March 22 Math 2306 sec. 51 Spring 2023

Section 11: Linear Mechanical Equations

Simple Harmonic Motion

We consider a flexible spring from which a mass is suspended. In the absence of any damping forces (e.g. friction, a dash pot, etc.), and free of any external driving forces, any initial displacement or velocity imparted will result in **free, undamped motion**–a.k.a. **simple harmonic motion**.

Harmonic Motion gif

Building an Equation: Hooke's Law

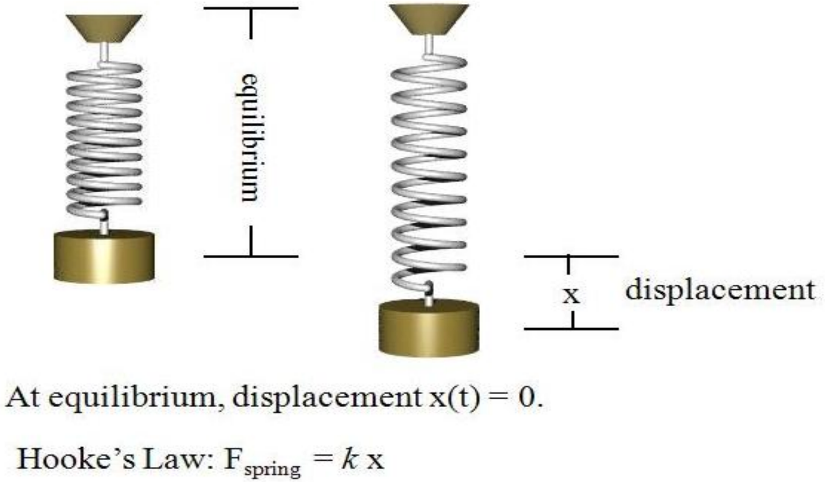

Figure: In the absence of any displacement, the system is at equilibrium. Displacement $x(t)$ is measured from equilibrium $x = 0$.

Building an Equation: Hooke's Law

Newton's Second Law: $F = ma$ (mass times acceleration)

$$a = \frac{d^2x}{dt^2} \quad \Longrightarrow \quad F = m\frac{d^2x}{dt^2}$$

Hooke's Law: $F = kx$ (proportional to displacement)

$mx'' = -kx \Rightarrow mx'' + kx = 0$

In standard form $x'' + \omega^2 x = 0$

where $\omega^2 = \frac{k}{m}$

2^{nd} order, linear, homogeneous, constant coef. ODE.

Displacment in Equilibrium

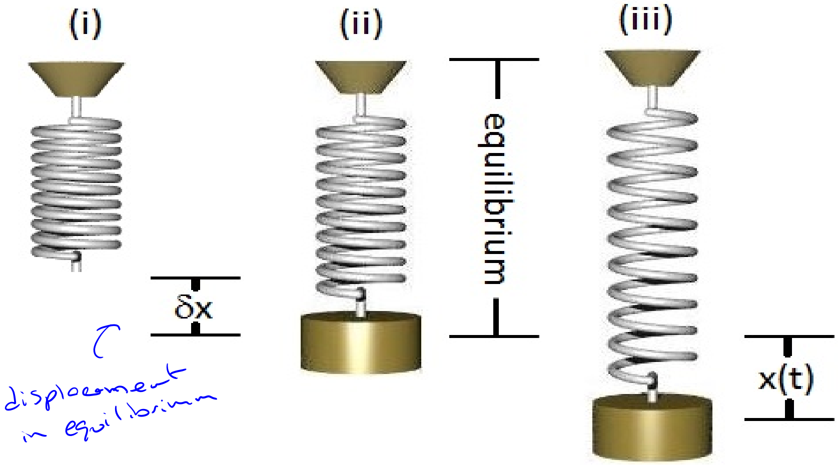

Figure: Spring only, versus spring-mass equilibrium, and spring-mass (nonzero) displacement

Obtaining the Spring Constant (US Customary Units)

If an object with weight W pounds stretches a spring δx feet in equilibrium, then by Hooke's law we compute the spring constant via the equation

$$W = k\delta x.$$

The units for k in this system of measure are lb/ft.

$$W = k\delta x \quad \Rightarrow \quad k = \frac{W}{\delta x} \qquad \frac{\text{lb}}{\text{ft}}$$

Obtaining the Mass (US Customary Units)

Note also that Weight = mass $\times$ acceleration due to gravity. Hence if we know the weight of an object, we can obtain the mass via

$$W = mg.$$

We typically take the approximation $g = 32$ ft/sec^2. The units for mass are lb sec^2/ft which are called slugs.

$$W = mg \implies m = \frac{W}{g}$$

Spring Constant and Mass (SI Units)

In SI units,

- Weight (force) would be in Newtons (N),
- Length would be in meters (m),
- Spring constant would be in N/m
- Mass would be in kilograms (kg)

It is customary to describe an object by its mass in kilograms. When we encounter such a description, we deduce the weight in Newtons

$$W = mg \quad \text{taking the approximation} \quad g = 9.8\,\text{m}/\text{sec}^2.$$

The *Circular Frequency* ω

Applying Hooke's law with the weight as force, we have

$$\frac{W}{m\delta x} = \frac{mg}{m\delta x} = \frac{k\delta x.}{m\delta x} \Rightarrow \frac{g}{\delta x} = \frac{k}{m}$$

We observe that the value ω can be deduced from δx by

$$\omega^2 = \frac{k}{m} = \frac{g}{\delta x}.$$

Provided that values for δx and g are used in appropriate units, ω is in units of per second.

