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Section 4.3: Linearly Independent Sets and Bases

Definition: A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in a vector space V is said to be **linearly independent** if the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \tag{1}$$

has only the trivial solutions $c_1 = c_2 = \cdots = c_p = 0$.

The set is **linearly dependent** if there exist a nontrivial solution (at least one of the weights c_i is nonzero).

If there is a nontrivial solution $c_1, \ldots, c_p$, then equation (1) is called a **linear dependence relation**.

Theorem

Theorem: Consider the ordered set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in a vector space V, where $p \geq 2$ and $\mathbf{v}_1 \neq \mathbf{0}$. This set is **linearly dependent** if and only if there is some $j > 1$ such that $\mathbf{v}_j$ is a linear combination of the preceding vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

This says that

- If one of the vectors, say $\mathbf{v}_j$ can be written as a linear combo of the ones that come before it, the set is linearly dependent, and
- if the set is linearly dependent, it must be possible to write one of the vectors as a linear combo of the others.

Example

Determine if the set is linearly dependent or independent in $\mathbb{P}_2$.

$\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ where $\mathbf{p}_1 = 1$, $\mathbf{p}_2 = 2t$, $\mathbf{p}_3 = t - 3$.

Note that $\vec{p}_3 = \frac{1}{2}\vec{p}_2 - 3\vec{p}_1$

Hence the set is linearly dependent.

we can state the linear dependence relation

$$3\vec{p}_1 - \frac{1}{2}\vec{p}_2 + \vec{p}_3 = \vec{0}$$

Definition (Basis)

Definition: Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a **basis** of H provided

(i) $\mathcal{B}$ is linearly independent, and

(ii) $H =$Span$(\mathcal{B})$.

We can think of a basis as a *minimal spanning set*. All of the *information* needed to construct vectors in H is contained in the basis, and none of this information is repeated.

Prelude to a Spanning Set Theorem

Example: Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ be vectors in a vector space V, and suppose that

(1) $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and
(2) $\mathbf{v}_3 = \mathbf{v}_1 - 2\mathbf{v}_2$.

Show that $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

We have to show that every vector in H can be written as a linear combination of $\vec{v}_1$ and $\vec{v}_2$. Let $\vec{h}$ be any vector in H.

Since $H = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ there are scalars c_1, c_2, c_3 such that

$$\vec{h} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$$

Using $\vec{v}_3 = \vec{v}_1 - 2\vec{v}_2$

$$\vec{h} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3(\vec{v}_1 - 2\vec{v}_2)$$

$$= (c_1 + c_3)\vec{v}_1 + (c_2 - 2c_3)\vec{v}_2$$

$$= k_1\vec{v}_1 + k_2\vec{v}_2 \quad \text{where} \quad k_1 = c_1 + c_3 \text{ and}$$

$$k_2 = c_2 - 2c_3$$

Thus every vector in H is a linear combination of $\vec{v}_1$ and $\vec{v}_2$, i.e.

$$H = \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}.$$

Theorem:

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ be a set in a vector space V and $H =$Span(S).

(a.) If one of the vectors in S, say $\mathbf{v}_k$ is a linear combination of the other vectors in S, then the subset of S obtained by eliminating $\mathbf{v}_k$ still spans H.

(b) If $H \neq \{\mathbf{0}\}$, then some subset of S is a basis for H.

If we start with a spanning set, we can eliminate *duplication* and arrive at a basis.

Column Space

Find a basis for the column space matrix B that is in reduced row echelon form

$$B = \begin{array}{c} \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \vec{b}_4 & \vec{b}_5 \end{array} \end{array}.$$

Name the columns

$B = [\vec{b}_1 \ \vec{b}_2 \ \vec{b}_3 \ \vec{b}_4 \ \vec{b}_5]$

By definition, $\operatorname{Col}(B) = \operatorname{Span}\{\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4, \vec{b}_5\}$.

Note $\vec{b}_2 = 4\vec{b}_1 \longrightarrow$ remove $\vec{b}_2$

$\vec{b}_4 = 2\vec{b}_1 - \vec{b}_3 \longrightarrow$ remove $\vec{b}_4$

A basis is $\{\vec{b}_1, \vec{b}_3, \vec{b}_5\}$.

We have a basis consisting of the pivot columns of B.

$$\mathrm{Col}(B) = \mathrm{Span}\left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \right\}.$$

Using the rref

Theorem: If $A = [\mathbf{a}_1 \cdots \mathbf{a}_n]$ and $B = [\mathbf{b}_1 \cdots \mathbf{b}_n]$ are row equivalent matrices, then Nul $A =$ Nul B. That is, the equations

$$A\mathbf{x} = \mathbf{0} \quad \text{and} \quad B\mathbf{x} = \mathbf{0}$$

have the same solution set.

Remark: This means that $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ and $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ have **exactly the same linear dependence relationships**!

