March 4 Math 3260 sec. 51 Spring 2024

Section 3.2: Properties of Determinants

Theorem:

Let A be an $n \times n$ matrix, and suppose the matrix B is obtained from A by performing a single elementary row operation. Then

(i) If B is obtained by adding a multiple of a row of A to another row of A (row replacement), then

$$\det(B) = \det(A).$$

(ii) If B is obtained from A by swapping any pair of rows (row swap) , then

$$\det(B) = -\det(A).$$

(iii) If B is obtained from A by scaling any row by the constant k (scaling), then

$$\det(B) = k\det(A).$$

Results on Determinants

Theorem

The $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$.

Theorem

For $n \times n$ matrix A, $\det(A^T) = \det(A)$.

Theorem

For $n \times n$ matrices A and B, $\det(AB) = \det(A)\det(B)$.

Remark: We showed last time that this last theorem implies $\det(A^{-1}) = (\det(A))^{-1}$ for any invertible matrix A.

Example

Let A be an $n \times n$ matrix, and suppose there exists invertible matrix P such that[1]

$$B = P^{-1}AP.$$

Show that

$$\det(B) = \det(A).$$

$$\det(B) = \det(P^{-1}AP)$$

$$= \det(P^{-1})\det(A)\det(P)$$

$$= \det(A)\det(P^{-1})\det(P)$$

$$= \det(A)\left(\det(P^{-1})\right)^{-1}\det(P)$$

mult of real numbers commutes

[1]The process of multiplying by P^{-1} on the left and P on the right is called a *similarly transform*. The matrices A and B are said to be *similar*.

$$= \det(A)(1)$$

$$= \det(A)$$

Section 3.3: Cramer's Rule, Volume, and Linear Transformations

Cramer's Rule is a method for solving some small linear systems of equations.

Notation:

For $n \times n$ matrix A and $\mathbf{b}$ in $\mathbb{R}^n$, let $A_i(\mathbf{b})$ be the matrix obtained from A by replacing the i^{th} column with the vector $\mathbf{b}$. That is

$$A_i(\mathbf{b}) = [\mathbf{a}_1 \cdots \mathbf{a}_{i-1}\ \mathbf{b}\ \mathbf{a}_{i+1} \cdots \mathbf{a}_n]$$

Example Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, then

$$A_3(\mathbf{b}) = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}$$

Cramer's Rule

Theorem:

Let A be an $n \times n$ nonsingular matrix. Then for any vector $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution of the system $A\mathbf{x} = \mathbf{b}$ is given by $\mathbf{x}$ where

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \quad i = 1, \ldots, n$$

Remark: The condition $\det(A) \neq 0$ is necessary for Cramer's rule to be a viable method. This allows for the solution to be given in terms of ratios of determinants.

Remark: If $\det(A) = 0$, the system may be consistent, but another method is required to make a determination.

Example

Determine whether Cramer's rule can be used to solve the system. If so, use it to solve the system.

$$\begin{aligned} 2x_1 &+ x_2 &= 9 \\ -x_1 &+ 7x_2 &= -3 \end{aligned}$$

State in matrix format

$A\vec{x} = \vec{b}$

$$\underbrace{\begin{bmatrix} 2 & 1 \\ -1 & 7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 9 \\ -3 \end{bmatrix}}_{\vec{b}}$$

$\det(A) = 2(7) - (1)(-1) = 15$ $\qquad \det(A) \neq 0$

$A_1(\vec{b}) = \begin{bmatrix} 9 & 1 \\ -3 & 7 \end{bmatrix}, \quad A_2(\vec{b}) = \begin{bmatrix} 2 & 9 \\ -1 & -3 \end{bmatrix}$

$$\det(A_1(\vec{b})) = 9(7) - (1)(-3) = 66$$

$$\det(A_2(\vec{b})) = 2(-3) - (-1)9 = 3$$

$$x_1 = \frac{\det(A_1(\vec{b}))}{\det(A)} = \frac{66}{15}$$

$$x_2 = \frac{\det(A_2(\vec{b}))}{\det(A)} = \frac{3}{15}$$

$$x_1 = \frac{22}{5}, \quad x_2 = \frac{1}{5}$$

Example

Determine whether Cramer's rule can be used to solve the system. If so, use it to solve the system.

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & 3x_3 & = & 3 \\ & & x_2 & + & 4x_3 & = & 3 \\ 5x_1 & + & 6x_2 & & & = & 4 \end{array}$$

State in the form $A\vec{x} = \vec{b}$.