Simple Harmonic Motion

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1 \tag{1}$$

Here, x_0 and x_1 are the initial position (relative to equilibrium) and velocity, respectively. The solution is

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) \tag{2}$$

called the **equation of motion.**

Caution: The phrase **equation of motion** is used differently by different authors.

Some use this phrase to refer the IVP (1). Others use it to refer to the **solution** to the IVP such as (2).

Simple Harmonic Motion

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t)$$

Characteristics of the system include

- the period $T = \frac{2\pi}{\omega}$,
- the frequency $f = \frac{1}{T} = \frac{\omega}{2\pi}$[1]
- the circular (or angular) frequency ω, and
- the amplitude or maximum displacement $A = \sqrt{x_0^2 + (x_1/\omega)^2}$

[1] Various authors call f the natural frequency and others use this term for ω.

Amplitude and Phase Shift

We can formulate the solution in terms of a single sine (or cosine) function. Letting

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) = A \sin(\omega t + \phi)$$

requires

$$A = \sqrt{x_0^2 + (x_1/\omega)^2},$$

and the **phase shift** ϕ must be defined by

$$\sin\phi = \frac{x_0}{A}, \quad \text{with} \quad \cos\phi = \frac{x_1}{\omega A}.$$

Amplitude and Phase Shift (alternative definition)

We can formulate the solution in terms of a single sine (or cosine) function. Letting

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) = A \cos(\omega t - \hat{\phi})$$

requires

$$A = \sqrt{x_0^2 + (x_1/\omega)^2},$$

and this **phase shift** $\hat{\phi}$ must be defined by

$$\cos \hat{\phi} = \frac{x_0}{A}, \quad \text{with} \quad \sin \hat{\phi} = \frac{x_1}{\omega A}.$$

Example

An object stretches a spring 6 inches in equilibrium. Assuming no driving force and no damping, set up the differential equation describing this system.

The ODE is $x'' + \omega^2 x = 0$ or $mx'' + kx = 0$

We need m, k or ω^2. We know

displacement in equilibrium $\delta x = 6$ in.

We can use $\omega^2 = \dfrac{g}{\delta x}$ with

$g = 32\ \text{ft}/\text{sec}^2$

Since $\delta x = 6\text{ in} = 6\text{in}\left(\dfrac{1\text{ft}}{12\text{in}}\right) = \dfrac{1}{2}\text{ ft}$

$$\omega^2 = \frac{32 \ ft/sec^2}{\frac{1}{2} ft} = 64 \ \frac{1}{sec^2}$$

The ODE is

$$x'' + 64x = 0$$

Example

A 4 pound weight stretches a spring 6 inches. The mass is released from a position 4 feet above equilibrium with an initial downward velocity of 24 ft/sec. Find the equation of motion, the period, amplitude, phase shift, and frequency of the motion. (Take $g = 32$ ft/sec^2.)

The model is $mx'' + kx = 0$ i.e. $x'' + \omega^2 x = 0$

We know that $\delta x = 6$ in so $\omega^2 = 64$, but let's find k and m. The weight $W = 4$ lb.

$W = mg$ and $W = k\,\delta x$

$$m = \frac{W}{g} = \frac{4\text{ lb}}{32\ \text{ft}/\text{sec}^2} = \frac{1}{8} \text{ slugs}$$

$$k = \frac{W}{\delta x} = \frac{4\,\text{lb}}{\frac{1}{2}\,\text{ft}} = 8\,\frac{\text{lb}}{\text{ft}}$$

$$\omega^2 = \frac{k}{m} = \frac{8\,\frac{\text{lb}}{\text{ft}}}{\frac{1}{8}\,\text{slugs}} = 64\,\frac{1}{\text{sec}^2}$$

The ODE is $\quad x'' + 64x = 0$

$x(0) = 4 \quad$ (4 ft above equilibrium)

$x'(0) = -24 \quad$ (24 ft/sec downward velocity)

Using r as the parameter, the characteristic eqn is

$$r^2 + 64 = 0$$

$r^2 = -64 \Rightarrow r = \pm\sqrt{-64} = \pm 8i$

complex case w/ $\alpha = 0$, $\beta = 8$

$x_1 = e^{0t}\cos(8t)$, $x_2 = e^{0t}\sin(8t)$

$x = c_1\cos(8t) + c_2\sin(8t)$

Apply IC $x' = -8c_1\sin(8t) + 8c_2\cos(8t)$

$x(0) = c_1\cos(0) + c_2\sin(0) = 4$

$c_1 = 4$

$x'(0) = -8c_1\sin(0) + 8c_2\cos(0) = -24$

$8c_2 = -24 \Rightarrow c_2 = \frac{-24}{8} = -3$

The equation of motion is

$$x = 4\cos(8t) - 3\sin(8t)$$

The period $T = \frac{2\pi}{\omega} = \frac{2\pi}{8} = \frac{\pi}{4}$ sec

Linear frequency $f = \frac{1}{T} = \frac{4}{\pi}$ $\frac{1}{\text{sec}}$

Amplitude $A = \sqrt{x_0^2 + \left(\frac{x_1}{\omega}\right)^2}$

$$= \sqrt{4^2 + (-3)^2} = 5$$

If $x(t) = A\sin(\omega t + \phi)$

$$x(t) = 5\sin(8t + \phi)$$

where $\sin\phi = \dfrac{x_0}{A} = \dfrac{4}{5}$ and

$$\cos\phi = \frac{x_1}{\omega A} = \frac{-3}{5}$$

$$\phi = \cos^{-1}\left(\frac{-3}{5}\right) \approx 2.21$$