March 13 Math 2306 sec. 52 Spring 2023

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials, e^{kx}
- sines and/or cosines, $\sin(kx)$, $\cos(kx)$ $\quad k$ - constant
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Method Basics: $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$

- Classify g as a certain *type*, and assume y_p is of this same type[1] with unspecified coefficients, A, B, C, etc.
- Substitute the assumed y_p into the ODE and collect like terms
- Match like terms on the left and right to get equations for the coefficients.
- Solve the resulting system to determine the coefficients for y_p.

[1] We will see shortly that our final conclusion on the format of y_p can depend on y_c.

Some Rules & Caveats

Rules of Thumb

- Polynomials include all powers from constant up to the degree.
- Where sines go, cosines follow and vice versa.
- Constants inside of sines, cosines, and exponentials (e.g., the “2” in e^{2x} or the “π” in $\sin(\pi x)$) are not undetermined. We don’t change those.

Caution

- The method is self correcting, but it’s best to get the set up correct.
- The form of y_p can depend on y_c. (More about this later.)

Initial Guesses

Let's go through several examples to determine what the form of y_p should be based on the right hand side $g(x)$.

At the moment, we're ignoring y_c. While **we can't do that in general**, we can always use these examples as the starting point for our set up.

We'll go through ten examples. Assume that we are dealing with an ODE that looks like

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

Examples of Forms of y_p based on g (Trial Guesses)

(a) $g(x) = 1$ (or really any nonzero constant)

$y_p = A$

(b) $g(x) = x - 7$ (1^{st} degree polynomial)

$y_p = Ax + B$

Examples of Forms of y_p based on g (Trial Guesses)

(c) $g(x) = 5x^2$ (2[nd] degree polynomial)

$$y_p = Ax^2 + Bx + C$$

(d) $g(x) = 3x^3 - 5$ (3[rd] degree polynomial)

$$y_p = Ax^3 + Bx^2 + Cx + D$$

Examples of Forms of y_p based on g (Trial Guesses)

(e) $g(x) = xe^{3x}$ (1[st] degree polynomial times e^{3x})

$$y_p = (Ax + B)e^{3x} = Axe^{3x} + Be^{3x}$$

(f) $g(x) = \cos(7x)$ (linear combo of cosine and sine of $7x$)

$$y_p = A\cos(7x) + B\sin(7x)$$

Examples of Forms of y_p based on g (Trial Guesses)

(g) $g(x) = \sin(2x) - \cos(4x)$ (two linear combos of sine/cosine)

$$y_p = A\sin(2x) + B\cos(2x) + C\sin(4x) + D\cos(4x)$$

(h) $g(x) = x^2\sin(3x)$ (linear combo 2^{nd} degree polynomial time sine and 2^{nd} degree poly times cosine)

$$y_p = (Ax^2 + Bx + C)\sin(3x) + (Dx^2 + Ex + F)\cos(3x)$$

Examples of Forms of y_p based on g (Trial Guesses)

(i) $g(x) = e^x \cos(2x)$ (linear combo of e^x cosine and e^x sine of $2x$)

$$y_p = Ae^x \cos(2x) + Be^x \sin(2x)$$

(j) $g(x) = xe^{-x} \sin(\pi x)$ (linear combo of 1^{st} poly times e^{-x} sine and 1^{st} poly times e^{-x} cosine)

$$y_p = (Ax+B)e^{-x} \sin(\pi x) + (Cx+D)e^{-x} \cos(\pi x)$$

The Superposition Principle

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x) + \ldots + g_k(x)$$

The principle of superposition for nonhomogeneous equations tells us that we can find y_p by considering separate problems

$$y_{p_1} \text{ solves } \quad a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x)$$

$$y_{p_2} \text{ solves } \quad a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_2(x),$$

and so forth.

Then $y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_k}$.

The Superposition Principle

Example: Determine the correct form of the particular solution using the method of undetermined coefficients for the ODE

$$y'' - 4y' + 4y = 6e^{-3x} + 16x^2$$

The principle of superposition says that we can consider the problem in parts.

1) Find y_{p_1} solving $y'' - 4y' + 4y = 6e^{-3x}$ $\boxed{y_{p_1} = Ae^{-3x}}$

2) Find y_{p_2} solving $y'' - 4y' + 4y = 16x^2$ $\boxed{y_{p_2} = Bx^2 + Cx + D}$

$$y_p = y_{p_1} + y_{p_2} \quad \Longrightarrow \quad \boxed{y_p = Ae^{-3x} + Bx^2 + Cx + D}$$

A Glitch!

What happens if the assumed form for y_p is part[2] of y_c? Consider applying the process to find a particular solution to the ODE

$$y'' - y' = 3e^x$$

$g(x) = 3e^x$

Set $y_p = Ae^x$

$y_p' = Ae^x$

$y_p'' = Ae^x$

substitute into ODE

$y_p'' - y_p' = 3e^x$

$Ae^x - Ae^x = 3e^x$

$0 = 3e^x$

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[2] A term in $g(x)$ is contained in a fundamental solution set of the associated homogeneous equation.

No value of A makes this true.

$y'' - y' = 3e^x$

Let's look @ y_c. y_c solves

$$y'' - y' = 0$$

Characteristic polynomial

$$m^2 - m = 0$$

$$m(m-1) = 0 \implies m = 0 \quad \text{or} \quad m = 1$$

$$y_1 = e^{0x} = 1, \quad y_2 = e^{1x} = e^x$$

$$y_c = c_1 + c_2 e^x$$

$y_p = Ae^x$ is part of y_c

The method can still be used. But we need to know how to modify our y_p to make the process work.

Cases: Comparing y_p to y_c

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x) + \ldots + g_k(x)$$

Consider one of the g's, say $g_i(x)$. We write out the guess for y_{p_i} and compare it to $y_c(x)$.

Case I: The guess for y_{p_i} **DOES NOT** have any like terms in common with y_c.

Then our guess for y_{p_i} will work as written. We do the substitution to find the A, B, etc.

Cases: Comparing y_p to y_c

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x) + \ldots + g_k(x)$$

Consider one of the g's, say $g_i(x)$. We write out the guess for y_{p_i} and compare it to $y_c(x)$.

Case II: The guess for y_{p_i} **DOES** have a like term in common with y_c.

Then we multiply our guess at y_{p_i} by x^n where n is the smallest positive integer such that our new guess $x^n y_{p_i}$ does not have any like terms in common with y_c. Then we take this new guess and substitute to find the A, B, etc.

Case II Examples

Find the general solution of the ODE.

$$y'' - y' = 3e^x$$

$y = y_c + y_p$

We found that $y_c = C_1 + C_2 e^x$

Find y_p.

$g(x) = 3e^x$

Set $y_p = Ae^x$ not correct, it duplicates part of y_c

Set $y_p = (Ae^x)x = Axe^x$ this is correct

$y'' - y' = 3e^x$

Substitute $y_p = Axe^x$

$y_p' = Ae^x + Axe^x$

$y_p'' = Ae^x + Ae^x + Axe^x$

$y_p'' - y_p' = 3e^x$

$2Ae^x + Axe^x - (Ae^x + Axe^x) = 3e^x$

$e^x(2A - A) + xe^x(A - A) = 3e^x$

$Ae^x = 3e^x$

$\Rightarrow A = 3.$

$y_p = Axe^x \Rightarrow y_p = 3xe^x$

The general solution

$$\boxed{y = c_1 + c_2 e^x + 3xe^x}$$