Ordinary Differential Equations

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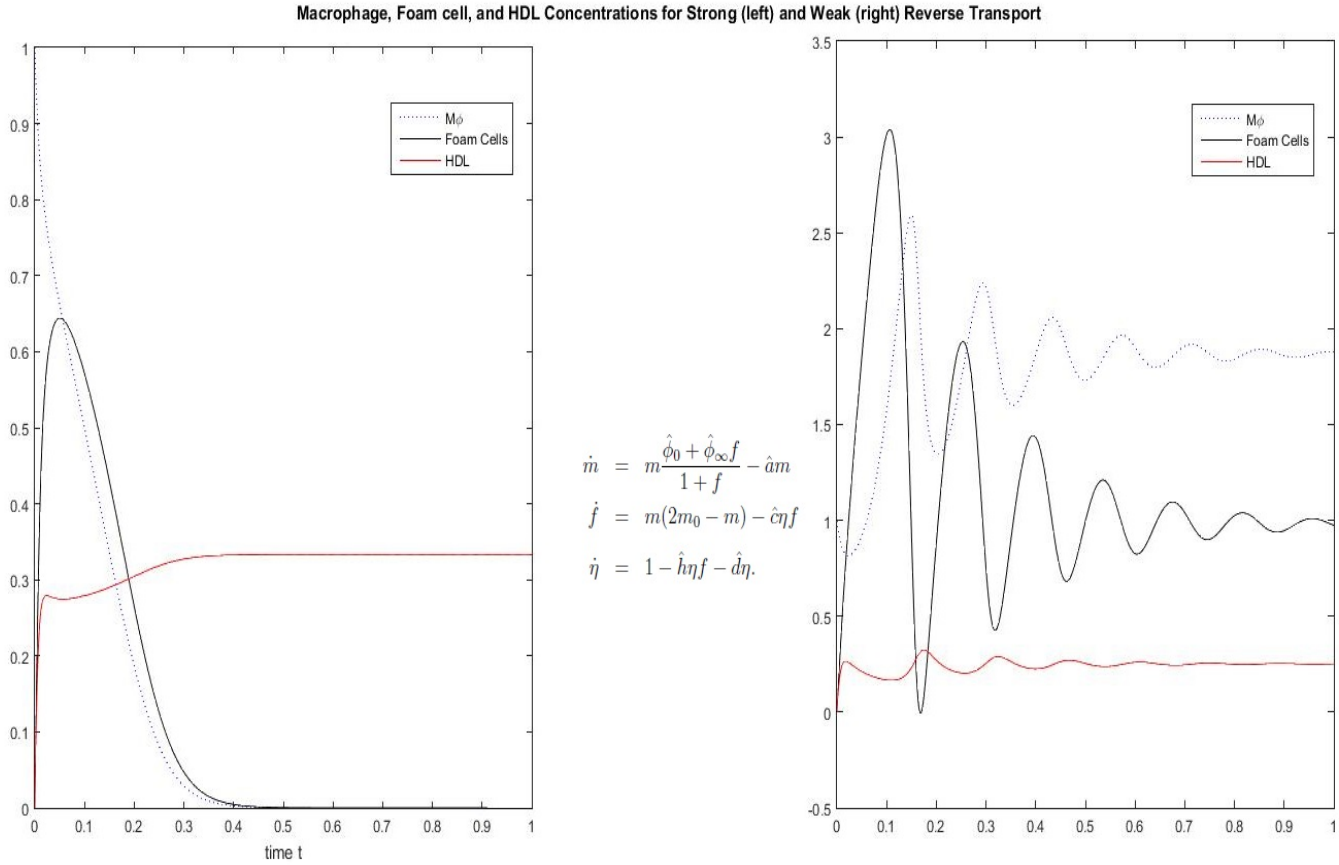

$$\begin{aligned}\dot{m} &= m\frac{\hat{\phi}_0 + \hat{\phi}_\infty f}{1+f} - \hat{a}m \\ \dot{f} &= m(2m_0 - m) - \hat{c}\eta f \\ \dot{\eta} &= 1 - \hat{h}\eta f - \hat{d}\eta.\end{aligned}$$

2023

MATH 2306: Ordinary Differential Equations

Lake Ritter, Kennesaw State University

This manuscript is a *text-like* version of the lecture slides I have created for use in my ODE classes at KSU. It is not intended to serve as a complete text or reference book, but is provided as a supplement to my students. I strongly advise against using this text as a substitute for regular class attendance. In particular, most of the computational details have been omitted. However, my own students are encouraged to read (and read ahead) as part of preparing for class.

