March 18 Math 3260 sec. 51 Spring 2022

Section 3.3: Cramer's Rule, Volume, and Linear Transformations

Definition: For $n \times n$ matrix A and $\mathbf{b}$ in $\mathbb{R}^n$, let $A_i(\mathbf{b})$ be the matrix obtained from A by replacing the i^{th} column with the vector $\mathbf{b}$. That is

$$A_i(\mathbf{b}) = [\mathbf{a}_1 \cdots \mathbf{a}_{i-1}\ \mathbf{b}\ \mathbf{a}_{i+1} \cdots \mathbf{a}_n]$$

Theorem: Let A be an $n \times n$ nonsingular matrix. Then for any vector $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution of the system $A\mathbf{x} = \mathbf{b}$ is given by $\mathbf{x}$ where

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \quad i = 1, \ldots, n$$

Application

In various engineering applications, electrical or mechanical components are often chosen to try to control the long term behavior of a system (e.g. adding a damper to kill off oscillatory behavior). Using *Laplace Transforms*, differential equations are converted into algebraic equations containing a parameter s. These give rise to systems of the form

$$\begin{array}{rcrcl} 3sX & - & 2Y & = & 4 \\ -6X & + & sY & = & 1 \end{array}$$

Determine the values of s for which the system is uniquely solvable. For such s, find the solution (X, Y) using Cramer's rule.

$$
\begin{aligned}
3sX &- 2Y = 4 \\
-6X &+ sY = 1
\end{aligned}
$$

In matrix format

$$\underbrace{\begin{bmatrix} 3s & -2 \\ -6 & s \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} X \\ Y \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 4 \\ 1 \end{bmatrix}}_{\vec{b}}$$

$\det(A) = 3s(s) - (-2)(-6) = 3s^2 - 12 = 3(s^2 - 4)$

The system has a unique solution if $\det(A) \neq 0$, so this requires $s \neq \pm 2$.

For $s \neq \pm 2$,

$$A_1(\vec{b}) = \begin{bmatrix} 4 & -2 \\ 1 & s \end{bmatrix} \qquad A_2(\vec{b}) = \begin{bmatrix} 3s & 4 \\ -6 & 1 \end{bmatrix}$$

$$\det(A_1(\vec{b})) = 4s + 2 \qquad \det(A_2(\vec{b})) = 3s + 24$$

$$\det(A) = 3(s^2 - 4)$$

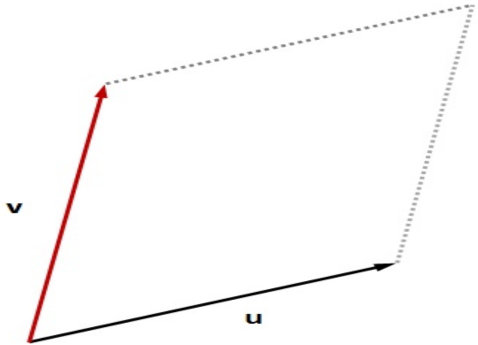
The solution will be

$$X = \frac{\det(A_1(\vec{b}))}{\det(A)} = \frac{2(2s+1)}{3(s^2-4)}$$

$$Y = \frac{\det(A_2(\vec{b}))}{\det(A)} = \frac{3(s+8)}{3(s^2-4)}$$

$$X = \frac{2(2s+1)}{3(s^2-4)}, \quad Y = \frac{s+8}{s^2-4}$$

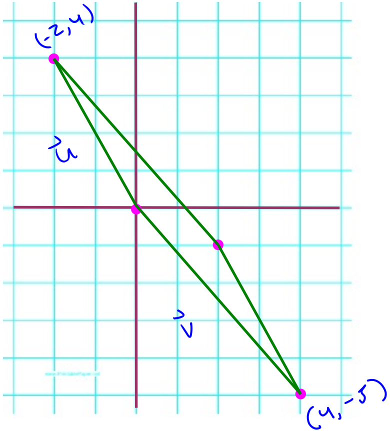
Area of a Parallelogram



Theorem: If **u** and **v** are nonzero, nonparallel vectors in $\mathbb{R}^2$, then the area of the parallelogram determined by these vectors is $|\det(A)|$ where $A = [\mathbf{u}\ \mathbf{v}]$.

