March 23 Math 3260 sec. 51 Spring 2022

Section 4.1: Vector Spaces and Subspaces

A **vector space** is a nonempty set V of objects called *vectors* together with two operations called *vector addition* and *scalar multiplication* that satisfy the following ten axioms: For all $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V, and for any scalars c and d

1. The sum $\mathbf{u}+\mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There exists a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each vector $\mathbf{u}$ there exists a vector $-\mathbf{u}$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. For each scalar c, $c\mathbf{u}$ is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=d(c\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$

Subspaces

Definition: A **subspace** of a vector space V is a subset H of V for which

a) The zero vector is in[1] H

b) H is closed under vector addition. (i.e. $\mathbf{u}, \mathbf{v}$ in H implies $\mathbf{u} + \mathbf{v}$ is in H)

c) H is closed under scalar multiplication. (i.e. $\mathbf{u}$ in H implies $c\mathbf{u}$ is in H)

[1] This is sometimes replaced with the condition that H is nonempty.

Definition: Linear Combination and Span

Definition Let V be a vector space and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ be a collection of vectors in V. A **linear combination** of the vectors is a vector $\mathbf{u}$

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

for some scalars $c_1, c_2, \ldots, c_p$.

Definition The **span**, $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$, is the subset of V consisting of all linear combinations of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$.

Example

On Monday, we looked at the set $H = \{(u_1, u_2) \in \mathbb{R}^2 \mid u_2 = 0\}$—i.e. the set of all vectors $(u_1, 0)$, and found it was a subspace of $\mathbb{R}^2$. Note that

$$H = \mathrm{Span}\{(1, 0)\}.$$

Theorem

Theorem: If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$, for $p \geq 1$, are vectors in a vector space V, then Span$\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$, is a subspace of V.

Remark 1: This is saying that a span is always a subspace.

Remark 2: $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is called the **subspace of V spanned by (or generated by)** $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.

Remark 3: If H is any subspace of V, a **spanning set** for H is any set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ such that $H =$Span$\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.

Example

$M_{2\times 2}$ denotes the set of all 2×2 matrices with real entries with regular addition and scalar multiplication of matrices. Consider the subset H of $M_{2\times 2}$

$$H = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a,\, b \in \mathbb{R} \right\}.$$

Show that H is a subspace of $M_{2\times 2}$ by finding a spanning set. That is, show that $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ for some appropriate vectors.

We can do this by taking an arbitrary element of H and writing it as a linear combination of fixed vectors (2x2 matrices).

Consider $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ in H for any a, b in $\mathbb{R}$.

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$$

$$= a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

This is a linear combo of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. So

$$H = \text{Span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

As a span, H is a subspace of $M_{2\times 2}$.

Section 4.2: Null & Column Spaces, Linear Transformations

Definition: Let A be an $m \times n$ matrix. The **null space** of A, denoted[2] by Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. That is

$$\text{Nul}\, A = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}.$$

We can say that Nul A is the subset of $\mathbb{R}^n$ that gets mapped to the zero vector under the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$.

[2] Some authors will write Null(A)—I tend to write two ells.

Example

Determine Nul A where $A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 7 \end{bmatrix}$.

We have to characterize all $\vec{x}$ in $\mathbb{R}^3$ such that $A\vec{x} = \vec{0}$. We can use the augmented matrix

$$[A \ \vec{0}] = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 1 & 2 & 7 & 0 \end{bmatrix} \quad -R_1 + R_2 \to R_2$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 4 & 0 \end{bmatrix} \quad \tfrac{1}{2}R_2 \to R_2 \quad \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

$x_1 = -3x_3$
$x_2 = -2x_3$, x_3 is free

For $\vec{x}$ in Nul(A)

$$\vec{x} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}.$$

Note $\text{Nul}(A) = \text{Span}\left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} \right\}$.

Theorem

Theorem: If A is an $m \times n$ matrix, then Nul(A) is a subspace of $\mathbb{R}^n$.

Note:

① Since $A\vec{0} = \vec{0}$, $\vec{0}$ is in Nul(A).

② If $\vec{u}$ and $\vec{v}$ are in Nul(A) then

$A\vec{u} = \vec{0}$ and $A\vec{v} = \vec{0}$.

Note that

$$A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0}$$

$\Rightarrow$ $\vec{u}+\vec{v}$ is in Nul(A).

③ If c is any scalar, note that

$$A(c\vec{u}) = cA\vec{u} = c\vec{0} = \vec{0}$$

Hence $c\vec{u}$ is in $\text{Nul}(A)$.

$\text{Nul}(A)$ has all the necessary properties to be a subspace of $\mathbb{R}^n$.

Example

For a given matrix, a spanning set for Nul$A$ gives an *explicit* description of this subspace. Find a spanning set for Nul A where

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 1 & 2 & 6 & -5 \end{bmatrix}.$$

We need to solve $A\vec{x} = \vec{0}$. where $\vec{x}$ is in $\mathbb{R}^4$.

Using an augmented matrix

$$[A \ \vec{0}] = \begin{bmatrix} 1 & 0 & 2 & -1 & 0 \\ 1 & 2 & 6 & -5 & 0 \end{bmatrix} \quad -R_1 + R_2 \to R_2$$

$$\begin{bmatrix} 1 & 0 & 2 & -1 & 0 \\ 0 & 2 & 4 & -6 & 0 \end{bmatrix} \quad \tfrac{1}{2} R_2 \to R_2$$

$$\begin{bmatrix} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & 2 & -3 & 0 \end{bmatrix}$$

$x_1 = -2x_3 + x_4$

$x_2 = -2x_3 + 3x_4$

x_3, x_4 are free

$$\vec{x} = \begin{bmatrix} -2x_3 + x_4 \\ -2x_3 + 3x_4 \\ x_3 \\ x_4 \end{bmatrix}.$$

$$\vec{x} = x_3 \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

An explicit description of Nul(A) is

$$\text{Nul}(A) = \text{Span}\left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Column Space

Definition: The **column space** of an $m \times n$ matrix A, denoted Col A, is the set of all linear combinations of the columns of A. If $A = [\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n]$, then

$$\mathrm{Col} A = \mathrm{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}.$$

Remark: Note that this corresponds to the set of solutions $\mathbf{b}$ of linear equations of the form $A\mathbf{x} = \mathbf{b}$. That is

$$\mathrm{Col} A = \{\mathbf{b} \in \mathbb{R}^m \mid \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^n\}.$$