

March 6 Math 3260 sec. 52 Spring 2024

Section 3.3: Cramer's Rule, Volume, and Linear Transformations

Notation:

For $n \times n$ matrix A and \mathbf{b} in \mathbb{R}^n , let $A_i(\mathbf{b})$ be the matrix obtained from A by replacing the i^{th} column with the vector \mathbf{b} . That is

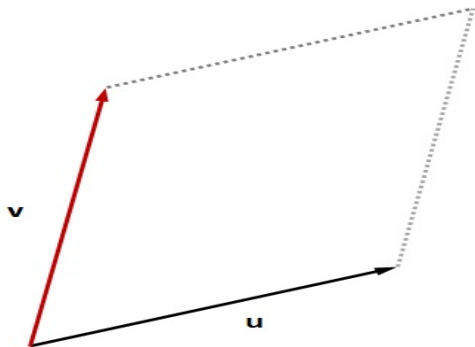
$$A_i(\mathbf{b}) = [\mathbf{a}_1 \cdots \mathbf{a}_{i-1} \ \mathbf{b} \ \mathbf{a}_{i+1} \cdots \mathbf{a}_n]$$

Theorem:

Let A be an $n \times n$ nonsingular matrix. Then for any vector \mathbf{b} in \mathbb{R}^n , the unique solution of the system $A\mathbf{x} = \mathbf{b}$ is given by \mathbf{x} where

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \quad i = 1, \dots, n$$

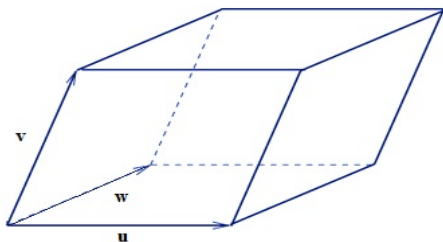
Area & Volume [▶ \(Video\)](#)



Theorem:

If \mathbf{u} and \mathbf{v} are nonzero, nonparallel vectors in \mathbb{R}^2 , then the area of the parallelogram determined by these vectors is $|\det(A)|$ where $A = [\mathbf{u} \ \mathbf{v}]$.

Volume of a Parallelepiped



Theorem:

If \mathbf{u} , \mathbf{v} , and \mathbf{w} are nonzero, non-collinear vectors in \mathbb{R}^3 , then the volume of the parallelepiped determined by these vectors is $|\det(A)|$ where $A = [\mathbf{u} \ \mathbf{v} \ \mathbf{w}]$.

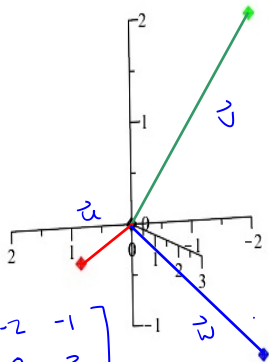
Example

Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(2, 3, 0)$, $(-2, 0, 2)$ and $(-1, 3, -1)$.

Let

$$\vec{u} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} \quad \text{and}$$

$$\vec{w} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$$



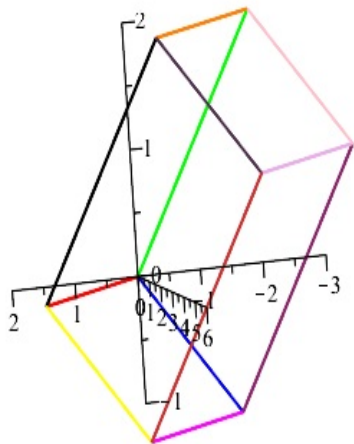
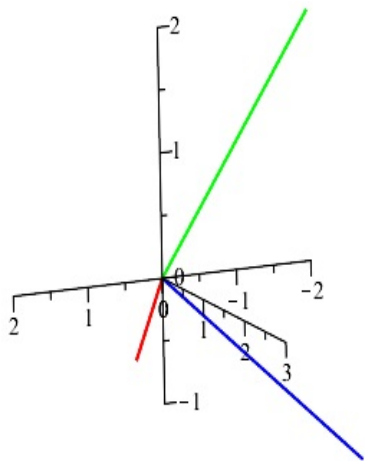
$$\text{Let } A = [\vec{u} \ \vec{v} \ \vec{w}] = \begin{bmatrix} 2 & -2 & -1 \\ 3 & 0 & 3 \\ 0 & 2 & -1 \end{bmatrix}$$

