pmatrix}0\\-1\end{pmatrix}\end{cases}\Rightarrow\begin{cases}\boldsymbol{a}=\begin{pmatrix}0\\-1\end{pmatrix}\\ \boldsymbol{b}=\begin{pmatrix}0\\0\end{pmatrix}\end{cases}$$

Hence, the particular solution is $\boldsymbol{y}_p(t)=t\boldsymbol{a}+\boldsymbol{b}=\begin{pmatrix}0\\-t\end{pmatrix}$, and the general solution is $\boldsymbol{y}(t)=c_1e^t\begin{pmatrix}1\\1\end{pmatrix}+c_2e^{-t}\begin{pmatrix}1\\-1\end{pmatrix}+\begin{pmatrix}0\\-t\end{pmatrix}$.

Example 5.26 (cont'd)

$$\boldsymbol{y}(t) = c_1 e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ -t \end{pmatrix}$$

Imposing the initial condition $\boldsymbol{y}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, we have

$$\begin{cases} c_1 + c_2 = 2 \\ c_1 - c_2 = -1 \end{cases} \Rightarrow \begin{cases} c_1 = 1/2 \\ c_2 = 3/2 \end{cases}$$

Therefore, the solution to the IVP is

$$\boldsymbol{y}(t) = \frac{1}{2} e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{3}{2} e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ -t \end{pmatrix}.$$

REMARK: this method is known as the method of undetermined coefficients.

Summary

Today we learned/reviewed

- more on nonhomogeneous linear systems
    - superposition principle and solution matrix
    - fundamental matrix
    - the method of variation of parameters
    - the method of undetermined coefficients

next time, we will learn

- Euler's method