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Section 1: Concepts and Terminology

Suppose $y = \phi(x)$ is a differentiable function. We know that $dy/dx = \phi'(x)$ is another (related) function.

For example, if $y = \cos(2x)$, then y is differentiable on $(-\infty, \infty)$. In fact,

$$\frac{dy}{dx} = -2\sin(2x).$$

Even dy/dx is differentiable with $d^2y/dx^2 = -4\cos(2x)$. Note that

$$\frac{d^2y}{dx^2} + 4y = 0.$$

The equation

$$\frac{d^2y}{dx^2} + 4y = 0.$$

is an example of a **differential equation**.

Questions: If we only started with the equation, how could we determine that $\cos(2x)$ satisfies it? Also, is $\cos(2x)$ the only possible function that y could be?

We will be able to answer these questions as we proceed.

Definition

A **Differential Equation** is an equation containing the derivative(s) of one or more dependent variables, with respect to one or more indendent variables.

Independent Variable: will appear as one that derivatives are taken with respect to.
Dependent Variable: will appear as one that derivatives are taken of.

$$\frac{dy}{dx} \qquad \frac{du}{dt} \qquad \frac{dx}{dr}$$

Identify the dependent and independent variables in these terms.

Classifications

Type: An **ordinary differential equation (ODE)** has exactly one independent variable[1]. For example

$$\frac{dy}{dx} - y^2 = 3x, \quad \text{or} \quad \frac{dy}{dt} + 2\frac{dx}{dt} = t, \quad \text{or} \quad y'' + 4y = 0$$

A **partial differential equation (PDE)** has two or more independent variables. For example

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}, \quad \text{or} \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

For those unfamiliar with the notation, ∂ is the symbol used when taking a derivative with respect to one variable keeping the remaining variable(s) constant. $\frac{\partial u}{\partial t}$ is read as the "partial derivative of u with respect to t."

[1] These are the subject of this course.

Classifications

Order: The order of a differential equation is the same as the highest order derivative appearing anywhere in the equation.

$$\frac{dy}{dx} - y^2 = 3x$$

$$y''' + (y')^4 = x^3$$

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$$

These are first, third, and second order equations, respectively.

Notations and Symbols

We'll use standard derivative notations:

$$\text{Leibniz:} \quad \frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \ldots \quad \frac{d^ny}{dx^n}, \quad \text{or}$$

$$\text{Prime \& superscripts:} \quad y', \quad y'', \quad \ldots \quad y^{(n)}.$$

Newton's **dot notation** may be used if the independent variable is time. For example if s is a position function, then

$$\text{velocity is } \frac{ds}{dt} = \dot{s}, \quad \text{and acceleration is } \frac{d^2s}{dt^2} = \ddot{s}$$

Notations and Symbols

An nth order ODE, with independent variable x and dependent variable y can always be expressed as an equation

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

where F is some real valued function of $n + 2$ variables.

Normal Form: If it is possible to isolate the highest derivative term, then we can write a **normal form** of the equation

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}).$$

Example

The equation $y'' + 4y = 0$ has the form $F(x, y, y', y'') = 0$ where

$$F(x, y, y', y'') = y'' + 4y.$$

This equation is second order. In normal form it is $y'' = f(x, y, y')$ where

$$f(x, y, y') = -4y.$$

Notations and Symbols

If $n = 1$ or $n = 2$, an equation in normal form would look like

$$\frac{dy}{dx} = f(x, y) \quad \text{or} \quad \frac{d^2y}{dx^2} = f(x, y, y').$$

Differential Form: A first order equation may appear in the form

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

Note that this can be rearranged into a couple of different normal forms

$$\frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)} \quad \text{or} \quad \frac{dx}{dy} = -\frac{N(x, y)}{M(x, y)}$$

Classifications

Linearity: An nth order differential equation is said to be **linear** if it can be written in the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

Note that each of the coefficients $a_0, \ldots, a_n$ and the right hand side g may depend on the independent variable but not on the dependent variable or any of its derivatives.

