March 18 Math 3260 sec. 51 Spring 2024

Section 4.1: Vector Spaces and Subspaces

Definition: Vector Space

A **vector space** is a nonempty set V of objects called *vectors* together with two operations called *vector addition* and *scalar multiplication* that satisfy the following ten axioms:

For all $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in V, and for any scalars c and d

1. The sum $\mathbf{u}+\mathbf{v}$ is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There exists a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each vector $\mathbf{u}$ there exists a vector $-\mathbf{u}$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. For each scalar c, $c\mathbf{u}$ is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=d(c\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$

Subspaces

Definition:

A **subspace** of a vector space V is a subset H of V for which

a) The zero vector is in[a] H

b) H is closed under vector addition. (i.e. $\mathbf{u}, \mathbf{v}$ in H implies $\mathbf{u} + \mathbf{v}$ is in H)

c) H is closed under scalar multiplication. (i.e. $\mathbf{u}$ in H implies $c\mathbf{u}$ is in H)

[a] This is sometimes replaced with the condition that H is nonempty.

Remark: A subspace is a vector space. If these three properties hold, it inherits the structure from its parent space.

Example

Determine which of the following is a subspace of $\mathbb{R}^2$.

1. The set $\{\mathbf{0}\}$ (the set containing just the zero vector).

Let's call it $H = \{\vec{0}\}$. Clearly H contains $\vec{0}$.

$\vec{0} + \vec{0} = \vec{0}$ and $c\vec{0} = \vec{0}$ for any scalar c.

H is closed under both operations.

It is a subspace of $\mathbb{R}^2$.

Example

Determine which of the following is a subspace of $\mathbb{R}^2$.

2. The set of all vectors of the form $\mathbf{u} = (1, u_2)$.

Let's again call it H.

$\vec{0} = (0,0) \neq (1, u_2)$ for any choice of u_2.

H doesn't contain the zero vector, hence it is not a subspace of $\mathbb{R}^2$.

Example

Let $S = \{\mathbf{p} \in \mathbb{P}_2 \mid \mathbf{p}(0) = 0 \text{ and } \mathbf{p}(1) = 0\}$. Show that S is a subspace of $\mathbb{P}_2$.

when $t=0$ $\vec{p}=0$

when $t=1$ $\vec{p}=0$

We need to show that S contains the zero vector and is closed under both operations.

Recall $\vec{0}(t) = 0 + 0t + 0t^2$.

Note $\vec{0}(0) = 0 + 0(0) + 0(0^2) = 0$ and

$\vec{0}(1) = 0 + 0(1) + 0(1^2) = 0$

Hence $\vec{0} \in S$.

Let $\vec{p}, \vec{q}$ be in S and let c be any scalar. Then $\vec{p}(0) = 0$, $\vec{p}(1) = 0$, $\vec{q}(0) = 0$ and $\vec{q}(1) = 0$. Note that

$$(\vec{p} + \vec{q})(0) = \vec{p}(0) + \vec{q}(0) = 0 + 0 = 0$$

$$(\vec{p} + \vec{q})(1) = \vec{p}(1) + \vec{q}(1) = 0 + 0 = 0$$

Hence $\vec{p} + \vec{q}$ is in S which is closed under vector addition.

Also, $(c\vec{p})(0) = c\vec{p}(0) = c\,(0) = 0$ and

$(c\vec{p})(1) = c\vec{p}(1) = c\,(0) = 0.$

Hence $c\vec{p}$ is in S which is closed under scalar multiplication.

As required we've shown that S is a subspace of $\mathbb{P}_2$.

Linear Combination and Span

Definition

Let V be a vector space and $\mathbf{v}_1$, $\mathbf{v}_2, \ldots, \mathbf{v}_p$ be a collection of vectors in V. A **linear combination** of these vectors is a vector $\mathbf{u}$ of the form

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

for some scalars $c_1, c_2, \ldots, c_p$.

Definition

The **span**, $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$, is the subset of V consisting of all linear combinations of the vectors $\mathbf{v}_1$, $\mathbf{v}_2, \ldots, \mathbf{v}_p$.

Span as Subspace

Theorem:

Let V be a vector space and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ be a nonempty set of vectors in V. Then $\mathsf{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a subspace of V.

Remarks

- The set $\mathsf{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is the subspace of V generated (or spanned) by the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$
- If H is any subspace of V, then a **spanning set** for H is any set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ such that $H = \mathsf{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

Example

$M_{2\times 2}$ denotes the set of all 2×2 matrices with real entries with regular matrix addition and scalar multiplication. Consider the subset H of $M_{2\times 2}$

$$H = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| \; a, b \in \mathbb{R} \right\}.$$

Show that H is a subspace of $M_{2\times 2}$ by finding a spanning set. That is, show that $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ for some appropriate vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

We want to write an arbitrary element of H as a linear combination of fixed vectors.

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$$

$$= a\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + b\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$$

This is a linear combo of $\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ and $\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$.

So $H = \text{Span}\left\{\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right\}$.

As a span, H is a subspace of $M_{2\times 2}$.

Example

Recall the set $S = \{\mathbf{p} \in \mathbb{P}_2 \mid \mathbf{p}(0) = 0 \text{ and } \mathbf{p}(1) = 0\}$. Argue that

$$S = \text{Span}\{t - t^2\}.$$

Let $\vec{p}(t) = p_0 + p_1 t + p_2 t^2$ be any element of S. We can find conditions on p_0, p_1 and p_2.

$$\vec{p}(0) = p_0 + p_1(0) + p_2(0^2) = p_0 = 0$$

$$\boxed{\Longrightarrow \quad p_0 = 0}$$

and $\quad \vec{p}(1) = p_1(1) + p_2(1^2) = 0$

$$\Rightarrow \quad p_1 + p_2 = 0$$

$$\Rightarrow \quad p_2 = -p_1$$

So $\vec{p}$ in S has the form

$$\vec{p}(t) = 0 + p_1 t + (-p_1) t^2$$
$$= p_1 (t - t^2)$$

That is, $\vec{p}$ is a linear combination of $t - t^2$. So $S = \text{Span}\{t - t^2\}$.

Span$\{t - t^2\}$

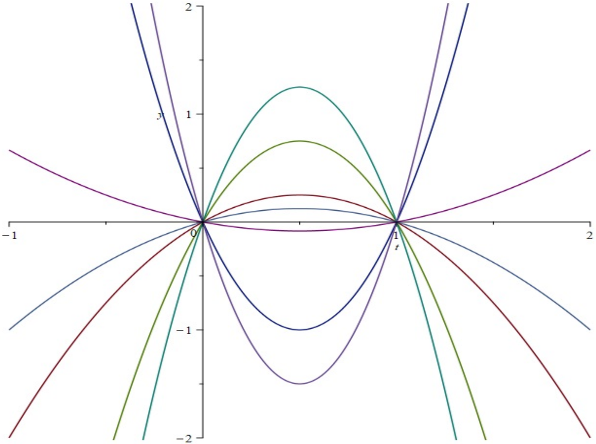

Figure: The graphs of various elements of Span$\{t - t^2\}$