March 20 Math 2306 sec. 52 Spring 2023

Section 10: Variation of Parameters

Variation of Parameters

$$y'' + P(x)y' + Q(x)y = g(x)$$

If $\{y_1, y_2\}$ is a fundamental solution set for the associated homogeneous equation, then the general solution is

$$y = y_c + y_p \quad \text{where}$$

$$y_c = c_1 y_1(x) + c_2 y_2(x), \quad \text{and} \quad y_p = u_1(x)y_1(x) + u_2(x)y_2(x).$$

Letting W denote the Wronskian of y_1 and y_2, the functions u_1 and u_2 are given by the formulas

$$u_1 = \int \frac{-y_2 g}{W}\,dx, \quad \text{and} \quad u_2 = \int \frac{y_1 g}{W}\,dx.$$

Solve the IVP

$$x^2y'' + xy' - 4y = 8x^2, \quad y(1) = 0, \quad y'(1) = 0$$

The complementary solution of the ODE is $y_c = c_1x^2 + c_2x^{-2}$.

We put the ODE in standard form and found that $g(x) = 8$. From the given y_c, we had $y_1 = x^2$, $y_2 = x^{-2}$, and we computed the Wronskian $W = -\frac{4}{x}$. We set up the integrals and got

$$u_1 = 2\ln(x) \quad \text{and} \quad u_2 = -\frac{1}{2}x^4.$$

This gave us

$$y_p = 2x^2\ln(x) - \frac{1}{2}x^2.$$

Now, we can finish the problem.

The general solution $y = y_c + y_p$

$$y = c_1 x^2 + c_2 x^{-2} + 2x^2 \ln x - \frac{1}{2} x^2$$

We can combine $c_1 x^2 - \frac{1}{2} x^2$, and take

$$y = k_1 x^2 + k_2 x^{-2} + 2x^2 \ln x$$

$y(1) = 0, \quad y'(1) = 0$ Apply the IC

$$y' = 2k_1 x - 2k_2 x^{-3} + 4x \ln x + 2x^2 \left(\frac{1}{x}\right)$$

$$y(1) = k_1 (1)^2 + k_2 (1)^{-2} + 2(1^2) \ln 1 = 0$$

$$k_1 + k_2 = 0$$

$$y'(1) = 2k_1(1) - 2k_2(1^{-3}) + 4(1)\ln 1 + 2(1) = 0$$

$$2k_1 - 2k_2 = -2$$

Solve

$$k_1 + k_2 = 0$$
$$k_1 - k_2 = -1$$

add $\quad 2k_1 = -1 \implies k_1 = \frac{-1}{2}$

subtract $\quad 2k_2 = 1 \implies k_2 = \frac{1}{2}$

The solution to the IVP is

$$\boxed{y = \frac{-1}{2}x^2 + \frac{1}{2}x^{-2} + 2x^2\ln x}$$

Example:

Solve the ODE $y'' + y = \tan x$.

The general solution $y = y_c + y_p$

Find y_c; y_c solves $y'' + y = 0$

Characteristic eqn $m^2 + 1 = 0$

$m^2 = -1 \Rightarrow m = \pm\sqrt{-1} = \pm i$

Complex w/ $\alpha = 0$, $\beta = 1$

$y_1 = e^{0x}\cos x$, $y_2 = e^{0x}\sin x$

$y_c = c_1 \cos x + c_2 \sin x$

To find y_p, we use Variation of parameters.

$y_1 = \cos x$, $y_2 = \sin x$, $g(x) = \tan x$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$u_1 = \int \frac{-g y_2}{W}\, dx = \int -\tan x \sin x \, dx$$

$$= -\int \frac{\sin^2 x}{\cos x}\, dx = -\int \frac{1 - \cos^2 x}{\cos x}\, dx$$

$$= \int (\cos x - \sec x)\,dx =$$

$$u_1 = \sin x - \ln|\sec x + \tan x|$$

$$u_2 = \int \frac{g\,y_1}{W}\,dx = \int \tan x \cos x\,dx$$

$$= \int \sin x\,dx = -\cos x$$

$y_1 = \cos x$, $y_2 = \sin x$ $\quad y_p = u_1 y_1 + u_2 y_2$

$$y_p = (\sin x - \ln|\sec x + \tan x|)\cos x + (-\cos x)\sin x$$

$= \sin x \cos x - \cos x \ln|\sec x + \tan x| - \sin x \cos x$

$\Rightarrow \quad y_p = -\cos x \ln|\sec x + \tan x|$

The general solution

$$y = c_1 \cos x + c_2 \sin x - \cos x \ln|\sec x + \tan x|$$

$y'' + y = \tan x$