Doing a cofactor expansion down
Column 2.

$$\det(A) = -2(-1)^{1+2} \begin{vmatrix} 3 & 3 \\ 0 & -1 \end{vmatrix} + 0 \star + 2(-1)^{3+2} \begin{vmatrix} 2 & -1 \\ 3 & 3 \end{vmatrix}$$

$$= 2(-3) - 2(6+3) = -6 - 18 = -24$$

$$\text{Volume} = |-24| = 24.$$



Section 4.1: Vector Spaces and Subspaces

Recall that we had defined \mathbb{R}^n as the set of all n -tuples of real numbers. We defined two operations, vector addition and scalar multiplication, and said that the following algebraic properties hold:

For every \mathbf{u} , \mathbf{v} , and \mathbf{w} in \mathbb{R}^n and scalars c and d

- (i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- (ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- (iii) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
- (iv) $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$
- (v) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
- (vi) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
- (vii) $c(d\mathbf{u}) = d(c\mathbf{u}) = (cd)\mathbf{u}$
- (viii) $1\mathbf{u} = \mathbf{u}$

We later saw that a set of $m \times n$ matrices with scalar multiplication and matrix addition satisfies the same set of properties.

Question: Are there other sets of objects with operations that share this same structure?

Definition: Vector Space

A **vector space** is a nonempty set V of objects called *vectors* together with two operations called *vector addition* and *scalar multiplication* that satisfy the following ten axioms:

For all \mathbf{u} , \mathbf{v} , and \mathbf{w} in V , and for any scalars c and d

1. The sum $\mathbf{u} + \mathbf{v}$ is in V .
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
4. There exists a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each vector \mathbf{u} there exists a vector $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. For each scalar c , $c\mathbf{u}$ is in V .
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $c(d\mathbf{u}) = d(c\mathbf{u}) = (cd)\mathbf{u}$.
10. $1\mathbf{u} = \mathbf{u}$

Remarks:

- ▶ V is more accurately called a *real vector space* when we assume that the relevant scalars are the real numbers.
- ▶ Property 1., $\mathbf{u} + \mathbf{v} \in V$, is called being **closed** under (or *with respect to*) vector addition.
- ▶ Property 6., $c\mathbf{u} \in V$, is called being **closed** under (or *with respect to*) scalar multiplication.
- ▶ A vector space has the same basic *algebraic structure* as \mathbb{R}^n
- ▶ These are **axioms**. That means they are assumed, not proven. However, we can use them to prove or disprove that some set with operations is actually a vector space.

An Example of a Vector Space: “P two”

“P two”

$$\mathbb{P}_2 = \left\{ \mathbf{p}(t) = p_0 + p_1 t + p_2 t^2 \mid p_0, p_1, p_2 \in \mathbb{R} \right\}.$$

Consider t to be some real variable, and consider the scalars to be \mathbb{R} . Let \mathbb{P}_2 be the set of all polynomials with real coefficients of degree at most two.

Examples of elements of \mathbb{P}_2 include things like

$$\mathbf{p}(t) = 1 + t - 3t^2, \quad \mathbf{q}(t) = -2 + 5t + 12t^2, \quad \text{and} \quad \mathbf{r}(t) = \pi + \frac{1}{\pi}t.$$

Remark: It doesn't make sense to state that \mathbb{P}_2 is a vector space until we define **scalar multiplication** and **vector addition**.

An Example of a Vector Space: “P two”

Let $\mathbf{p}(t) = p_0 + p_1t + p_2t^2$ and $\mathbf{q}(t) = q_0 + q_1t + q_2t^2$ be polynomials in \mathbb{P}_2 and c be a scalar. We define the two operations as follows:

$$\text{Scalar Multiplication: } (c\mathbf{p})(t) = c\mathbf{p}(t) = cp_0 + cp_1t + cp_2t^2.$$

$$\begin{aligned} \text{Vector Addition: } (\mathbf{p} + \mathbf{q})(t) &= \mathbf{p}(t) + \mathbf{q}(t) \\ &= (p_0 + q_0) + (p_1 + q_1)t + (p_2 + q_2)t^2. \end{aligned}$$

Remark: It can be shown that \mathbb{P}_2 with these operations satisfies the ten vector space axiom. Note, this means that

the polynomials ARE vectors.