March 27 Math 2306 sec. 51 Spring 2023

Section 11: Linear Mechanical Equations

Driven Motion

We can consider the application of an external driving force (with or without damping). Assume a time dependent force $f(t)$ is applied to the system. The ODE governing displacement becomes

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx + f(t), \quad \beta \geq 0.$$

Divide out m and let $F(t) = f(t)/m$ to obtain the nonhomogeneous equation

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)$$

Forced Undamped Motion and Resonance

Consider the case $F(t) = F_0 \cos(\gamma t)$ or $F(t) = F_0 \sin(\gamma t)$, and $\lambda = 0$. Two cases arise

$$\text{(1)} \quad \gamma \neq \omega, \quad \text{and (2)} \quad \gamma = \omega.$$

Taking the sine case, the DE is

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

with complementary solution

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

Note that

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Using the method of undetermined coefficients, the **first guess** to the particular solution is

$x_p = A\cos(\gamma t) + B\sin(\gamma t)$ Suppose $\gamma \neq \omega$

This would be correct.

The position would look like

$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) + A\cos(\gamma t) + B\sin(\gamma t)$

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

Note that

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Using the method of undetermined coefficients, the **first guess** to the particular solution is

$x_p = A\cos(\gamma t) + B\sin(\gamma t)$ Suppose $\gamma = \omega$

This would not be correct

$$x_p = (A\cos(\omega t) + B\sin(\omega t))t$$
$$= At\cos(\omega t) + Bt\sin(\omega t)$$

The position

$$x(t) = c_1\cos(\omega t) + c_2\sin(\omega t) + At\cos(\omega t) + Bt\sin(\omega t)$$

Forced Undamped Motion and Resonance

For $F(t) = F_0 \sin(\gamma t)$ starting from rest at equilibrium:

at equilibrium

$$\text{Case (1):} \quad x'' + \omega^2 x = F_0 \sin(\gamma t), \quad x(0) = 0, \quad x'(0) = 0$$

from rest

$$x(t) = \frac{F_0}{\omega^2 - \gamma^2} \left(\sin(\gamma t) - \frac{\gamma}{\omega} \sin(\omega t) \right)$$

If $\gamma \approx \omega$, the amplitude of motion could be rather large!

Pure Resonance

Case (2): $\quad x'' + \omega^2 x = F_0 \sin(\omega t), \quad x(0) = 0, \quad x'(0) = 0$

$$x(t) = \frac{F_0}{2\omega^2}\sin(\omega t) - \frac{F_0}{2\omega} t\cos(\omega t)$$

Note that the amplitude, α, of the second term is a function of t:

$$\alpha(t) = \frac{F_0 t}{2\omega}$$

which grows without bound!

Forced Motion and Resonance Applet

Choose "Elongation diagram" to see a plot of displacement. Try exciter frequencies close to ω.

Section 12: LRC Series Circuits

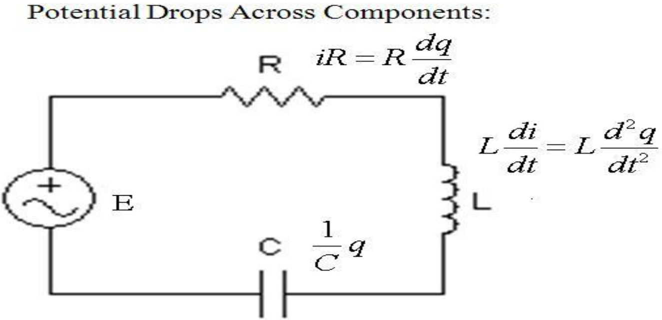

Figure: Kirchhoff's Law: The charge q on the capacitor satisfies $Lq'' + Rq' + \frac{1}{C}q = E(t)$.

This is a second order, linear, constant coefficient nonhomogeneous (if $E \neq 0$) equation.

