March 25 Math 3260 sec. 52 Spring 2022

Section 4.2: Null & Column Spaces, Row Space, Linear Transformations

Definition: Let A be an $m \times n$ matrix. The **null space** of A, denoted by Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. That is

$$\text{Nul}\, A = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}.$$

Theorem: If A is an $m \times n$ matrix, then $\text{Nul}(A)$ is a subspace of $\mathbb{R}^n$.

Column Space

Definition: The **column space** of an $m \times n$ matrix A, denoted Col A, is the set of all linear combinations of the columns of A. If $A = [\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n]$, then

$$\text{Col} A = \text{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}.$$

Theorem: The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

Corollary: Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.

Example

Find a matrix A such that $W = \text{Col } A$ where

$$W = \left\{ \begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}.$$

We need to recognize elements of W in the form $A\vec{x}$ for some matrix A.

$$\begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} = \begin{bmatrix} 6a \\ a \\ -7a \end{bmatrix} + \begin{bmatrix} -b \\ b \\ 0 \end{bmatrix}$$

$$= a \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

Hence $W = \mathrm{Col}(A)$ where $A = \begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix}$.

Row Space

Definition: The **row space**, denoted Row A, of an $m \times n$ matrix A is the subspace of $\mathbb{R}^n$ spanned by the rows of A.

Theorem If two matrices A and B are row equivalent, then their row spaces are the same.

This says that row operations don't change the row space.

Example

Find two spanning sets for Row(A) given

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 9 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We can take one set from A and one from the rref

$$\text{Row}(A) = \text{Span}\left\{ \begin{bmatrix} 2 \\ 4 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -5 \\ 7 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ -8 \\ 6 \end{bmatrix} \right\}$$

and

$$\text{Row}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 9 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Example: Comparing Col(A) and Nul(A)

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$$

(a) If Col A is a subspace of $\mathbb{R}^k$, what is k?

The columns are in $\mathbb{R}^3$, so $k=3$.

(b) If Nul A is a subspace of $\mathbb{R}^k$, what is k?

$\vec{x}$ is in Nul(A) if $A\vec{x} = \vec{0}$ this requires

$\vec{x}$ to be in $\mathbb{R}^4$ So $k=4$.

Example Continued...

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$$

(c) Is **u** in Nul A? Could **u** be in Col A?

$\vec{u}$ is in Nul(A) if $A\vec{u} = \vec{0}$.

$$A\vec{u} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \vec{0}$$ so $\vec{u}$ is <u>not</u> in Nul(A).

$\vec{u}$ could not be in Col(A). Col(A) is a subspace of $\mathbb{R}^3$; $\vec{u}$ is in $\mathbb{R}^4$.

Example Continued...

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$$

(d) Is $\mathbf{v}$ in Col A? Could $\mathbf{v}$ be in Nul A?

$\vec{v}$ is in $\mathbb{R}^3$ so it can't be in Nul(A).

$\vec{v}$ is in Col(A) if $A\vec{x} = \vec{v}$ is consistent. Using the augmented matrix

$$[A \ \vec{v}], \quad \xrightarrow{\text{rref}} \quad \begin{bmatrix} 1 & 0 & 9 & 0 & 5 \\ 0 & 1 & -5 & 0 & -30/17 \\ 0 & 0 & 0 & 1 & 1/17 \end{bmatrix}$$

The last column is not a pivot column,

hence $A\vec{x} = \vec{v}$ is consistent.

So $\vec{v}$ is in $\mathrm{Col}(A)$.

Linear Transformation

Definition: Let V and W be vector spaces. A linear transformation $T : V \longrightarrow W$ is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W such that

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for every $\mathbf{u}, \mathbf{v}$ in V, and

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for every $\mathbf{u}$ in V and scalar c.

Example

Let $C^1(\mathbb{R})$ denote the set of all real valued functions that are differentiable and $C^0(\mathbb{R})$ the set of all continuous real valued functions. Note that differentiation is a linear transformation. That is

$$D : C^1(\mathbb{R}) \longrightarrow C^0(\mathbb{R}), \quad D(f) = f'$$

satisfies the two conditions in the previous definition.

We know from calculus that if f and g are differentiable and c is a scalar, then

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x) \quad \text{and} \quad \frac{d}{dx}(cf(x)) = cf'(x).$$

Characterize the subset of $C^1(\mathbb{R})$ such that $Df = 0$.

This is the set of constant functions.

Range and Kernel

Definition: The **range** of a linear transformation $T : V \longrightarrow W$ is the set of all vectors in W of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in V. (The set of all images of elements of V.)

A column space is a **range**.

Definition: The **kernel** of a linear transformation $T : V \longrightarrow W$ is the set of all vectors $\mathbf{x}$ in V such that $T(\mathbf{x}) = \mathbf{0}$. (The analog of the null space of a matrix.)

A null space is a **kernel**.

Range & Kernel as Subspaces

Theorem: Given linear transformation $T : V \longrightarrow W$,

- the range of T is a subspace of W,
- and the kernel of T is a subspace of V.

Example

Consider $T : C^1(\mathbb{R}) \longrightarrow C^0(\mathbb{R})$ defined by

$$T(f) = \frac{df}{dx} + \alpha f(x), \quad \alpha \text{ a fixed constant.}$$

(a) Express the equation that a function y must satisfy if y is in the kernel of T.

y is in the kernel of T if

$T(y) = 0$. Well, $T(y) = \frac{dy}{dx} + \alpha y$.

The equation is $\frac{dy}{dx} + \alpha y = 0$.

Example: $T : C^1(\mathbb{R}) \longrightarrow C^0(\mathbb{R})$

(b) Show that for any scalar c, $y = ce^{-\alpha x}$ is in the kernel of T.

If y is in the kernel, then

$$\frac{dy}{dx} + \alpha y = 0.$$

If $y = ce^{-\alpha x}$ then $\frac{dy}{dx} = c(-\alpha e^{-\alpha x}) = -\alpha c e^{-\alpha x}$

$$\frac{dy}{dx} + \alpha y = -\alpha c e^{-\alpha x} + \alpha(ce^{-\alpha x})$$
$$= -\alpha c e^{-\alpha x} + \alpha c e^{-\alpha x} = 0$$

$\Rightarrow y = ce^{-\alpha x}$ is in the kernel of T.