Examples (Linear -vs- Nonlinear)

$$y'' + 4y = 0$$

$$t^2 \frac{d^2x}{dt^2} + 2t\frac{dx}{dt} - x = e^t$$

Convince yourself that the top two equations are linear.

$$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 = x^3$$

$$u'' + u' = \cos u$$

The terms $(dy/dx)^4$ and $\cos u$ make these nonlinear.

Solution of $F(x, y, y', \ldots, y^{(n)}) = 0$ (*)

Definition: A function ϕ defined on an interval I[2] and possessing at least n continuous derivatives on I is a **solution** of (*) on I if upon substitution (i.e. setting $y = \phi(x)$) the equation reduces to an identity.

Definition: An **implicit solution** of (*) is a relation $G(x, y) = 0$ provided there exists at least one function $y = \phi$ that satisfies both the differential equation (*) and this relation.

[2] The interval is called the *domain of the solution* or the *interval of definition*.

Function -vs- Solution

The interval of defintion has to be an interval.

Consider $y' = -y^2$. Clearly $y = \frac{1}{x}$ solves the DE. The interval of defintion can be $(-\infty, 0)$, or $(0, \infty)$—or any interval that doesn't contain the origin. But it can't be $(-\infty, 0) \cup (0, \infty)$ because this isn't an interval!

Often, we'll take I to be the largest, or one of the largest, possible interval. It may depend on other information.

