March 14 Math 3260 sec. 52 Spring 2022

Section 3.1: Introduction to Determinants

For an $n \times n$ matrix A, we defined a number $\det(A)$ called the **determinant** of the matrix. This number is a function of the entries in the matrix and it was defined so that

- if $\det(A) \neq 0$, then A is nonsingular (a.k.a invertible), and
- if $\det(A) = 0$, then A is singular.

2×2 and 3×3 Cases

2×2 $\quad \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a_{12}.$

3×3 $\quad \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} =$

$$= a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Minors & Cofactors

Definition: Let $n \geq 2$ and let $A = [a_{ij}]$ be an $n \times n$ matrix. The i, j^{th} **minor** of A is

$$M_{ij} = \det(A_{ij}),$$

where A_{ij} is the $(n-1) \times (n-1)$ matrix obtained from A by removing the i^{th} row and the j^{th} column.

The i, j^{th} **cofactor** of A is $C_{ij} = (-1)^{i+j} M_{ij}$.

Using the notation of cofactor and minors, the determinant of a 3×3 matrix A has the simple formula

$$\begin{aligned} \det(A) &= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} \\ &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \end{aligned}$$

The Determinant Defined

Definition: For $n \geq 2$, the **determinant** of the $n \times n$ matrix $A = [a_{ij}]$ is the number

$$\begin{aligned} \det(A) &= a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} M_{1j} \end{aligned} \tag{1}$$

The expression in equation (1) is called a **cofactor expansion.**

Example

Find all values of x such that A is singular where

$$A = \begin{bmatrix} 3-x & 2 & 1 \\ 0 & 2-x & 4 \\ 0 & 3 & 1-x \end{bmatrix}$$

We can use det(A) since det(A) = 0 when A is singular.

$$\det(A) = (3-x)\det\begin{bmatrix} 2-x & 4 \\ 3 & 1-x \end{bmatrix} - 2\underbrace{\det\begin{bmatrix} 0 & 4 \\ 0 & 1-x \end{bmatrix}}_{=0} + 1\underbrace{\det\begin{bmatrix} 0 & 2-x \\ 0 & 3 \end{bmatrix}}_{=0}$$

$$= (3-x)\left[(2-x)(1-x) - 12\right]$$

$$= (3-x)\left(2 - 3x + x^2 - 12\right)$$

$$= (3-x)\left(x^2 - 3x - 10\right)$$

$$= (3-x)(x-5)(x+2)$$

A is singular when $\det(A) = 0$. Set $\det(A) = 0$

$$0 = (3-x)(x-5)(x+2) \Rightarrow x = 3,\ x = 5,\ \text{or } x = -2.$$

Hence A is singular if $x = 3$, $x = 5$, or if $x = -2$.

Theorem:

The determinant of an $n \times n$ matrix can be computed by cofactor expansion across any row or down any column.

We can fix any row i of a matrix A and then

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

Or, we can fix any column j of a matrix A and then

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

Triangular Matrices

Definition: The $n \times n$ matrix $A = [a_{ij}]$ is said to be **upper triangular** if $a_{ij} = 0$ for all $i > j$.

It is said to be **lower triangular** if $a_{ij} = 0$ for all $j > i$. A matrix that is both upper and lower triangular is a diagonal matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

Upper Triangular **Lower Triangular**

Determinants of Triangular[1] Matrices

Theorem: For $n \geq 2$, the determinant of an $n \times n$ triangular matrix is the product of its diagonal entries.

That is, if $A = [a_{ij}]$ is a triangular matrix, then

$$\det(A) = a_{11}a_{22}a_{33}\cdots a_{nn}$$

[1] We'll use the catch-all name **triangular matrix** to refer to a matrix that is either uppper triangular or lower triangular.

Example: Compute $\det(A)$

(i) $A = \begin{bmatrix} -1 & 3 & 4 & 0 & 2 \\ 0 & 2 & -3 & 0 & -4 \\ 0 & 0 & 3 & 7 & 5 \\ 0 & 0 & 0 & -4 & 6 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix}$

$\det(A) = (-1)(2)(3)(-4)(6)$

$= 144$

upper triangular

(ii) $A = \begin{bmatrix} 7 & 0 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 4 & 2 & 2 & 2 \end{bmatrix}$

$\det(A) = 7(6)(2)(2)$

$= 168$

lower triangular

Section 3.2: Properties of Determinants

Theorem: Let A be an $n \times n$ matrix, and suppose the matrix B is obtained from A by performing a single elementary row operation[2]. Then

(i) If B is obtained by adding a multiple of a row of A to another row of A (row replacement), then
$$\det(B) = \det(A).$$

(ii) If B is obtained from A by swapping any pair of rows (row swap) , then
$$\det(B) = -\det(A).$$

(iii) If B is obtained from A by scaling any row by the constant k (scaling), then
$$\det(B) = k\det(A).$$

[2] If "row" is replaced by "column" in any of the operations, the conclusions still follow.

Example: Using Row Operations

Use row operations to obtain a triangular matrix, and then find the determinant.

$$\begin{bmatrix} 2 & 5 & -7 & 3 \\ 0 & 2 & 4 & -2 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{bmatrix} = A$$

Lets get an ref and use its determinant to find det(A).

Effect

$R_1 + R_4 \to R_4$

① no change

$$\begin{bmatrix} 2 & 5 & -7 & 3 \\ 0 & 2 & 4 & -2 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{bmatrix}$$

$\frac{1}{2} R_2 \to R_2$

② factor $\frac{1}{2}$

$$\begin{bmatrix} 2 & 5 & -7 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{bmatrix}$$

$-3R_2 + R_3 \to R_3$

$$\begin{bmatrix} 2 & 5 & -7 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & -3 & 1 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$\begin{bmatrix} 2 & 5 & -7 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

③ no change

④ factor -1

Call this matrix B.

$$\det(B) = 2(1)(-3)(5) = -30$$

From our row ops

$$\det(B) = \frac{1}{2}(-1)\det(A)$$

$$\Longrightarrow \quad \det(A) = -2\det(B) = -2(-30) = 60.$$

Some Theorems:

Theorem: The $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$.

Theorem: For $n \times n$ matrix A, $\det(A^T) = \det(A)$.

Theorem: For $n \times n$ matrices A and B, $\det(AB) = \det(A)\det(B)$.

Example

Show that if A is an $n \times n$ invertible matrix, then

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

If A is invertible, then $\det(A) \neq 0$. Also

$A^{-1}A = I$. Hence $\det(A^{-1}A) = \det(I) = 1$

Using the last theorem on the previous slide,

$$\det(A^{-1}A) = \det(A^{-1})\det(A) = 1.$$

$$\Longrightarrow \quad \det(A^{-1}) = \frac{1}{\det(A)}.$$

these are scalars

Example

Let A be an $n \times n$ matrix, and suppose there exists invertible matrix P such that

$$B = P^{-1}AP.$$

Show that

$$\det(B) = \det(A).$$

$$\det(B) = \det(P^{-1}AP)$$

$$= \det(P^{-1})\det(A)\det(P)$$

these are scalars

$$= \det(P^{-1})\det(P)\det(A)$$

$$= \underbrace{\frac{1}{\det(P)}\det(P)}_{1}\det(A)$$

$$= \det(A).$$