March 24 Math 2306 sec. 51 Spring 2023

Section 11: Linear Mechanical Equations

The position $x(t)$ at the time t of an object of mass m attached to a spring with spring constant k subject to **simple harmonic motion** (i.e., undamped and unforced) is governed by

$$mx'' + kx = 0 \quad x(0) = x_0, \quad x'(0) = x_1$$

where x_0 and x_1 are the initial position (relative to equilibrium) and initial velocity, respectively. In standard form, the equation can be written in terms of the circular frequency ω as

$$x'' + \omega^2 x = 0 \quad \text{where} \quad \omega^2 = \frac{k}{m}.$$

Next, we want to consider an added damping force.

Free Damped Motion

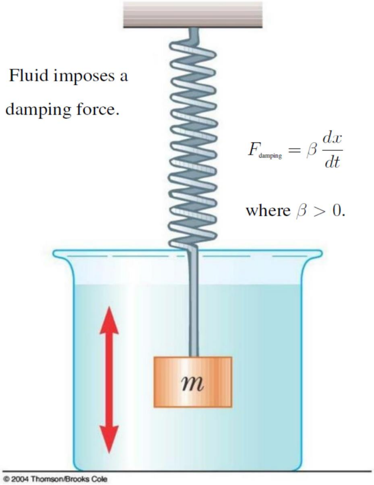


Figure: If a damping force is added, we'll assume that this force is proportional to the instantaneous velocity.

Free Damped Motion

Now we wish to consider an added force corresponding to damping—friction, a dashpot, air resistance.

Total Force = Force of spring + Force of damping

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx \quad \Longrightarrow \quad \frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

where

$$2\lambda = \frac{\beta}{m} \quad \text{and} \quad \omega = \sqrt{\frac{k}{m}}.$$

Three qualitatively different solutions can occur depending on the nature of the roots of the characteristic equation

$$r^2 + 2\lambda r + \omega^2 = 0 \quad \text{with roots} \quad r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}.$$

Case 1: $\lambda^2 > \omega^2$ Overdamped

$$x(t) = e^{-\lambda t}\left(c_1 e^{t\sqrt{\lambda^2-\omega^2}} + c_2 e^{-t\sqrt{\lambda^2-\omega^2}}\right)$$

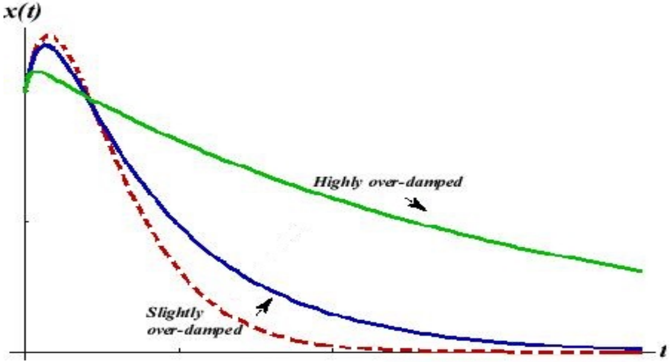


Figure: Two distinct real roots. No oscillations. Approach to equilibrium may be slow.

Case 2: $\lambda^2 = \omega^2$ Critically Damped

$$x(t) = e^{-\lambda t}(c_1 + c_2 t)$$

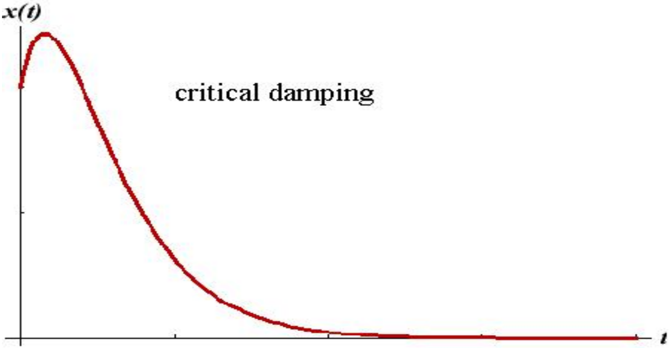


Figure: One real root. No oscillations. Fastest approach to equilibrium.