Theorem:

The pivot columns of a matrix A form a basis of Col A.

Caveat: This means we can use row reduction to identify a basis, but the vectors we obtain will be from the original matrix A. (As illustrated in the following example.)

Find a basis for Col A

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}.$$

Our basis vectors will be the pivot columns, so I just need to know which columns are pivot columns.

Find a basis for Col A

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}. \quad \xrightarrow{rref} \quad \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From the rref, the pivot columns are 1, 3, and 5. So a basis for Col(A) is

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 2 \\ 8 \end{bmatrix} \right\}.$$

$$\text{Col}(A) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 2 \\ 8 \end{bmatrix} \right\}.$$

Basis for a Row Space

Theorem: If two matrices A and B are row equivalent, then their row spaces are the same.

This tells us that a basis for the row space of an $m \times n$ matrix A is the nonzero rows of its rref.

Find a basis for Row(A)

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}. \quad \xrightarrow{rref} \quad \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From the rref a basis for Row(A) is

$$\left\{ \begin{bmatrix} 1 \\ 4 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

$$\text{Row}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 4 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Find bases for Nul A and Col A

$$A = \begin{bmatrix} 1 & 0 & 3 & -2 \\ 2 & 1 & 5 & 1 \end{bmatrix}$$

Let's get an rref.

$$\begin{bmatrix} 1 & 0 & 3 & -2 \\ 2 & 1 & 5 & 1 \end{bmatrix} \xrightarrow{-2R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 5 \end{bmatrix}$$

The pivot columns are columns 1 and 2.

A basis for Col (A) is $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

For the null space, we need solutions to $A\vec{x} = \vec{0}$. The rref for $[A \ \vec{0}]$ is

$$\begin{bmatrix} 1 & 0 & 3 & -2 & 0 \\ 0 & 1 & -1 & 5 & 0 \end{bmatrix}$$

$$x_1 = -3x_3 + 2x_4$$
$$x_2 = x_3 - 5x_4$$
$$x_3, x_4 - \text{free}$$

For $\vec{x}$ in $\text{Nul}(A)$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_3 + 2x_4 \\ x_3 - 5x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

A basis for $\text{Nul}(A)$ is $\left\{ \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$

Bases for $\text{Col}(A)$, $\text{Row}(A)$, and $\text{Nul}(A)$

Given a matrix A, find the rref. Then

- The pivot columns of the original matrix A give a basis for $\text{Col}(A)$.
- The nonzero rows of $\text{rref}(A)$ give a basis for $\text{Row}(A)$.
- Use the rref to solve $A\mathbf{x} = \mathbf{0}$ to identify a basis for $\text{Nul}(A)$.

Example

If A is an invertible $n \times n$ matrix, then we know[1] that (1) the columns are linearly independent, and (2) the columns span $\mathbb{R}^n$. Use this to determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$ where

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}.$$

If we set $A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$, then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$ if and only if A is nonsingular.

We can use the determinant.

[1] from our large theorem on invertible matrices from section 2.3

$$\det(A) = \det\begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{bmatrix}$$

$$= 3\begin{vmatrix} 1 & 1 \\ 7 & 5 \end{vmatrix} - 6\begin{vmatrix} -4 & -2 \\ 1 & 1 \end{vmatrix}$$

$$= 3(-2) - 6(-2) = -6 + 12 = 6$$

Since $\det(A) \neq 0$, A is invertible.

Hence $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$.

Standard Basis in $\mathbb{R}^n$

The columns of the $n \times n$ identity matrix provide an obvious basis for $\mathbb{R}^n$. This is called the **standard basis** for $\mathbb{R}^n$. For example, the standard bases in $\mathbb{R}^2$ and $\mathbb{R}^3$ are

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \quad \text{and} \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{respectively.}$$

Other Vector Spaces

The set $\{1, t, t^2, t^3\}$ is a basis[2] for $\mathbb{P}_3$.

Notice that for any vector $\mathbf{p}$ in $\mathbb{P}_3$,

$$\mathbf{p}(t) = p_0 1 + p_1 t + p_2 t^2 + p_3 t^3.$$

This is a linear combination of 1, t, t^2, and t^3. We already know that the zero polynomial

$$\mathbf{0}(t) = 0 1 + 0 t + 0 t^2 + 0 t^3.$$

That is, the equation

$$c_0 + c_1 t + c_2 t^2 + c_3 t^3 = 0 \quad \Leftrightarrow \quad c_0 = c_1 = c_2 = c_3 = 0$$

[2]The set $\{1, t, \ldots, t^n\}$ is called the **standard basis** for $\mathbb{P}_n$

Other Vector Spaces

The set $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for $M_{2\times 2}$.

The exercise is left to the reader. It must be shown that

- every matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be written as a linear combination of these vectors and
- this is a linearly independent set.