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}}_{\vec{b}}$$

first column

$$\det(A) = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$$

$$= 1(-1)^{1+1}\begin{vmatrix} 1 & 4 \\ 6 & 0 \end{vmatrix} + 0 \left| \quad \right| + 5(-1)^{1+3}\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix}$$

$$= (0-24) + 5(8-3) = -24+25 = 1$$

$$A_1(\vec{b}) = \begin{bmatrix} 3 & 2 & 3 \\ 3 & 1 & 4 \\ 4 & 6 & 0 \end{bmatrix} \quad \det(A_1(\vec{b})) = 4\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} - 6\begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix}$$

$$= 4(8-3) - 6(12-9) = 2$$

$$A_2(\vec{b}) = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 3 & 4 \\ 5 & 4 & 0 \end{bmatrix} \quad \det(A_2(\vec{b})) = 1\begin{vmatrix} 3 & 4 \\ 4 & 0 \end{vmatrix} + 5\begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix}$$

$$= 1(-16) + 5(3) = -1$$

$$A_3(\vec{b}) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 5 & 6 & 4 \end{bmatrix} \quad \det(A_3(\vec{b})) = 1\begin{vmatrix} 1 & 3 \\ 6 & 4 \end{vmatrix} + 5\begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix}$$

$$= (4-18) + 5(6-3) = 1$$

$$x_1 = \frac{\det(A_1(\vec{b}))}{\det(A)} = \frac{2}{1}$$

$$x_2 = \frac{\det(A_2(\vec{b}))}{\det(A)} = \frac{-1}{1}$$

$$x_3 = \frac{\det(A_3(\vec{b}))}{\det(A)} = \frac{1}{1}$$

$$(x_1, x_2, x_3) = (2, -1, 1)$$

Application: Laplace Transforms

In various engineering applications, electrical or mechanical components are often chosen to try to control the long term behavior of a system (e.g. adding a damper to kill off oscillatory behavior). Using *Laplace Transforms*, differential equations are converted into algebraic equations containing a parameter s. These give rise to systems of the form

$$\begin{array}{rcrcl} 3sX & - & 2Y & = & 4 \\ -6X & + & sY & = & 1 \end{array}$$

Determine the values of s for which the system is uniquely solvable. For such s, find the solution (X, Y) using Cramer's rule.

$$\begin{aligned} 3sX &- 2Y = 4 \\ -6X &+ sY = 1 \end{aligned}$$

write as $A\vec{x} = \vec{b}$

$$\underbrace{\begin{bmatrix} 3s & -2 \\ -6 & s \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} X \\ Y \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 4 \\ 1 \end{bmatrix}}_{\vec{b}}$$

$\det(A) = 3s(s) - (-6)(-2) = 3s^2 - 12 = 3(s^2 - 4)$

Note, $\det(A) = 0 \Rightarrow 3(s^2 - 4) = 0$

$3(s-2)(s+2) = 0 \Rightarrow s = 2$ or $s = -2$.

The system is uniquely solvable for all

$s \neq \pm 2$.

$$\begin{bmatrix} 3s & -2 \\ -6 & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

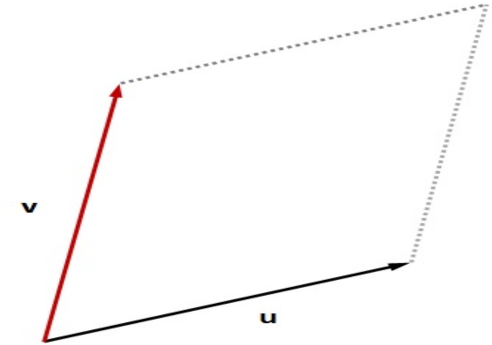
$$A_1(\vec{b}) = \begin{bmatrix} 4 & -2 \\ 1 & s \end{bmatrix} \qquad \det(A_1(\vec{b})) = 4s+2$$

$$A_2(\vec{b}) = \begin{bmatrix} 3s & 4 \\ -6 & 1 \end{bmatrix} \qquad \det(A_2(\vec{b})) = 3s+24$$

For $s \neq \pm 2$,

$$x = \frac{4s+2}{3(s^2-4)} \quad \text{and} \quad y = \frac{3s+24}{3(s^2-4)}$$

Area & Volume (Video)

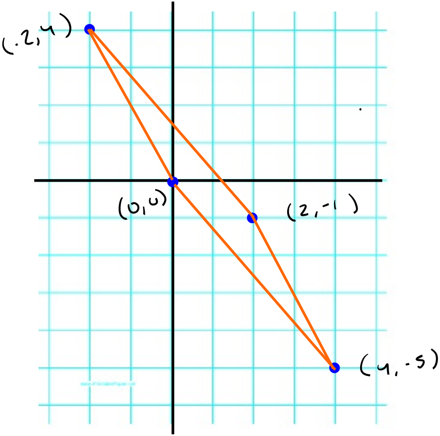


Theorem:

If $\mathbf{u}$ and $\mathbf{v}$ are nonzero, nonparallel vectors in $\mathbb{R}^2$, then the area of the parallelogram determined by these vectors is $|\det(A)|$ where $A = [\mathbf{u}\ \mathbf{v}]$.

Example

Find the area of the parallelogram with vertices $(0,0)$, $(-2,4)$, $(4,-5)$, and $(2,-1)$.



Let $\vec{u} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$

and $\vec{v} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$

If $A = [\vec{u} \;\; \vec{v}]$,

then Area $= |\det(A)|$

$$A = \begin{bmatrix} -2 & 4 \\ 4 & -5 \end{bmatrix}$$

$$\det(A) = (-2)(-5) - 4(4) =$$

$$= 10 - 16 \quad = -6$$

The area is 6.