March 15 Math 2306 sec. 51 Spring 2023

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Method Basics: $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$

- Classify g as a certain *type*, and assume y_p is of this same type[1] with unspecified coefficients, A, B, C, etc.
- Compare y_p to y_c and modify as needed.
- Substitute the assumed y_p into the ODE and collect like terms
- Match like terms on the left and right to get equations for the coefficients.
- Solve the resulting system to determine the coefficients for y_p.

[1] We will see shortly that our final conclusion on the format of y_p can depend on y_c.

Cases: Comparing y_p to y_c

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x) + \ldots + g_k(x)$$

Consider one of the g's, say $g_i(x)$. We write out the guess for y_{p_i} and compare it to $y_c(x)$.

Case I: The guess for y_{p_i} **DOES NOT** have any like terms in common with y_c.

Then our guess for y_{p_i} will work as written. We do the substitution to find the A, B, etc.

Cases: Comparing y_p to y_c

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Case II: The guess for y_{p_i} **DOES** have a like term in common with y_c.

Then we multiply our guess at y_{p_i} by x^n where n is the smallest positive integer such that our new guess $x^n y_{p_i}$ does not have any like terms in common with y_c. Then we take this new guess and substitute to find the A, B, etc.

Find the **form** of the particular solution.

This is a *set up only* problem. Don't bother solving for the A, B, etc.

$$y'' - 4y' + 4y = \sin(4x) + xe^{2x}$$

Find y_c: y_c solves $y'' - 4y' + 4y = 0$

Charac. eqn. $m^2 - 4m + 4 = 0$

$(m-2)^2 = 0 \Rightarrow m = 2$ double root

$y_1 = e^{2x}$, $y_2 = xe^{2x}$ $y_c = c_1 e^{2x} + c_2 x e^{2x}$

Let's let y_{p_1} solve $y'' - 4y' + 4y = \sin(4x)$

and y_{p_2} solve $y'' - 4y' + 4y = x e^{2x}$

$$y'' - 4y' + 4y = \sin(4x) + xe^{2x}$$

$y_c = c_1 e^{2x} + c_2 x e^{2x}$

For y_{p_1}: $g_1(x) = \sin(4x)$

$y_{p_1} = A\sin(4x) + B\cos(4x)$ ✓

no terms common w/ y_c

For y_{p_2}: $g_2(x) = xe^{2x}$

✗ $y_{p_2} = (Cx + D)e^{2x} = Cxe^{2x} + De^{2x}$ duplicates y_c

✗ $y_{p_2} = x(Cx + D)e^{2x} = Cx^2e^{2x} + Dxe^{2x}$ like term w/ y_c

$y_{p_2} = x^2(Cx + D)e^{2x}$

$\checkmark \quad y_{p_2} = Cx^3e^{2x} + Dx^2e^{2x}$ no like terms w/ y_c

Finally, $y_p = A\sin(4x) + B\cos(4x) + Cx^3e^{2x} + Dx^2e^{2x}$

Find the form of the particular solution.

$$y''' - y'' + y' - y = \cos x + x^4$$

Find y_c: y_c solves $y''' - y'' + y' - y = 0$

Characteristic eqn: $m^3 - m^2 + m - 1 = 0$

factor by grouping

$$m^2(m-1) + (m-1) = 0$$

$$(m-1)(m^2+1) = 0$$

$m - 1 = 0 \Rightarrow m = 1$

$y_1 = e^x$

$m^2 + 1 = 0 \Rightarrow m^2 = -1$ $\alpha \pm i\beta$

$m = \pm\sqrt{-1} = \pm i$

$\alpha = 0, \ \beta = 1$

$y_1 = e^x$, $y_2 = \cos x$, $y_3 = \sin x$

$$y_c = C_1 e^x + C_2 \cos x + C_3 \sin x$$

$$y''' - y'' + y' - y = \cos x + x^4$$

Let y_{p_1} solve $y''' - y'' + y' - y = \cos x$

and y_{p_2} solve $y''' - y'' + y' - y = x^4$

y_{p_1} : $g_1(x) = \cos x$

✗ $y_{p_1} = A\cos x + B\sin x$

like terms w/ y_c

✓ $y_{p_1} = x(A\cos x + B\sin x)$

$= Ax\cos x + Bx\sin x$

correct form

$$y_c = C_1 e^x + C_2 \cos x + C_3 \sin x$$

$$y''' - y'' + y' - y = \cos x + x^4$$

y_{p_2} : $g_2(x) = x^4$

$$y_{p_2} = Cx^4 + Dx^3 + Ex^2 + Fx + G$$

Then

$$y_p = Ax\cos x + Bx\sin x + Cx^4 + Dx^3 + Ex^2 + Fx + G$$

Solve the IVP

$$y'' - y = 4e^{-x} \quad y(0) = -1, \quad y'(0) = 1$$

Find the general solution $y = y_c + y_p$

Find y_c: y_c solves $y'' - y = 0$

Char. eqn. $m^2 - 1 = 0$

$(m-1)(m+1) = 0$

$m = 1$ or $m = -1$

$y_1 = e^{x}, \quad y_2 = e^{-x}.$

$y_c = c_1 e^{x} + c_2 e^{-x}$

$y'' - y = 4e^{-x}$

Find y_p using undetermined coeff.

$g(x) = 4e^{-x}$

✗ $y_p = Ae^{-x}$ nope, duplicates part of y_c

✓ $y_p = x(Ae^{-x}) = Axe^{-x}$

Substitute:

$y_p = Axe^{-x}$

$y_p' = Ae^{-x} - Axe^{-x}$

$y_p'' = -Ae^{-x} - Ae^{-x} + Axe^{-x} = -2Ae^{-x} + Axe^{-x}$

$y_p'' - y_p = 4e^{-x}$

$$-2Ae^{-x} + Axe^{-x} - (Axe^{-x}) = 4e^{-x}$$

$$e^{-x}(-2A) + xe^{-x}(A - A) = 4e^{-x}$$

$$-2Ae^{-x} = 4e^{-x}$$

$$-2A = 4 \Rightarrow A = -2$$

$$y_p = -2xe^{-x}$$

The general solution

$$y = c_1 e^{x} + c_2 e^{-x} - 2xe^{-x}$$

$y(0) = -1, \quad y'(0) = 1$ Apply these IC

$$y' = c_1 e^{x} - c_2 e^{-x} - 2e^{-x} + 2x e^{-x}$$

$$y(0) = c_1 e^0 + c_2 e^0 - 2(0)e^0 = -1 \Rightarrow c_1 + c_2 = -1$$

$$y'(0) = c_1 e^0 - c_2 e^0 - 2e^0 + 2(0)e^0 = 1 \Rightarrow c_1 - c_2 - 2 = 1$$

Solve

$$c_1 + c_2 = -1$$
$$c_1 - c_2 = 3$$

add $\quad 2c_1 = 2 \Rightarrow c_1 = 1$

subtract $\quad 2c_2 = -4 \Rightarrow c_2 = -2$

$$y = c_1 e^{x} + c_2 e^{-x} - 2x e^{-x}$$

The solution to the IVP is

$$\boxed{y = e^{x} - 2e^{-x} - 2x e^{-x}}$$