March 29 Math 2306 sec. 52 Spring 2023

Section 13: The Laplace Transform

A quick word about functions of 2-variables:

Suppose $G(s,t)$ is a function of two independent variables (s and t) defined over some rectangle in the plane $a \le t \le b$, $c \le s \le d$. If we compute an integral with respect to one of these variables, say t,

$$\int_\alpha^\beta G(s,t)\,dt$$

- the result is a function of the remaining variable s, and
- the variable s is treated as a constant while integrating with respect to t.

For Example...

Assume that $s \neq 0$ and $b > 0$. Compute the integral

$$\int_0^b e^{-st}\,dt = \frac{1}{-s} e^{-st} \Big|_0^b = \frac{1}{-s} e^{-sb} - \frac{1}{-s} e^{-s(0)}$$

$$= \frac{1}{s} - \frac{1}{s} e^{-sb}$$

Treat s like a constant

Integral Transform

An **integral transform** is a mapping that assigns to a function $f(t)$ another function $F(s)$ via an integral of the form

$$\int_a^b K(s,t)f(t)\,dt.$$

- The function K is called the **kernel** of the transformation.
- The limits a and b may be finite or infinite.
- The integral may be improper so that convergence/divergence must be considered.
- This transform is **linear** in the sense that

$$\int_a^b K(s,t)(\alpha f(t)+\beta g(t))\,dt = \alpha\int_a^b K(s,t)f(t)\,dt + \beta\int_a^b K(s,t)g(t)\,dt.$$

The Laplace Transform

Definition: Let $f(t)$ be defined on $[0, \infty)$. The Laplace transform of f is denoted and defined by

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s).$$

The domain of the transformation $F(s)$ is the set of all s such that the integral is convergent.

Note: The **kernel** for the Laplace transform is $K(s,t) = e^{-st}$.

Limits at Infinity e^{-st}

If $s > 0$, evaluate

$$\lim_{t\to\infty} e^{-st} = 0$$

$-st < 0$ $\quad$ $-st \to -\infty$

If $s < 0$, evaluate

$$\lim_{t\to\infty} e^{-st} = \infty$$

$-st > 0$ $\quad$ $-st \to +\infty$

Find[1] the Laplace transform of $f(t) = 1$.

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1 \, dt$$

If $s=0$, the integral is

$$\int_0^\infty 1 \, dt = \lim_{b\to\infty} \int_0^b dt = \lim_{b\to\infty} t \Big|_0^b = \lim_{b\to\infty} b = \infty$$

The integral diverges. Zero is not in the domain of $\mathcal{L}\{1\}$.

[1] Unless stated otherwise, the domain for each example is $[0, \infty)$.

For $s \neq 0$

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt$$

$$= \lim_{b \to \infty} \int_0^b e^{-st}\,dt$$

$$= \lim_{b \to \infty} \frac{1}{s} - \frac{1}{s} e^{-sb}$$

Convergence requires $s > 0$

For $s > 0$

$$\mathcal{L}\{1\} = \lim_{b \to \infty} \frac{1}{s} - \frac{1}{s} e^{-sb} = \frac{1}{s} - 0$$

$$\boxed{\mathscr{L}\{1\} = \frac{1}{s} \quad \text{for} \quad s > 0}$$

Find the Laplace transform of $f(t) = t$.

$$\mathscr{L}\{t\} = \int_0^\infty e^{-st}\, t\, dt$$

for $s=0$, we get

$$\int_0^\infty t\, dt = \frac{t^2}{2}\Big|_0^\infty = \infty \quad , \quad \text{divergent}$$

zero is not in the domain.

