March 18 Math 3260 sec. 52 Spring 2022

Section 3.3: Cramer's Rule, Volume, and Linear Transformations

Definition: For $n \times n$ matrix A and $\mathbf{b}$ in $\mathbb{R}^n$, let $A_i(\mathbf{b})$ be the matrix obtained from A by replacing the i^{th} column with the vector $\mathbf{b}$. That is

$$A_i(\mathbf{b}) = [\mathbf{a}_1 \cdots \mathbf{a}_{i-1}\ \mathbf{b}\ \mathbf{a}_{i+1} \cdots \mathbf{a}_n]$$

Theorem: Let A be an $n \times n$ nonsingular matrix. Then for any vector $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution of the system $A\mathbf{x} = \mathbf{b}$ is given by $\mathbf{x}$ where

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \quad i = 1, \ldots, n$$

Application

In various engineering applications, electrical or mechanical components are often chosen to try to control the long term behavior of a system (e.g. adding a damper to kill off oscillatory behavior). Using *Laplace Transforms*, differential equations are converted into algebraic equations containing a parameter s. These give rise to systems of the form

$$\begin{array}{rcrcl} 3sX & - & 2Y & = & 4 \\ -6X & + & sY & = & 1 \end{array}$$

Determine the values of s for which the system is uniquely solvable. For such s, find the solution (X, Y) using Cramer's rule.

$$
\begin{aligned}
3sX &- 2Y &= 4\\
-6X &+ sY &= 1
\end{aligned}
$$

Write this in matrix format

$$\underbrace{\begin{bmatrix} 3s & -2 \\ -6 & s \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} X \\ Y \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 4 \\ 1 \end{bmatrix}}_{\vec{b}}$$

$\det(A) = 3s(s) - (-2)(-6) = 3s^2 - 12 = 3(s^2-4)$

A is nonsingular when $\det(A) \neq 0$. The system has a unique solution for all $s \neq \pm 2$.

For $s \neq \pm 2$

$$A_1(\vec{b}) = \begin{bmatrix} 4 & -2 \\ 1 & s \end{bmatrix}, \quad A_2(\vec{b}) = \begin{bmatrix} 3s & 4 \\ -6 & 1 \end{bmatrix}.$$

$$\det(A_1(\vec{b})) = 4s+2 \qquad \det(A_2(\vec{b})) = 3s+24$$

$$\det(A) = 3(s^2-4)$$

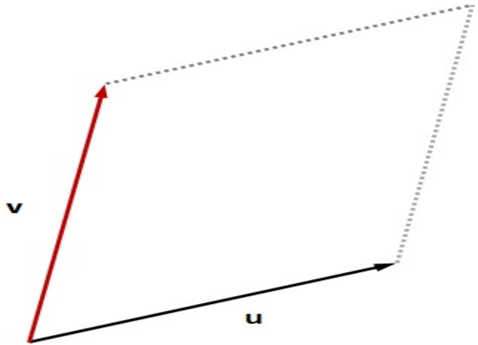
The solution

$$X = \frac{\det(A_1(\vec{b}))}{\det(A)} = \frac{2(2s+1)}{3(s^2-4)}.$$

$$Y = \frac{\det(A_2(\vec{b}))}{\det(A)} = \frac{3(s+8)}{3(s^2-4)} = \frac{s+8}{s^2-4}$$

$$X = \frac{2(2s+1)}{3(s^2-4)}, \quad Y = \frac{s+8}{s^2-4}.$$

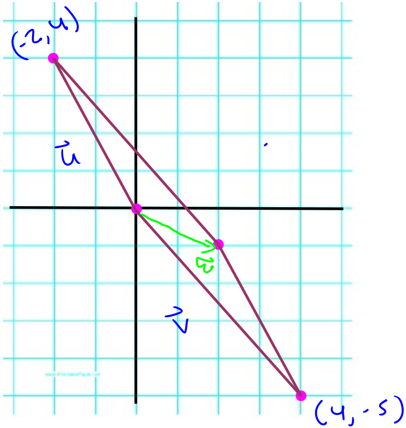
Area of a Parallelogram



Theorem: If **u** and **v** are nonzero, nonparallel vectors in $\mathbb{R}^2$, then the area of the parallelogram determined by these vectors is $|\det(A)|$ where $A = [\mathbf{u}\ \mathbf{v}]$.

