

## Section 3.1: Introduction to Determinants

Recall that a  $2 \times 2$  matrix is invertible if and only if the number called its **determinant** is nonzero. We had

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

We wish to extend the concept of determinant to  $n \times n$  matrices in general. And we wish to do so in such a way that invertibility holds if and only if the determinant is nonzero.

## Determinant $3 \times 3$ case:

Suppose we start with a  $3 \times 3$  invertible matrix. And suppose that  $a_{11} \neq 0$ . We can multiply the second and third rows by  $a_{11}$  and begin row reduction.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{bmatrix} \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \end{bmatrix}$$

## Determinant $3 \times 3$ case continued...

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \end{bmatrix}$$

If  $A \sim I$ , one of the entries in the 2, 2 or the 3, 2 position must be nonzero. Let's assume it is the 2, 2 entry. Continue row reduction to get

$$A \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & a_{11}\Delta \end{bmatrix}.$$

Again, if  $A$  is invertible, it must be that the bottom right entry is nonzero. That is

$$\Delta \neq 0.$$

Note that if  $\Delta = 0$ , the rref of  $A$  is not  $I$ — $A$  would be singular.

## Determinant $3 \times 3$ case continued...

With a little rearrangement, we have

$$\begin{aligned}\Delta &= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + \\ &\quad + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11}\det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\end{aligned}$$

The number  $\Delta$  will be called the **determinant** of  $A$ .

## Definitions: Minors

Let  $n \geq 2$ . For an  $n \times n$  matrix  $A$ , let  $A_{ij}$  denote the  $(n - 1) \times (n - 1)$  matrix obtained from  $A$  by deleting the  $i^{\text{th}}$  row and the  $j^{\text{th}}$  column of  $A$ .

For example, if

$$A = \begin{bmatrix} -1 & 3 & 2 & 0 \\ 4 & 4 & 0 & -3 \\ -2 & 1 & 7 & 2 \\ 3 & 0 & -1 & 6 \end{bmatrix} \quad \text{then} \quad A_{23} = \begin{bmatrix} -1 & 3 & 0 \\ -2 & 1 & 2 \\ 3 & 0 & 6 \end{bmatrix}.$$

**Definition:** The  $i, j^{\text{th}}$  **minor** of the  $n \times n$  matrix  $A$  is the number

$$M_{ij} = \det(A_{ij}).$$

*A<sub>ij</sub> is a matrix*

*M<sub>ij</sub> is a number*

## Definitions: Cofactor

**Definition:** Let  $A$  be an  $n \times n$  matrix with  $n \geq 2$ . The  $i, j^{\text{th}}$  **cofactor** of  $A$  is the number

$$C_{ij} = (-1)^{i+j} M_{ij}.$$

**Example:** Find the three minors  $M_{11}$ ,  $M_{12}$ ,  $M_{13}$  and find the 3 cofactors  $C_{11}$ ,  $C_{12}$ ,  $C_{13}$  of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

need  $A_{11}$ ,  $A_{12}$ ,  $A_{13}$

$$A_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

$$A_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}, \quad A_{13} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

## (Example Continued...)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$M_{11} = \det(A_{11}) = a_{22}a_{33} - a_{32}a_{23}$$

$$M_{12} = \det(A_{12}) = a_{21}a_{33} - a_{31}a_{23}$$

$$M_{13} = \det(A_{13}) = a_{21}a_{32} - a_{31}a_{22}$$

$$C_{11} = (-1)^{1+1} M_{11} = (-1)^2 M_{11} = M_{11} = a_{22}a_{33} - a_{32}a_{23}$$

$$C_{12} = (-1)^{1+2} M_{12} = (-1)^3 M_{12} = -M_{12} = -(a_{21}a_{33} - a_{31}a_{23})$$

$$C_{13} = (-1)^{1+3} M_{13} = (-1)^4 M_{13} = M_{13} = a_{21}a_{32} - a_{31}a_{22}$$

## Observation:

Comparison with the determinant of the  $3 \times 3$  matrix, we can note that

$$\begin{aligned}\Delta &= a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \\ &= a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}\end{aligned}$$



## Definition: Determinant

For  $n \geq 2$ , the **determinant** of the  $n \times n$  matrix  $A = [a_{ij}]$  is the number

$$\begin{aligned}\det(A) &= a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n} \\ &= \sum_{j=1}^n (-1)^{1+j} a_{1j} M_{1j}\end{aligned}$$

(We'll call such a sum a **cofactor expansion**.)

## Example: Evaluate the determinant

$$A = \begin{bmatrix} -1 & 3 & 0 \\ -2 & 1 & 2 \\ 3 & 0 & 6 \end{bmatrix}$$

$$\det(A) = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

$$\det(A) = (-1) \det \begin{bmatrix} 1 & 2 \\ 0 & 6 \end{bmatrix} - (3) \det \begin{bmatrix} -2 & 2 \\ 3 & 6 \end{bmatrix} + (0) \det \begin{bmatrix} -2 & 1 \\ 3 & 0 \end{bmatrix}$$

$$= -1(1 \cdot 6 - 0 \cdot 2) - 3(-2 \cdot 6 - 3 \cdot 2) + 0(-2 \cdot 0 - 3 \cdot 1)$$

$$= -6 - 3(-18) + 0(-3)$$

$$= -6 + 54$$

$$= 48$$

## Theorem:

The determinant of an  $n \times n$  matrix can be computed by cofactor expansion across any row or down any column.

We can fix any row  $i$  of a matrix  $A$  and then

$$\det(A) = \sum_{j=1}^n (-1)^{i+j} a_{ij} M_{ij}$$

Or, we can fix any column  $j$  of a matrix  $A$  and then

$$\det(A) = \sum_{i=1}^n (-1)^{i+j} a_{ij} M_{ij}$$

## Example:

Find the determinant of the matrix

$$A = \begin{bmatrix} -1 & 3 & 4 & 0 \\ 0 & 0 & -3 & 0 \\ -2 & 1 & 2 & 2 \\ 3 & 0 & -1 & 6 \end{bmatrix} \quad \text{Using row 2}$$
$$\det(A) = \underbrace{a_{21}}_0 C_{21} + \underbrace{a_{22}}_0 C_{22} + a_{23} C_{23} + \underbrace{a_{24}}_0 C_{24}$$

$$\det(A) = 0 C_{21} + 0 C_{22} + (-3) C_{23} + 0 \cdot C_{24}$$

$$C_{23} = (-1)^{2+3} M_{23} = -\det(A_{23}) = -\det \begin{bmatrix} -1 & 3 & 0 \\ -2 & 1 & 2 \\ 3 & 0 & 6 \end{bmatrix}$$

From the last example, we have

$$C_{23} = -(48) = -48$$

$$\det(A) = -3(-48) = 144$$