For $s \neq 0$

$$\mathscr{L}\{t\} = \int_0^\infty e^{-st}\, t\, dt$$

$u = t, \quad du = dt$

$dv = e^{-st}\, dt$

$$= \frac{-1}{s} e^{-st} t \Big|_0^\infty - \int_0^\infty \frac{-1}{s} e^{-st}\, dt \qquad v = \frac{-1}{s} e^{-st}$$

Convergence requires $s > 0$

$$\mathcal{L}\{t\} = \frac{1}{s} \underbrace{\int_0^\infty e^{-st}\, dt}_{\mathcal{L}\{1\}}$$

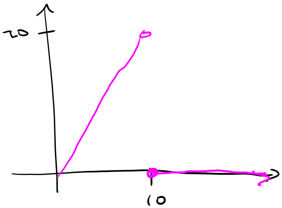
$$\mathcal{L}\{t\} = \frac{1}{s}\left(\frac{1}{s}\right) = \frac{1}{s^2}, \quad s > 0$$

$$\boxed{\mathcal{L}\{t\} = \frac{1}{s^2}, \quad s > 0}$$

A piecewise defined function

Find the Laplace transform of f defined by

$$f(t) = \begin{cases} 2t, & 0 \le t < 10 \\ 0, & t \ge 10 \end{cases}$$



$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^{10} e^{-st} f(t)\,dt + \int_{10}^\infty e^{-st} f(t)\,dt$$

$$= \int_0^{10} e^{-st}(2t)\,dt + \int_{10}^\infty e^{-st}(0)\,dt$$

For $s=0$ we get

$$\int_0^{10} 2t\,dt = t^2\Big|_0^{10} = 100$$

For $s\neq 0$, we get

$$2\int_0^{10} e^{-st}\,t\,dt =$$

$u = t$, $du = dt$

$dv = e^{-st}\,dt$

$v = \frac{-1}{s} e^{-st}$

$$2\left(\frac{-1}{s} e^{-st} t \Big|_0^{10} - \int_0^{10} \frac{-1}{s} e^{-st}\,dt\right)$$

$$= 2\left(\frac{-1}{s} e^{-10s}(10) - 0 + \frac{1}{s}\int_0^{10} e^{-st}\,dt\right)$$

$$= 2\left(\frac{-10}{s}e^{-10s} + \frac{1}{s}\left(\frac{-1}{s}\right)e^{-st}\Big|_0^{10}\right)$$

$$= 2\left(\frac{-10}{s}e^{-10s} - \frac{1}{s^2}\left(e^{-10s} - e^{0}\right)\right)$$

$$= \frac{-20}{s}e^{-10s} + \frac{2}{s^2} - \frac{2}{s^2}e^{-10s}$$

$$f(t) = \begin{cases} 2t, & 0 \leq t < 10 \\ 0, & t \geq 10 \end{cases}$$

$$\mathcal{L}\{f(t)\} = \begin{cases} \frac{2}{s^2} - \frac{2}{s^2}e^{-10s} - \frac{20}{s}e^{-10s}, & s \neq 0 \\ 100, & s = 0 \end{cases}$$

The Laplace Transform is a Linear Transformation

Some basic results include:

- $\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$
- $\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0$
- $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$ for $n = 1, 2, \ldots$
- $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$
- $\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s > 0$
- $\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \quad s > 0$

Evaluate the Laplace transform $\mathscr{L}\{f(t)\}$ if

$$\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s > 0$$

(a) $f(t) = \cos(\pi t)$

Here, $k = \pi$

$$\mathscr{L}\{\cos(\pi t)\} = \frac{s}{s^2+\pi^2}, \quad s > 0$$

Evaluate the Laplace transform $\mathscr{L}\{f(t)\}$ if

$\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0$

$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$ for n =

$\mathscr{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$

(b) $f(t) = 2t^4 - e^{-5t} + 3$

$$\mathscr{L}\{2t^4 - e^{-5t} + 3\} = 2\mathscr{L}\{t^4\} - \mathscr{L}\{e^{-5t}\} + 3\mathscr{L}\{1\}$$

$$= 2\frac{4!}{s^{4+1}} - \frac{1}{s-(-5)} + 3\frac{1}{s}$$

$$= \frac{48}{s^5} - \frac{1}{s+5} + \frac{3}{s}, \quad s > 0$$