March 16 Math 3260 sec. 52 Spring 2022

Section 3.3: Cramer's Rule, Volume, and Linear Transformations

Cramer's Rule is a method for solving a square system $A\mathbf{x} = \mathbf{b}$ by use of determinants. While it is impractical for large systems, it provides a fast method for some small systems (say 2×2 or 3×3).

Definition: For $n \times n$ matrix A and $\mathbf{b}$ in $\mathbb{R}^n$, let $A_i(\mathbf{b})$ be the matrix obtained from A by replacing the i^{th} column with the vector $\mathbf{b}$. That is

$$A_i(\mathbf{b}) = [\mathbf{a}_1 \cdots \mathbf{a}_{i-1}\ \mathbf{b}\ \mathbf{a}_{i+1} \cdots \mathbf{a}_n]$$

Example

Consider $A = \begin{bmatrix} 2 & 1 \\ -1 & 7 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 9 \\ -3 \end{bmatrix}$. Find

- $A_1(\mathbf{b})$ and $A_2(\mathbf{b})$, and

$A_1(\vec{b}) = \begin{bmatrix} 9 & 1 \\ -3 & 7 \end{bmatrix}$, $A_2(\vec{b}) = \begin{bmatrix} 2 & 9 \\ -1 & -3 \end{bmatrix}$

- $\det(A)$, $\det(A_1(\mathbf{b}))$, and $\det(A_2(\mathbf{b}))$.

$\det(A) = 2(7) - (-1)(1) = 14 + 1 = 15$

$\det(A_1(\vec{b})) = 9(7) - (-3)(1) = 63 + 3 = 66$

$\det(A_2(\vec{b})) = 2(-3) - (-1)9 = -6 + 9 = 3$

Cramer's Rule

Theorem: Let A be an $n \times n$ nonsingular matrix. Then for any vector $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution of the system $A\mathbf{x} = \mathbf{b}$ is given by $\mathbf{x}$ where

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \quad i = 1, \dots, n$$

Example

Determine whether Cram er's rule can be used to solve the system. If so, use it to solve the system.

$$\begin{array}{rcrcr} 2x_1 & + & x_2 & = & 9 \\ -x_1 & + & 7x_2 & = & -3 \end{array}$$

Restate this as a matrix equation $A\vec{x} = \vec{b}$.

$$\underbrace{\begin{bmatrix} 2 & 1 \\ -1 & 7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 9 \\ -3 \end{bmatrix}}_{\vec{b}}$$

$\det(A) = 15 \neq 0 \Rightarrow A$ is nonsingular

$A_1(\vec{b}) = \begin{bmatrix} 9 & 1 \\ -3 & 7 \end{bmatrix}$, $\det(A_1(\vec{b})) = 66$

$$A_2(\vec{b}) = \begin{bmatrix} 2 & 9 \\ -1 & -3 \end{bmatrix}, \quad \det(A_2(\vec{b})) = 3$$

By Cramer's rule, the solution

$$x_1 = \frac{\det(A_1(\vec{b}))}{\det(A)} = \frac{66}{15}$$

$$x_2 = \frac{\det(A_2(\vec{b}))}{\det(A)} = \frac{3}{15}$$

$$\Rightarrow \quad x_1 = \frac{22}{5}, \quad x_2 = \frac{1}{5}$$

Notation : $\det(A) = |A|$