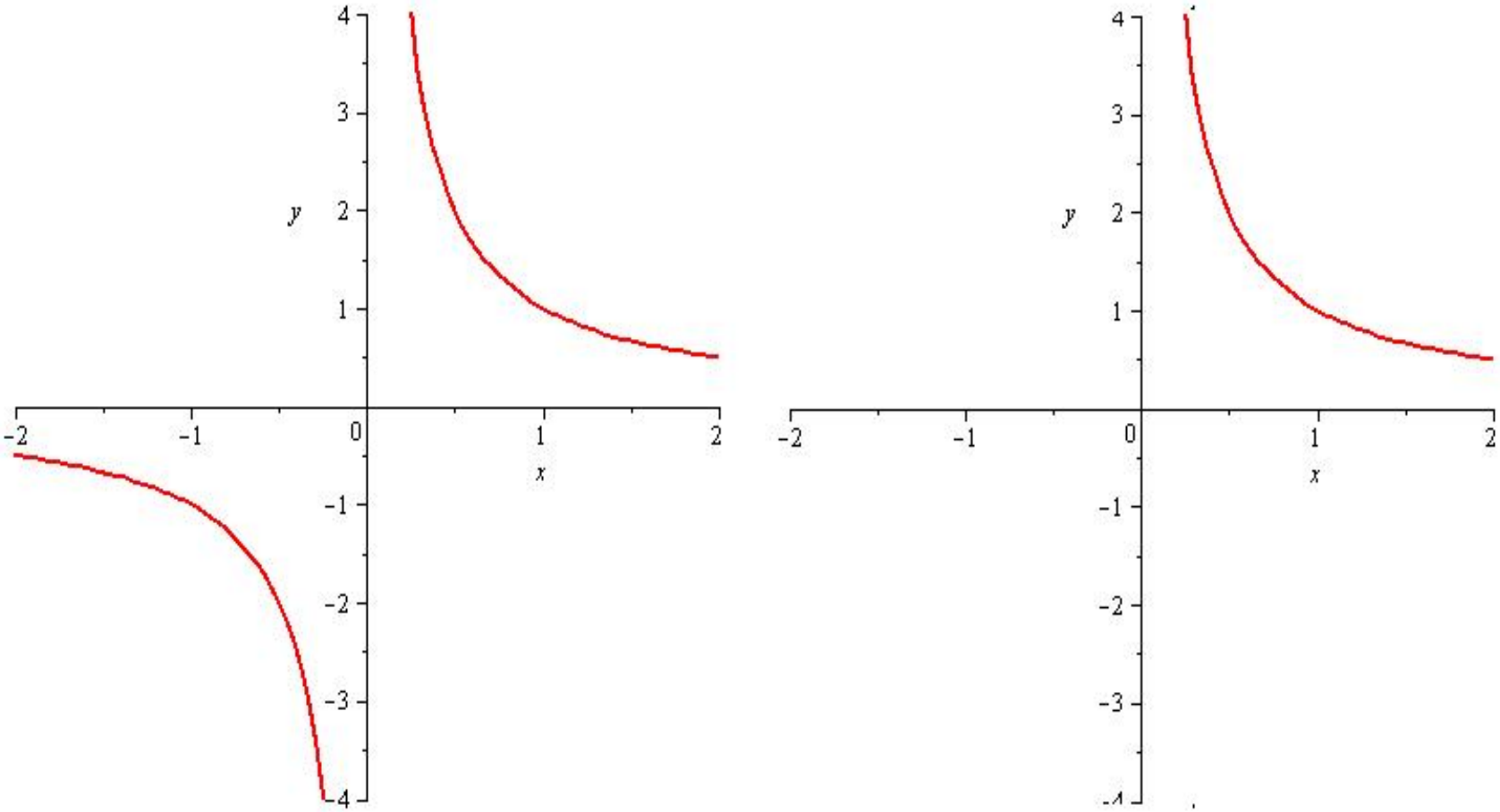


Figure: Left: Plot of $f(x) = \frac{1}{x}$ as a **function**. Right: A plot of $f(x) = \frac{1}{x}$ as a possible **solution** of an ODE.

Note that for any choice of constants c_1 and c_2, $y = c_1 x + \frac{c_2}{x}$ is a solution of the differential equation

$$x^2 y'' + xy' - y = 0$$

We have

$$y' = c_1 - \frac{c_2}{x^2}, \quad \text{and} \quad y'' = \frac{2c_2}{x^3}$$

So

$$\begin{aligned} x^2 y'' + xy' - y &= x^2 \left(\frac{2c_2}{x^3}\right) + x\left(c_1 - \frac{c_2}{x^2}\right) - \left(c_1 x + \frac{c_2}{x}\right) \\ &= \frac{2c_2}{x} + c_1 x - \frac{c_2}{x} - c_1 x - \frac{c_2}{x} \\ &= (2c_2 - c_2 - c_2)\frac{1}{x} + (c_1 - c_1)x \\ &= 0 \end{aligned}$$

as required.

Some Terms

- A **parameter** is an unspecified constant such as c_1 and c_2 in the last example.
- A **family of solutions** is a collection of solution functions that only differ by a parameter.
- An n**-parameter family of solutions** is one containing n parameters (e.g. $c_1 x + \frac{c_2}{x}$ is a 2 parameter family).
- A **particular solution** is one with no arbitrary constants in it.
- The **trivial solution** is the simple constant function $y = 0$.
- An **integral curve** is the graph of one solution (perhaps from a family).

Systems of ODEs

Sometimes we want to consider two or more dependent variables that are functions of the same independent variable. The ODEs for the dependent variables can depend on one another. Some examples of relevant situations are

- predator and prey
- competing species
- two or more masses attached to a system of springs
- two or more composite fluids in attached tank systems

Such systems can be **linear** or **nonlinear**.

Example of Nonlinear System

$$\begin{aligned} \frac{dx}{dt} &= -\alpha x + \beta xy \\ \frac{dy}{dt} &= \gamma y - \delta xy \end{aligned}$$

This is known as the **Lotka-Volterra** predator-prey model. $x(t)$ is the population (density) of predators, and $y(t)$ is the population of prey. The numbers α, β, γ and δ are nonnegative constants.
This model is built on the assumptions that

- in the absence of predation, prey increase exponentially
- in the absence of predation, predators decrease exponentially,
- predator-prey interactions increase the predator population and decrease the prey population.

