# March 15 Math 2306 sec. 52 Spring 2023

## Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0y = g(x)$$

where $g$ comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

# Method Basics: $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$

- Classify $g$ as a certain *type*, and assume $y_p$ is of this same type[1] with unspecified coefficients, $A$, $B$, $C$, etc.
- Compare $y_p$ to $y_c$ and modify as needed.
- Substitute the assumed $y_p$ into the ODE and collect like terms
- Match like terms on the left and right to get equations for the coefficients.
- Solve the resulting system to determine the coefficients for $y_p$.

[1] We will see shortly that our final conclusion on the format of $y_p$ can depend on $y_c$.

# Cases: Comparing $y_p$ to $y_c$

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x) + \ldots + g_k(x)$$

Consider one of the $g$'s, say $g_i(x)$. We write out the guess for $y_{p_i}$ and compare it to $y_c(x)$.

**Case I:** The guess for $y_{p_i}$ **DOES NOT** have any like terms in common with $y_c$.

Then our guess for $y_{p_i}$ will work as written. We do the substitution to find the $A$, $B$, etc.

# Cases: Comparing $y_p$ to $y_c$

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x) + \ldots + g_k(x)$$

Consider one of the $g$'s, say $g_i(x)$. We write out the guess for $y_{p_i}$ and compare it to $y_c(x)$.

**Case II:** The guess for $y_{p_i}$ **DOES** have a like term in common with $y_c$.

Then we multiply our guess at $y_{p_i}$ by $x^n$ where $n$ is the smallest positive integer such that our new guess $x^n y_{p_i}$ does not have any like terms in common with $y_c$. Then we take this new guess and substitute to find the $A$, $B$, etc.

# Find the **form** of the particular solution.

This is a *set up only* problem. Don't bother solving for the $A$, $B$, etc.

$$y'' - 4y' + 4y = \sin(4x) + xe^{2x}$$

Find $y_c$: $y_c$ solves $y'' - 4y' + 4y = 0$

Characteristic eqn. $m^2 - 4m + 4 = 0$

$(m-2)^2 = 0 \Rightarrow m = 2$ double root

$y_1 = e^{2x}$, $y_2 = xe^{2x}$.

