March 1 Math 2306 sec. 51 Spring 2023

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Examples (from last time)

$$y'' - 4y' + 4y = 8x + 1, \quad g(x) = 8x + 1$$

We guessed that $y_p = Ax + B$ and found that $A = 2$ and $B = \frac{9}{4}$ worked. So a particular solution was

$$y_p = 2x + \frac{9}{4}.$$

$$y'' - 4y' + 4y = 6e^{-3x}, \quad g(x) = 6e^{-3x}$$

We guessed that $y_p = Ae^{-3x}$ and found that $A = \frac{6}{25}$ worked. So a particular solution was

$$y_p = \frac{6}{25}e^{-3x}.$$

Classifying the RHS

$$y'' - 4y' + 4y = 16x^2$$

Question: Is $g(x) = 16x^2$ a **monomial** or a **polynomial**?

- Setting $y_p = Ax^2$ failed (led to conflicting conditions on A).
- We have to account for all *like terms* that can arise from taking derivatives.
- The proper classification for g in this context is as a 2^{nd} degree polynomial.
- The correct form for the particular solution is

$$\boxed{y_p = Ax^2 + Bx + C}$$

Solve an Initial Value Problem

$$y'' - 4y' + 4y = 16x^2, \quad y(0) = 0, \quad y'(0) = 0.$$

The general solution looks like $y = y_c + y_p$

y_c solves $y'' - 4y' + 4y = 0$

$y_c = c_1 y_1 + c_2 y_2$

Charact. eqn $m^2 - 4m + 4 = 0$

$(m-2)^2 = 0 \Rightarrow m = 2$ double root

$y_1 = e^{2x}$, $y_2 = x e^{2x}$

$$\boxed{y_c = c_1 e^{2x} + c_2 x e^{2x}}$$

$$y'' - 4y' + 4y = 16x^2$$

$$\boxed{y_p = Ax^2 + Bx + C}$$

Find y_p using the method of undetermined coeff.

$y_p' = 2Ax + B$

$y_p'' = 2A$

$$y_p'' - 4y_p' + 4y_p = 16x^2$$

$$2A - 4(2Ax + B) + 4(Ax^2 + Bx + C) = 16x^2$$

$$4Ax^2 + (-8A + 4B)x + (2A - 4B + 4C) = 16x^2$$

$$4Ax^2 + (-8A+4B)x + (2A-4B+4C) = 16x^2 + 0x + 0$$

Match like terms:

$$4A = 16 \Rightarrow A = 4$$

$$-8A + 4B = 0 \Rightarrow 4B = 8A \Rightarrow B = 2A = 8$$

$$2A - 4B + 4C = 0$$

$$C \Rightarrow 4C = -2A + 4B = -2(4) + 4(8) = 24$$

$$C = 6$$

$$\boxed{y_p = Ax^2 + Bx + C}$$

We found

$A = 4$, $B = 8$, $C = 6$

$$\boxed{y_p = 4x^2 + 8x + C}$$

$$\boxed{y_c = c_1 e^{2x} + c_2 x e^{2x}}$$

The general solution to the ODE is

$$y = c_1 e^{2x} + c_2 x e^{2x} + 4x^2 + 8x + 6$$

Apply $y(0) = 0$, $y'(0) = 0$

$$y' = 2c_1 e^{2x} + c_2 e^{2x} + 2c_2 x e^{2x} + 8x + 8$$

$$y(0) = c_1 e^0 + c_2 \cdot 0 \cdot e^0 + 4(0)^2 + 8(0) + 6 = 0$$

$$c_1 + 6 = 0$$

$$c_1 = -6$$

$$y'(0) = 2c_1 e^0 + c_2 e^0 + 2c_2 \cdot 0 \cdot e^0 + 8(0) + 8 = 0$$

$$2c_1 + c_2 + 8 = 0$$

$$c_2 = -8 - 2c_1$$

$$= -8 - 2(-6) = 4$$

The solution to the IVP is

$$y = 4xe^{2x} - 6e^{2x} + 4x^2 + 8x + 6$$

General Form: sines and cosines

$$y'' - y' = 20\sin(2x)$$

If we assume that $y_p = A\sin(2x)$, taking two derivatives would lead to the equation

$$-4A\sin(2x) - 2A\cos(2x) = 20\sin(2x).$$

This would require (matching coefficients of sines and cosines)

$$-4A = 20 \quad \text{and} \quad -2A = 0.$$

This is impossible as it would require $-5 = 0$!

General Form: sines and cosines

We must think of our equation $y'' - y' = 20\sin(2x)$ as

$$y'' - y' = 20\sin(2x) + 0\cos(2x).$$

The correct format for y_p is

$$y_p = A\sin(2x) + B\cos(2x).$$

Two Rules of Thumb

- Polynomials include all powers from constant up to the degree.
- Where sines go, cosines follow and vice versa.