Example of a Linear System

$$\frac{di_2}{dt} = -2i_2 - 2i_3 + 60$$
$$\frac{di_3}{dt} = -2i_2 - 5i_3 + 60$$

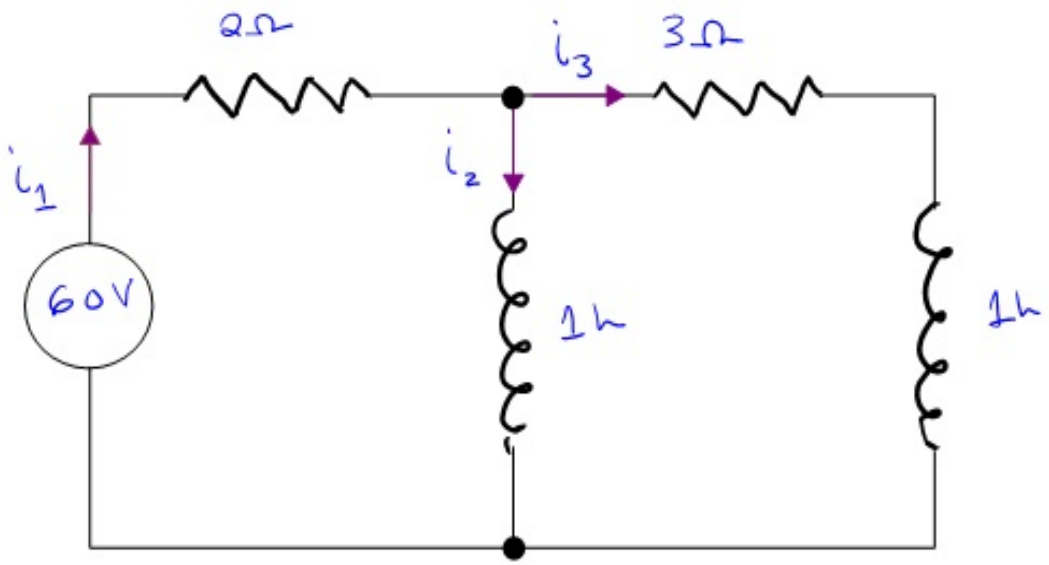


Figure: Electrical Network of resistors and inductors showing currents i_2 and i_3 modeled by this system of equations.

Systems of ODEs

Solving a system means finding all dependent variables as functions of the independent variable.

Example: Show that the pair of functions $i_2(t) = 30 - 24e^{-t} - 6e^{-6t}$ and $i_3(t) = 12e^{-t} - 12e^{-6t}$ are a solution to the system

$$\begin{aligned} \frac{di_2}{dt} &= -2i_2 - 2i_3 + 60 \\ \frac{di_3}{dt} &= -2i_2 - 5i_3 + 60 \end{aligned}$$

Exercise left to the reader.

Systems of ODEs

There are various approaches to solving a system of differential equations. These can include

- elimination (try to eliminate a dependent variable),
- matrix techniques,
- Laplace transforms[3]
- numerical approximation techniques

[3] We'll consider this later.

Section 2: Initial Value Problems

An initial value problem consists of an ODE with additional conditions.

For Example: Solve the equation [4]

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}) \tag{1}$$

subject to the *initial conditions*

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, y^{(n-1)}(x_0) = y_{n-1}. \tag{2}$$

The problem (1)–(2) is called an *initial value problem* (IVP).
Note that y and its derivatives are evaluated at the same initial x value of x_0.

[4] on some interval I containing x_0.

Examples for $n = 1$ or $n = 2$

First order case:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

Second order case:

$$\frac{d^2y}{dx^2} = f(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = y_1$$

Example

Given that $y = c_1 x + \frac{c_2}{x}$ is a 2-parameter family of solutions of $x^2 y'' + xy' - y = 0$, solve the IVP

$$x^2 y'' + xy' - y = 0, \quad y(1) = 1, \quad y'(1) = 3$$

Satisfying the initial conditions will require certain values for c_1 and c_2.

$$y(1) = c_1(1) + \frac{c_2}{1} = 1 \quad \Longrightarrow \quad c_1 + c_2 = 1$$

$$y'(1) = c_1 - \frac{c_2}{1^2} = 3 \quad \Longrightarrow \quad c_1 - c_2 = 3$$

Solving this algebraic system, one finds that $c_1 = 2$ and $c_2 = -1$. So the solution to the IVP is

