March 2 Math 3260 sec. 51 Spring 2022

Section 3.1: Introduction to Determinants

Recall that a 2×2 matrix is invertible if and only if the number called its **determinant** is nonzero. We had

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

We wish to extend the concept of determinant to $n \times n$ matrices in general. And we wish to do so in such a way that invertibility holds if and only if the determinant is nonzero.

Determinant 3×3 case:

Suppose we start with a 3×3 invertible matrix. And suppose that $a_{11} \neq 0$. We can multiply the second and third rows by a_{11} and begin row reduction.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{bmatrix} \sim$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \end{bmatrix}$$

Determinant 3×3 case continued...

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \end{bmatrix}$$

If $A \sim I$, one of the entries in the $2,2$ or the $3,2$ position must be nonzero. Let's assume it is the $2,2$ entry. Continue row reduction to get

$$A \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & a_{11}\Delta \end{bmatrix}.$$

Again, if A is invertible, it must be that the bottom right entry is nonzero. That is

$$\Delta \neq 0.$$

Note that if $\Delta = 0$, the rref of A is not I—A would be singular.

Determinant 3×3 case continued...

With a little rearrangement, we have

$$\begin{aligned}\Delta &= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + \\ &+ \quad a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11}\text{det}\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\text{det}\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\text{det}\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\end{aligned}$$

The number Δ will be called the **determinant** of A.

Definitions: Minors

Let $n \geq 2$. For an $n \times n$ matrix A, let A_{ij} denote the $(n-1) \times (n-1)$ matrix obtained from A by deleting the i^{th} row and the j^{th} column of A.

For example, if

$$A = \begin{bmatrix} -1 & 3 & 2 & 0 \\ 4 & 4 & 0 & -3 \\ -2 & 1 & 7 & 2 \\ 3 & 0 & -1 & 6 \end{bmatrix} \quad \text{then} \quad A_{23} = \begin{bmatrix} -1 & 3 & 0 \\ -2 & 1 & 2 \\ 3 & 0 & 6 \end{bmatrix}.$$

A_{ij} is a matrix

Definition: The i, j^{th} **minor** of the $n \times n$ matrix A is the number

$$M_{ij} = \det(A_{ij}).$$

M_{ij} is a number

Definitions: Cofactor

Definition: Let A be an $n \times n$ matrix with $n \geq 2$. The i, j^{th} **cofactor** of A is the number

$$C_{ij} = (-1)^{i+j} M_{ij}.$$

Example: Find the three minors M_{11}, M_{12}, M_{13} and find the 3 cofactors C_{11}, C_{12}, C_{13} of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

$M_{11} = \det(A_{11})$, $M_{13} = \det(A_{13})$

$M_{12} = \det(A_{12})$,

$A_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$, $A_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$, $A_{13} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$

(Example Continued...)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

$M_{11} = \det(A_{11}) = a_{22}a_{33} - a_{32}a_{23}$

$M_{12} = \det(A_{12}) = a_{21}a_{33} - a_{31}a_{23}$

$M_{13} = \det(A_{13}) = a_{21}a_{32} - a_{31}a_{22}$

$C_{11} = (-1)^{1+1}M_{11} = (-1)^{2}M_{11} = M_{11} = a_{22}a_{33} - a_{32}a_{23}$

$C_{12} = (-1)^{1+2}M_{12} = (-1)^{3}M_{12} = -(a_{21}a_{33} - a_{31}a_{23})$

$C_{13} = (-1)^{1+3}M_{13} = (-1)^{4}M_{13} = a_{21}a_{32} - a_{31}a_{22}$

Observation:

Comparison with the determinant of the 3×3 matrix, we can note that

$$\Delta = a_{11}\text{det}\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\text{det}\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\text{det}\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

Definition: Determinant

For $n \geq 2$, the **determinant** of the $n \times n$ matrix $A = [a_{ij}]$ is the number

$$\begin{aligned}\det(A) &= a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} M_{1j}\end{aligned}$$

(Well call such a sum a **cofactor expansion.**)

Example: Evaluate the determinant

$$A = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{bmatrix} -1 & 3 & 0 \\ -2 & 1 & 2 \\ 3 & 0 & 6 \end{bmatrix} \end{array}$$

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\det(A) = (-1)\det\begin{bmatrix} 1 & 2 \\ 0 & 6 \end{bmatrix} - 3\det\begin{bmatrix} -2 & 2 \\ 3 & 6 \end{bmatrix} + 0\det\begin{bmatrix} -2 & 1 \\ 3 & 0 \end{bmatrix}$$

$$= -1(1\cdot 6 - 0\cdot 2) - 3(-2\cdot 6 - 3\cdot 2) + 0(-2\cdot 0 - 3\cdot 1)$$

$$= -(6) - 3(-18) + 0(-3) = -6 + 54 = 48$$

Theorem:

The determinant of an $n \times n$ matrix can be computed by cofactor expansion across any row or down any column.

We can fix any row i of a matrix A and then

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

Or, we can fix any column j of a matrix A and then

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

Example:

Find the determinant of the matrix

$$A = \begin{bmatrix} -1 & 3 & 4 & 0 \\ 0 & 0 & -3 & 0 \\ -2 & 1 & 2 & 2 \\ 3 & 0 & -1 & 6 \end{bmatrix}$$

Using row 2

$$\det(A) = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} + a_{24}C_{24}$$

$$= 0 \cdot C_{21} + 0 \cdot C_{22} + (-3)C_{23} + 0 \cdot C_{24}$$

What is C_{23}?

$$C_{23} = (-1)^{2+3} M_{23} = (-1)^{2+3} \det(A_{23})$$

$$A_{23} = \begin{bmatrix} -1 & 3 & 0 \\ -2 & 1 & 2 \\ 3 & 0 & 6 \end{bmatrix}.$$

We just found $\det(A_{23}) = 48$ in the last example

$$\det(A) = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} + a_{24}C_{24}$$

$$= 0 \cdot C_{21} + 0 \cdot C_{22} + (-3)\,C_{23} + 0 \cdot C_{24}$$

$$= -3(-1)^{2+3}(48) = 144$$