Example

Find the area of the parallelogram with vertices $(0,0)$, $(-2,4)$, $(4,-5)$, and $(2,-1)$.



$\vec{u} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ $\quad \vec{v} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$

Let $A = [\vec{u} \ \vec{v}]$

$= \begin{bmatrix} -2 & 4 \\ 4 & -5 \end{bmatrix}$

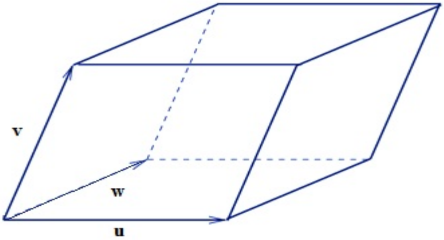
$\text{Area} = |\det(A)|$

$= |10 - 16| = 6$

$$\vec{u} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\vec{w} - \vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} - \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2+2 \\ -1-4 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \end{bmatrix},$$

Volume of a Parallelopiped



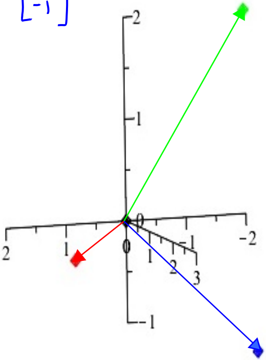
Theorem: If **u**, **v**, and **w** are nonzero, non-collinear vectors in $\mathbb{R}^3$, then the volume of the parallelopiped determined by these vectors is $|\det(A)|$ where $A = [\mathbf{u}\ \mathbf{v}\ \mathbf{w}]$.

Example

Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(2, 3, 0)$, $(-2, 0, 2)$ and $(-1, 3, -1)$.

Let $\vec{u} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$

and Let $A = [\vec{u}\ \vec{v}\ \vec{w}]$



$$A = \begin{bmatrix} 2 & -2 & -1 \\ 3 & 0 & 3 \\ 0 & 2 & -1 \end{bmatrix}$$

Doing a cofactor expansion across row 2

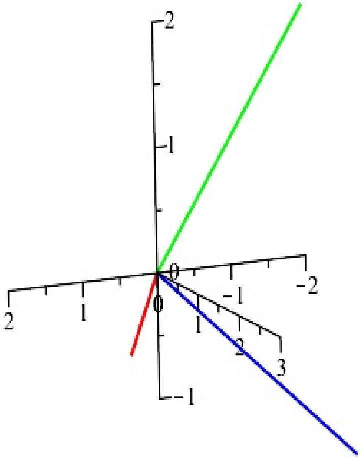
$$\det(A) = a_{21} C_{21} + a_{22} C_{22} + a_{23} C_{23}$$

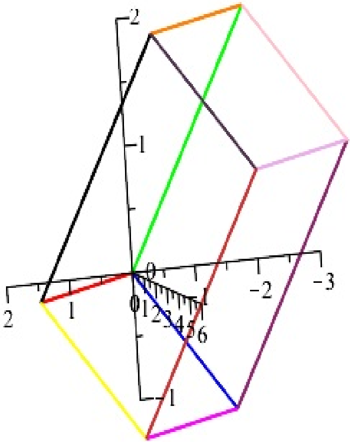
$$(-1)^3 M_{21} \qquad (-1)^{2+2} M_{22} \qquad (-1)^5 M_{23}$$

$$= -3 \begin{vmatrix} -2 & -1 \\ 2 & -1 \end{vmatrix} + 0 \begin{vmatrix} \; \end{vmatrix} - 3 \begin{vmatrix} 2 & -2 \\ 0 & 2 \end{vmatrix}$$

$$= -3(2+2) - 3(4-0) = -12 - 12 = -24$$

The volume $V = |\det(A)| = 24$


2
1
0
-1
2
1
0
-1
-2
1
2
3
-1


2
1
0
-1
2
1
0
-2
-3
1
2
3
4
5
6
-1