March 17 Math 2306 sec. 51 Spring 2023

Section 10: Variation of Parameters

We are still considering nonhomogeneous, linear ODEs. Consider equations of the form

$$y'' + y = \tan x, \quad \text{or} \quad x^2y'' + xy' - 4y = e^x.$$

The method of undetermined coefficients is not applicable to either of these.

- The first equation has constant coefficient left side, but the tangent is not the right kind of right hand side.
- The second equation has an exponential right side, but the left side isn't constant coefficient.

We need another approach.

Variation of Parameters

For the equation in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = g(x),$$

suppose $\{y_1(x),\ y_2(x)\}$ is a fundamental solution set for the associated homogeneous equation. We seek a particular solution of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where u_1 and u_2 are functions we will determine (in terms of y_1, y_2 and g).

$y_c = c_1y_1 + c_2y_2$

This method is called **variation of parameters**.

Variation of Parameters: Derivation of y_p

$$y'' + P(x)y' + Q(x)y = g(x)$$

Set $\quad y_p = u_1(x)y_1(x)+u_2(x)y_2(x)$

$y_p = u_1 y_1 + u_2 y_2$

$y_p' = u_1 y_1' + u_2 y_2' + u_1' y_1 + u_2' y_2$

2nd eqn: $\quad u_1' y_1 + u_2' y_2 = 0$

Remember that $\quad y_i'' + P(x)y_i' + Q(x)y_i = 0, \quad$ for $i = 1, 2$

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p' = u_1 y_1' + u_2 y_2'$$

$$y_p'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''$$

Sub $\quad y_p'' + P(x) y_p' + Q(x) y_p = g(x)$

$$u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2'' + P(x)\left(u_1 y_1' + u_2 y_2'\right) + Q(x)\left(u_1 y_1 + u_2 y_2\right) = g(x)$$

Collect u_1', u_2', u_1 and u_2

$$u_1' y_1' + u_2' y_2' + u_1 \left(\underbrace{y_1'' + P(x) y_1' + Q(x) y_1}_{0}\right) +$$

$$u_2\underbrace{\left(y_2'' + P(x)y_2' + Q(x)y_2\right)}_{=0} = g(x)$$

This reduces to

$$u_1' y_1' + u_2' y_2' = g(x)$$

We need to solve the system

$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = g(x)$$

We can state this using matrices

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g \end{bmatrix}$$

Using Cramer's rule

$$u_1' = \frac{W_1}{W} \quad \text{and} \quad u_2' = \frac{W_2}{W}$$

where

$$W_1 = \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix} = -g y_2, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix} = y_1 g$$

and

$$W = W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$u_1 = \int \frac{-g y_2}{W}\,dx \quad \text{and} \quad u_2 = \int \frac{g y_1}{W}\,dx$$

Variation of Parameters

$$y'' + P(x)y' + Q(x)y = g(x)$$

If $\{y_1, y_2\}$ is a fundamental solution set for the associated homogeneous equation, then the general solution is

$$y = y_c + y_p \quad \text{where}$$

$$y_c = c_1y_1(x) + c_2y_2(x), \quad \text{and} \quad y_p = u_1(x)y_1(x) + u_2(x)y_2(x).$$

Letting W denote the Wronskian of y_1 and y_2, the functions u_1 and u_2 are given by the formulas

$$u_1 = \int \frac{-y_2g}{W}\,dx, \quad \text{and} \quad u_2 = \int \frac{y_1g}{W}\,dx.$$

Solve the IVP

$$x^2y'' + xy' - 4y = 8x^2, \quad y(1) = 0, \quad y'(1) = 0$$

The complementary solution of the ODE is $y_c = c_1x^2 + c_2x^{-2}$.

Use variation of parameters to find y_p. $x > 0$

$y_p = u_1y_1 + u_2y_2$

Standard form: $y'' + \frac{1}{x}y' - \frac{4}{x^2}y = 8$

$g(x) = 8$, $y_1 = x^2$ $y_2 = x^{-2}$

$W = \begin{vmatrix} x^2 & x^{-2} \\ 2x & -2x^{-3} \end{vmatrix} = x^2(-2x^{-3}) - 2x(x^{-2}) = \frac{-2}{x} - \frac{2}{x} = \frac{-4}{x}$

$$u_1 = \int \frac{-g y_2}{W}\,dx = -\int \frac{8x^{-2}}{\frac{-4}{x}}\,dx = 2\int x^{-1}\,dx = 2\ln x$$

$$u_2 = \int \frac{g y_1}{W}\,dx = \int \frac{8x^2}{\frac{-4}{x}}\,dx = -2\int x^3\,dx = \frac{-2}{4}x^4$$

$$y_p = u_1 y_1 + u_2 y_2$$
$$= 2\ln x\,(x^2) - \frac{1}{2}x^4(x^{-2})$$

$$\boxed{y_p = 2x^2\ln x - \frac{1}{2}x^2}$$

we'll finish next time.