Case 3: $\lambda^2 < \omega^2$ Underdamped

$$x(t) = e^{-\lambda t}\left(c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t)\right), \quad \omega_1 = \sqrt{\omega^2 - \lambda^2}$$

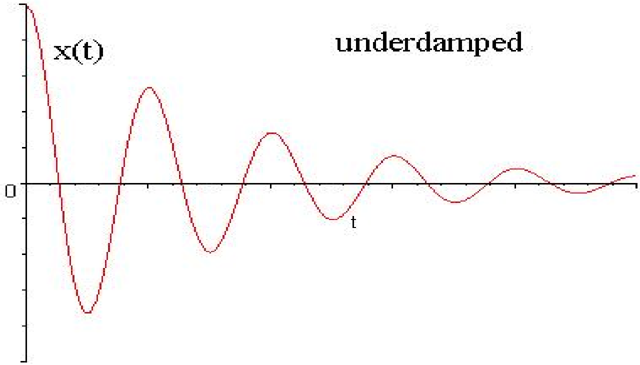
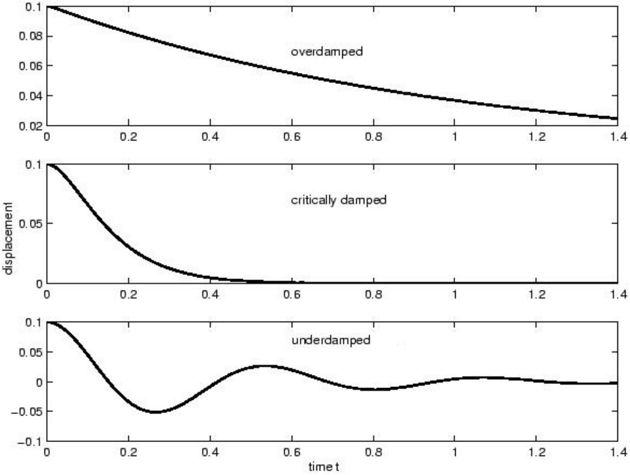


Figure: Complex conjugate roots. Oscillations occur as the system approaches (resting) equilibrium.

Comparison of Damping



2 real

one real

complex

Figure: Comparison of motion for the three damping types.

Example

A 2 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 10 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped.

The ODE is $mx'' + \beta x' + kx = 0$

$m = 2\,kg$

$k = 12\ N/m$

$\beta = 10\ N/m/sec$

$2x'' + 10x' + 12x = 0$

Standard form

$x'' + 5x' + 6x = 0$

The characteristic eqn is

$$r^2 + 5r + 6 = 0$$

$$(r+2)(r+3) = 0 \Rightarrow r = -2 \text{ or } r = -3$$

Two real roots $\Rightarrow$ the system is over damped.

Note $\omega^2 = \frac{k}{m} = \frac{12}{2} = 6$

$$2\lambda = \frac{\beta}{m} = \frac{10}{2} = 5 \Rightarrow \lambda = \frac{5}{2}$$

Is $\lambda^2 - \omega^2 > 0$?

$$\lambda^2 = \left(\frac{5}{2}\right)^2 = \frac{25}{4}, \quad \omega^2 = 6 = \frac{24}{4}$$

$$\lambda^2 - \omega^2 = \frac{25}{4} - \frac{24}{4} = \frac{1}{4} > 0$$

Example

A 3 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 12 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped. If the mass is released from the equilibrium position with an upward velocity of 1 m/sec, solve the resulting initial value problem.

$$mx'' + \beta x' + kx = 0$$

$m = 3$

$\beta = 12$

$k = 12$

$$3x'' + 12x' + 12x = 0$$

Standard form

$$x'' + 4x' + 4x = 0$$

Characteristic eqn

$$r^2 + 4r + 4 = 0$$

$$(r+2)^2 = 0 \implies r = -2 \quad \text{double root}$$

One root $\Rightarrow$ the system is critically damped.

$$x_1 = e^{-2t}, \quad x_2 = te^{-2t}$$

The general solution

$$x = c_1 e^{-2t} + c_2 t e^{-2t}$$

App's $\quad x(0)=0 \quad$ (at equilibrium)

$x'(0) = 1 \quad$ ($1\,\frac{m}{sec}$ upward

$$x' = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t}$$

$$x(0) = c_1 e^0 + c_2 (0) e^0 = 0 \quad \Rightarrow \quad c_1 = 0$$

$$x'(0) = -2c_1 e^0 + c_2 e^0 - 2c_2 (0) e^0 = 1$$

$$c_2 = 1$$

The position

$$x(t) = t e^{-2t}$$