$y_c = c_1 e^{2x} + c_2 x e^{2x}$

$$y'' - 4y' + 4y = \sin(4x) + xe^{2x}$$

$y_c = c_1 e^{2x} + c_2 x e^{2x}$

Let $y_{p_1}$ solve $y'' - 4y' + 4y = \sin(4x)$

and $y_{p_2}$ solve $y'' - 4y' + 4y = xe^{2x}$

For $y_{p_1}$, $g_1(x) = \sin(4x)$

$y_{p_1} = A\sin(4x) + B\cos(4x)$ ✓ no like terms w/ $y_c$

For $y_{p_2}$, $g_2(x) = xe^{2x}$

✗ $y_{p_2} = (Cx + D)e^{2x} = Cxe^{2x} + De^{2x}$ Duplicates $y_c$

✗ $y_{p_2} = x(Cx + D)e^{2x} = Cx^2e^{2x} + Dxe^{2x}$ duplicates $y_c$

$$y_c = c_1 e^{2x} + c_2 x e^{2x}$$

$$\checkmark \quad y_{p_2} = x^2 (Cx + D) e^{2x} = Cx^3 e^{2x} + Dx^2 e^{2x}$$

Finally, $y_p$ has the form

$$y_p = A\sin(4x) + B\cos(4x) + Cx^3 e^{2x} + Dx^2 e^{2x}$$

# Find the form of the particular solution.

$$y''' - y'' + y' - y = \cos x + x^4$$

Find $y_c$: $y_c$ solves $y''' - y'' + y' - y = 0$

Characteristic eqn $m^3 - m^2 + m - 1 = 0$

factor by grouping $m^2(m-1) + (m-1) = 0$

$(m-1)(m^2+1) = 0$

$m - 1 = 0 \Rightarrow m = 1$ , $m^2 + 1 = 0 \Rightarrow m^2 = -1$

$y_1 = e^x$ $m = \pm\sqrt{-1} = \pm i$ $\alpha \pm i\beta$

$\alpha = 0$ $\beta = 1$

$y_2 = e^{0x} \cos(1x) = \cos x$

$y_3 = e^{0x} \sin(1x) = \sin x$

$$y_c = C_1 e^x + C_2 \cos x + C_3 \sin x$$

$$y''' - y'' + y' - y = \cos x + x^4$$

Let $y_{p_1}$ solve $y''' - y'' + y' - y = \cos x$

and $y_{p_2}$ solve: $y''' - y'' + y' - y = x^4$

For $y_{p_1}$, $g_1(x) = \cos x$

$y_{p_1} = A \cos x + B \sin x$ nope! duplicates $y_c$

$y_{p_1} = x(A \cos x + B \sin x)$

$\checkmark$ $y_{p_1} = Ax\cos x + Bx\sin x$ correct

$$y_c = C_1 e^x + C_2 \cos x + C_3 \sin x$$

For $y_{p_2}$, $g_2(x) = x^4$

$$y_{p_2} = Cx^4 + Dx^3 + Ex^2 + Fx + G$$

$\checkmark$ no like terms w/ $y_c$

Finally

$$y_p = Ax\cos x + Bx\sin x + Cx^4 + Dx^3 + Ex^2 + Fx + G$$

# Solve the IVP

$$y'' - y = 4e^{-x} \quad y(0) = -1, \quad y'(0) = 1$$

Find the general solution $y = y_c + y_p$

Find $y_c$ that solves $y'' - y = 0$

Charac. eqn $m^2 - 1 = 0$

$$(m-1)(m+1) = 0$$

$$m = 1 \quad \text{or} \quad m = -1$$

$$y_1 = e^{x} \quad , \quad y_2 = e^{-x}$$

$$y_c = c_1 e^{x} + c_2 e^{-x}$$

$y'' - y = 4e^{-x}$

Find $y_p$ using undetermined coef.

$g(x) = 4e^{-x}$

$y_p = Ae^{-x}$ ✗ duplicates part of $y_c$

$y_p = x(Ae^{-x}) = Axe^{-x}$ ✓

Substitute

$y_p = Axe^{-x}$

$y_p' = Ae^{-x} - Axe^{-x}$

$y_p'' = -Ae^{-x} - Ae^{-x} + Axe^{-x} = -2Ae^{-x} + Axe^{-x}$

$$y_p'' - y_p = 4e^{-x}$$

$$-2Ae^{-x} + Axe^{-x} - Axe^{-x} = 4e^{-x}$$

$$e^{-x}(-2A) + xe^{-x}(A - A) = 4e^{-x}$$

$$-2Ae^{-x} = 4e^{-x}$$

$$-2A = 4 \Rightarrow A = -2$$

$$y_p = Axe^{-x} \Rightarrow y_p = -2xe^{-x}$$

The general solution to the ODE is

$$y = c_1 e^{x} + c_2 e^{-x} - 2xe^{-x}$$

Now, apply $y(0) = -1$ and $y'(0) = 1$

$$y' = c_1 e^{x} - c_2 e^{-x} - 2e^{-x} + 2xe^{-x}$$

$y(0) = c_1 e^0 + c_2 e^0 - 2(0) e^0 = -1 \Rightarrow c_1 + c_2 = -1$

$y'(0) = c_1 e^0 - c_2 e^0 - 2e^0 + 2(0)e^0 = 1 \Rightarrow c_1 - c_2 - 2 = 1$

Solve $\quad c_1 + c_2 = -1$

$\quad\quad\;\; c_1 - c_2 = 3$

add $\quad 2c_1 = 2 \Rightarrow c_1 = 1$

subtract $\quad 2c_2 = -4 \Rightarrow c_2 = -2$

The solution to the IVP is

$$y = e^x - 2e^{-x} - 2xe^{-x}$$