LRC Series Circuit (Free Electrical Vibrations)

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$

If the applied force $E(t) = 0$, then the **electrical vibrations** of the circuit are said to be **free.** These are categorized as

$$\begin{array}{lrcll} \textbf{overdamped if} & R^2 - 4L/C & > & 0, & \text{(2 real roots)} \\ \textbf{critically damped if} & R^2 - 4L/C & = & 0, & \text{(1 real root)} \\ \textbf{underdamped if} & R^2 - 4L/C & < & 0. & \text{(complex roots)} \end{array}$$

Steady and Transient States

Given a nonzero applied voltage $E(t)$, we obtain an IVP with nonhomogeneous ODE for the charge q

$$Lq'' + Rq' + \frac{1}{C}q = E(t), \quad q(0) = q_0, \quad q'(0) = i_0.$$

From our basic theory of linear equations we know that the solution will take the form

$$q(t) = q_c(t) + q_p(t).$$

$i_c = \frac{dq_c}{dt}$

transient state current

The function of q_c is influenced by the initial state (q_0 and i_0) and will decay exponentially as $t \to \infty$. Hence q_c is called the **transient state charge** of the system.

Steady and Transient States

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$$q(t) = q_c(t) + q_p(t).$$

$i_p = \frac{dq_p}{dt}$ steady state current

The function q_p is independent of the initial state but depends on the characteristics of the circuit (L, R, and C) and the applied voltage E. q_p is called the **steady state charge** of the system.

Example

An LRC series circuit has inductance 0.5 h, resistance 10 ohms, and capacitance $4 \cdot 10^{-3}$ f. Find the steady state current of the system if the applied force is $E(t) = 5\cos(10t)$.

The model is $Lq'' + Rq' + \frac{1}{C} q = E$

$L = \frac{1}{2}$ h

$R = 10\,\Omega$

$C = 4 \cdot 10^{-3}$ f

$$\frac{1}{2} q'' + 10q' + \frac{1}{4 \cdot 10^{-3}} q = 5\cos(10t)$$

The steady state current

$i_p = \frac{dq_p}{dt}$ this is what we want

$$\frac{1}{4 \cdot 10^{-3}} = \frac{10^3}{4} = \frac{1000}{4} = 250$$

Standard form: $q'' + 20q' + 500q = 10\cos(10t)$

Identify q_c:

Charact. eqn $m^2 + 20m + 500 = 0$

$$m^2 + 20m + 100 - 100 + 500 = 0$$

$$(m+10)^2 + 400 = 0$$

$$(m+10)^2 = -400$$

$$m + 10 = \pm\sqrt{-400}$$

$$m = -10 \pm 20i$$

$q_1 = e^{-10t}\cos(20t)$, $q_2 = e^{-10t}\sin(20t)$

$$q_c = c_1 e^{-10t}\cos(20t) + c_2 e^{-10t}\sin(20t)$$

$$q'' + 20q' + 500q = 10\cos(10t)$$

Find q_p using undetermined coefficients

500 $\quad q_p = A\cos(10t) + B\sin(10t)$

20 $\quad q_p' = -10A\sin(10t) + 10B\cos(10t)$

1 $\quad q_p'' = -100A\cos(10t) - 100B\sin(10t)$

$$q_p'' + 20q_p' + 500q_p = 10\cos(10t)$$

$$\cos(10t)\left(-100A + 200B + 500A\right) + \sin(10t)\left(-100B - 200A + 500B\right)$$

$$= 10\cos(10t) + 0\cdot\sin(10t)$$

Matching

$$400A + 200B = 10 \qquad\qquad 40A + 20B = 1$$
$$-200A + 400B = 0 \quad \Rightarrow \quad -20A + 40B = 0$$

$$\begin{array}{r} 40A + 20B = 1 \\ -40A + 80B = 0 \\ \hline 100B = 1 \end{array} \qquad B = \frac{1}{100}$$

$$40B = 20A \Rightarrow A = 2B = \frac{2}{100}$$

$$q_p = \frac{2}{100}\cos(10t) + \frac{1}{100}\sin(10t)$$

This is the steady state charge.

The steady state current

$$i_p = \frac{-2}{10}\sin(10t) + \frac{1}{10}\cos(10t)$$