March 21 Math 3260 sec. 52 Spring 2022

Section 4.1: Vector Spaces and Subspaces

Recall that we had defined $\mathbb{R}^n$ as the set of all n-tuples of real numbers. We defined two operations, vector addition and scalar multiplication, and said that the following algebraic properties hold:

For every $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^n$ and scalars c and d

(i) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(ii) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$

(iii) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$

(iv)[1] $\mathbf{u}+(-\mathbf{u})=-\mathbf{u}+\mathbf{u}=\mathbf{0}$

(v) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$

(vi) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$

(vii) $c(d\mathbf{u})=d(c\mathbf{u})=(cd)\mathbf{u}$

(viii) $1\mathbf{u}=\mathbf{u}$

[1] The term $-\mathbf{u}$ denotes $(-1)\mathbf{u}$.

Definition: Vector Space

A **vector space** is a nonempty set V of objects called *vectors* together with two operations called *vector addition* and *scalar multiplication* that satisfy the following ten axioms: For all $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in V, and for any scalars c and d

1. The sum $\mathbf{u} + \mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
4. There exists a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each vector $\mathbf{u}$ there exists a vector $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. For each scalar c, $c\mathbf{u}$ is in V.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $c(d\mathbf{u}) = d(c\mathbf{u}) = (cd)\mathbf{u}$.
10. $1\mathbf{u} = \mathbf{u}$

Remarks

- V is more accurately called a *real vector space* when we assume that the relevant scalars are the real numbers.
- Property 1. is that V is **closed** under (a.k.a. with respect to) vector addition.
- Property 6. is that V is **closed** under scalar multiplication.
- A vector space has the same basic *structure* as $\mathbb{R}^n$
- These are **axioms.** We assume (not "prove") that they hold for vector space V. However, they can be used to **prove or disprove** that a given set (with operations) is actually a vector space.

Examples of Vector Spaces

For an integer $n \geq 0$, $\mathbb{P}_n$ denotes the set of all polynomials with real coefficients of degree at most n. That is

$$\mathbb{P}_n = \{\mathbf{p}(t) = p_0 + p_1 t + \cdots + p_n t^n \mid p_0, p_1, \ldots, p_n \in \mathbb{R}\},$$

where addition[2] and scalar multiplication are defined by

$$(\mathbf{p} + \mathbf{q})(t) = \mathbf{p}(t) + \mathbf{q}(t) = (p_0 + q_0) + (p_1 + q_1)t + \cdots + (p_n + q_n)t^n,$$

$$(c\mathbf{p})(t) = c\mathbf{p}(t) = cp_0 + cp_1 t + \cdots + cp_n t^n.$$

[2] $\mathbf{q}(t) = q_0 + q_1 t + \cdots + q_n t^n$

Example

What is the zero vector $\mathbf{0}$ in $\mathbb{P}_n$?

Let $\mathbf{0}(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$. Find the values of $a_0, \ldots, a_n$.

We know that for any vector $\vec{p}$ in $\mathbb{P}_n$,

$\vec{0} + \vec{p} = \vec{p}$.

$$(\vec{0}+\vec{p})(t) = \vec{0}(t) + \vec{p}(t)$$

$$= (a_0+p_0) + (a_1+p_1)t + (a_2+p_2)t^2 + \cdots + (a_n+p_n)t^n$$

$$= p_0 + p_1 t + p_2 t^2 + \cdots + p_n t^n$$

$\Rightarrow \quad a_0 + p_0 = p_0 \quad \Rightarrow \quad a_0 = 0$

$a_1 + p_1 = p_1 \quad \Rightarrow \quad a_1 = 0$

* recall a_i and p_i are real numbers

$$\vdots$$

$$a_n + p_n = 0 \quad \Rightarrow \quad a_n = 0$$

That is $a_i = 0$ for all $i = 0, \ldots, n$.

$$\vec{0}(t) = 0 + 0t + 0t^2 + \ldots + 0t^n = 0$$

Example

If $\mathbf{p}(t) = p_0 + p_1 t + \cdots + p_n t^n$, what is the vector $-\mathbf{p}$?

Let $-\mathbf{p}(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n$. Find the values of $c_0, \ldots, c_n$.