Example

Find the area of the parallelogram with vertices $(0,0)$, $(-2,4)$, $(4,-5)$, and $(2,-1)$.



Let $\vec{u} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$

Set $A = [\vec{u}\ \vec{v}]$

$= \begin{bmatrix} -2 & 4 \\ 4 & -5 \end{bmatrix}$

The area $= |\det(A)|$

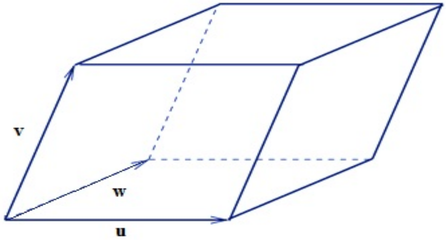
$= |(-2)(-5) - 4\cdot 4|$

$= |10 - 16| = 6$

Let $\vec{w} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $\vec{u} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$

$$\vec{w} - \vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} - \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2+2 \\ -1-4 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$$

Volume of a Parallelopiped



Theorem: If **u**, **v**, and **w** are nonzero, non-collinear vectors in $\mathbb{R}^3$, then the volume of the parallelopiped determined by these vectors is $|\det(A)|$ where $A = [\mathbf{u}\ \mathbf{v}\ \mathbf{w}]$.

Example

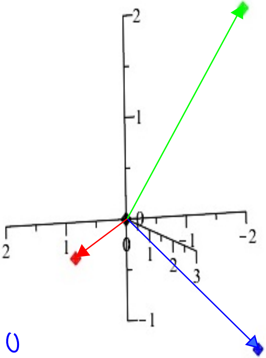
Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(2,3,0)$, $(-2,0,2)$ and $(-1,3,-1)$.

Let $\vec{u} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$

and set

$$A = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -1 \\ 3 & 0 & 3 \\ 0 & 2 & -1 \end{bmatrix}$$



$$\det(A) = a_{11}\underbrace{C_{11}}_{(-1)^2 M_{11}} + a_{21}\underbrace{C_{21}}_{(-1)^3 M_{21}} + \overbrace{a_{31}}^{0}\underbrace{C_{31}}_{(-1)^4 M_{31}}$$

down column 1

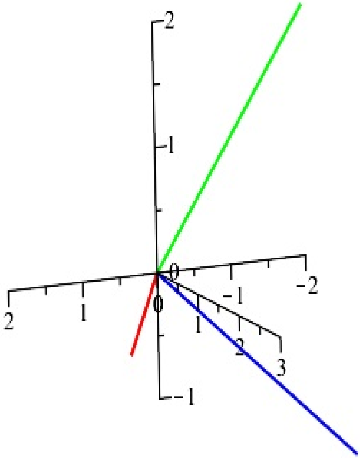
$$= 2 \begin{vmatrix} 0 & 3 \\ 2 & -1 \end{vmatrix} - 3 \begin{vmatrix} -2 & -1 \\ 2 & -1 \end{vmatrix} + 0 \begin{vmatrix} & \\ & \end{vmatrix}$$

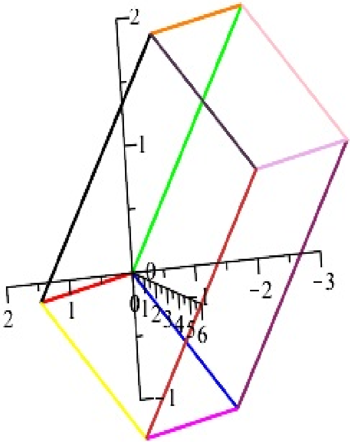
$$= 2(0-6) - 3(2+2)$$

$$= -12 - 12$$

$$= -24$$

The volume $V = |\det(A)| = |-24| = 24$


2
1
0
−1
−2
2
1
0
1
2
3
−1


2
1
0
−2
−3
2
1
0
1
2
3
4
5
6
−1