We know that $(-\vec{p} + \vec{p})(t) = \vec{0}(t)$

$$(-\vec{p} + \vec{p})(t) = -\vec{p}(t) + \vec{p}(t)$$
$$= (c_0 + p_0) + (c_1 + p_1)t + (c_2 + p_2)t^2 + \ldots + (c_n + p_n)t^n$$
$$= 0 + 0t + 0t^2 + \ldots + 0t^n$$

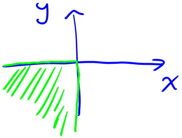
$$c_0 + p_0 = 0 \Rightarrow c_0 = -p_0$$
$$c_1 + p_1 = 0 \Rightarrow c_1 = -p_1$$
$$\vdots$$

$$c_n + \dot{p}_n = 0 \Rightarrow c_n = -p_n$$

$$\Rightarrow -\vec{p}(t) = -p_0 - p_1 t - p_2 t^2 - \dots - p_n t^n$$

A set that is not a Vector Space

Let $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \mid x \le 0, y \le 0 \right\}$ with regular vector addition and scalar multiplication in $\mathbb{R}^2$. Note V is the third quadrant in the xy-plane.



(1) Does property 1. hold for V?

Let $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ be in V.

So $x, y, a, b \le 0$.

$\vec{u} + \vec{v} = \begin{bmatrix} x+a \\ y+b \end{bmatrix}$. $x+a \le 0$ and $y+b \le 0$ so

$\vec{u}+\vec{v}$ is in V.

V is closed under vector addition.

A set that is not a Vector Space

Let $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \mid x \leq 0, y \leq 0 \right\}$ with regular vector addition and scalar multiplication in $\mathbb{R}^2$. Note V is the third quadrant in the xy-plane.

(2) Does property 6. hold for V?

Consider $\vec{u} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$. This is in V.

Consider $c\vec{u}$ for $c = -1$

$c\vec{u} = -1\vec{u} = -1 \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. which is not in V.

V is not closed under scalar multiplication.

V is not a vector space.

Theorem

Let V be a vector space. For each $\mathbf{u}$ in V and scalar c

$$\begin{aligned} 0\mathbf{u} &= \mathbf{0} \\ c\mathbf{0} &= \mathbf{0} \\ -1\mathbf{u} &= -\mathbf{u} \end{aligned}$$

Subspaces

Definition: A **subspace** of a vector space V is a subset H of V for which

a) The zero vector is in[3] H

b) H is closed under vector addition. (i.e. $\mathbf{u}, \mathbf{v}$ in H implies $\mathbf{u} + \mathbf{v}$ is in H)

c) H is closed under scalar multiplication. (i.e. $\mathbf{u}$ in H implies $c\mathbf{u}$ is in H)

[3]This is sometimes replaced with the condition that H is nonempty.

Example

Determine which of the following is a subspace of $\mathbb{R}^2$.

(a) The set of all vectors of the form $\mathbf{u} = (u_1, 0)$.

Let's call this subset H. $H = \{(u_1, u_2) \text{ in } \mathbb{R}^2 \mid u_2 = 0\}$.

we have to determine if

① $\vec{0}$ is in H
② H is closed under vector addition, and
③ H is closed under scalar multiplication.

Is $\vec{0} = (0,0)$ in H? Yes, its second component is zero.

Let $\vec{u} = (u_1, 0)$ and $\vec{v} = (v_1, 0)$ be in H.

$\vec{u}+\vec{v} = (u_1+v_1,\ 0+0) = (u_1+v_1,\ 0)$.

This has 2^{nd} component zero, so $\vec{u}+\vec{v}$ is in H.

H is closed under vector addition.

Let c be any scalar, and consider

$c\vec{u} = c(u_1, 0) = (cu_1, c\cdot 0) = (cu_1, 0)$.

This has 2^{nd} component zero, hence it is in H. H is closed under scalar multiplication.

H has all three properties. It is a subspace of $\mathbb{R}^2$.

* H is the x-axis in $\mathbb{R}^2$.

Example continued

(b) The set of all vectors of the form $\mathbf{u} = (u_1, 1)$.

Let's call this G,

$$G = \{ (u_1, u_2) \text{ in } \mathbb{R}^2 \mid u_2 = 1 \}.$$

Is $\vec{0}$ in G? No, hence G is not a subspace of $\mathbb{R}^2$.

G doesn't satisfy the other properties either.