$$y = 2x - \frac{1}{x}.$$

Graphical Interpretation

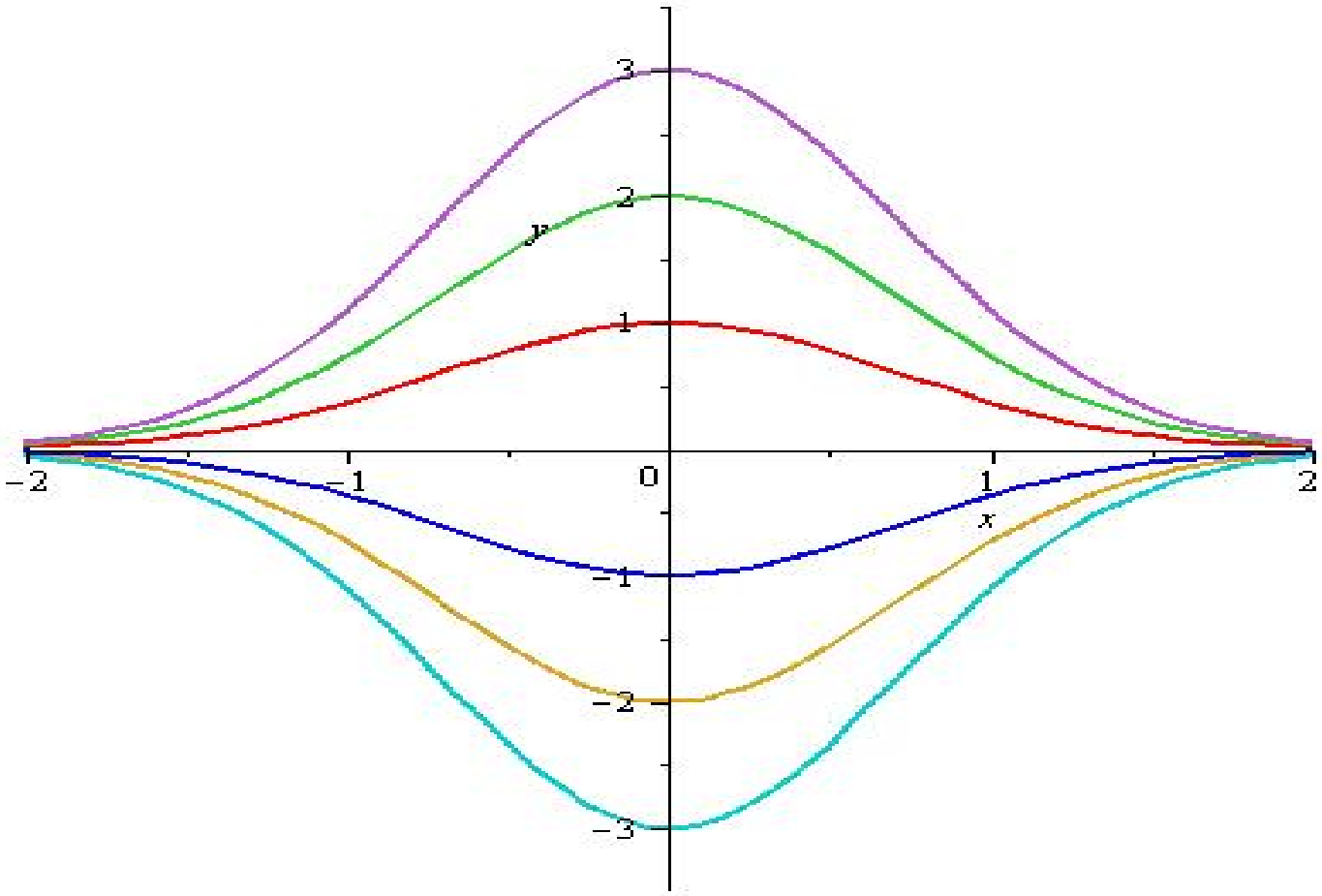


Figure: Each curve solves $y' + 2xy = 0$, $y(0) = y_0$. Each colored curve corresponds to a different value of y_0

A Numerical Solution

Consider a first order initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

In the coming sections, we'll see methods for solving some of these problems analytically (e.g. by hand). The method will depend on the type of equation. But not all ODEs are readily solved by hand. We can ask whether we can at least obtain an approximation to the solution, for example as a table of values or in the form of a curve. In general, the answer is that we can get such an approximation. Various algorithms have been developed to do this. We're going to look at a method known as **Euler's Method**.

The strategy behind Euler's method is to construct the solution starting with the known initial point (x_0, y_0) and using the tangent line to *find* the next point on the solution curve.

Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

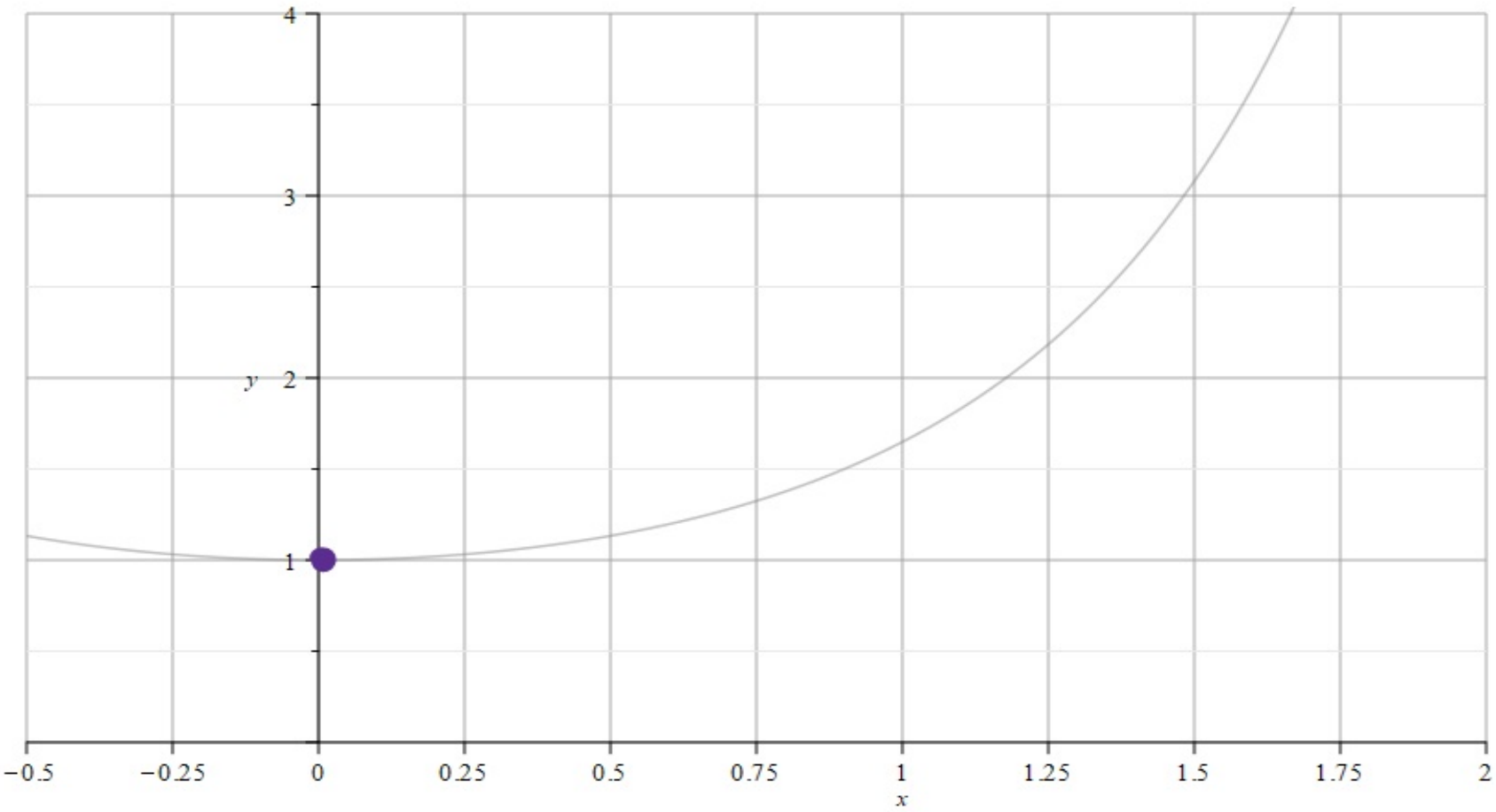


Figure: We know that the point $(x_0, y_0) = (0, 1)$ is on the curve. And the slope of the curve at $(0, 1)$ is $m_0 = f(0, 1) = 0 \cdot 1 = 0$.
Note: The gray curve is the true solution to this IVP. It's shown for reference.

Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

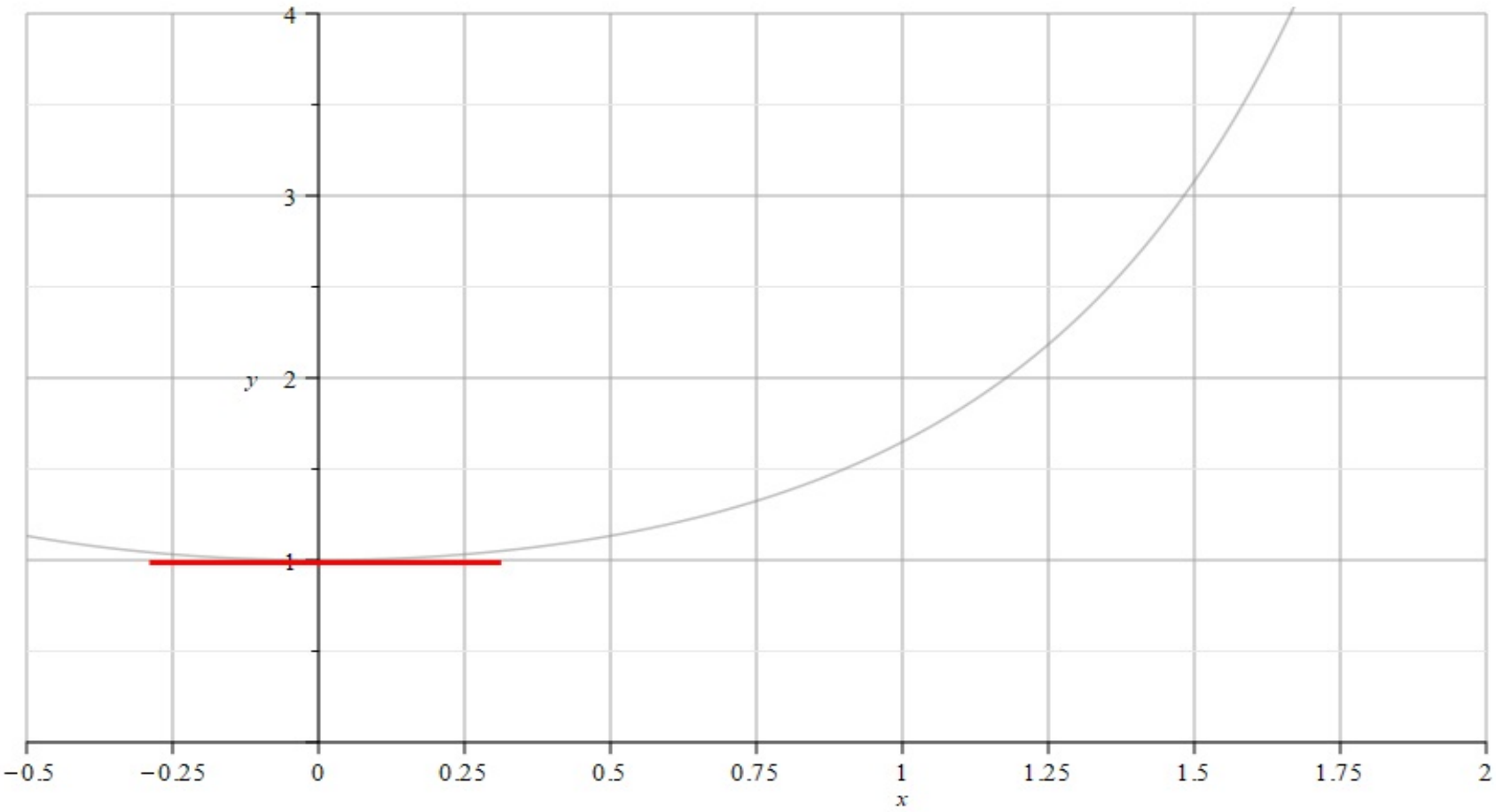


Figure: So we draw a little tangent line (we know the point and slope). Then we increase x, say $x_1 = x_0 + h$, and approximate the solution value $y(x_1)$ with the value on the tangent line y_1. So $y_1 \approx y(x_1)$.

Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

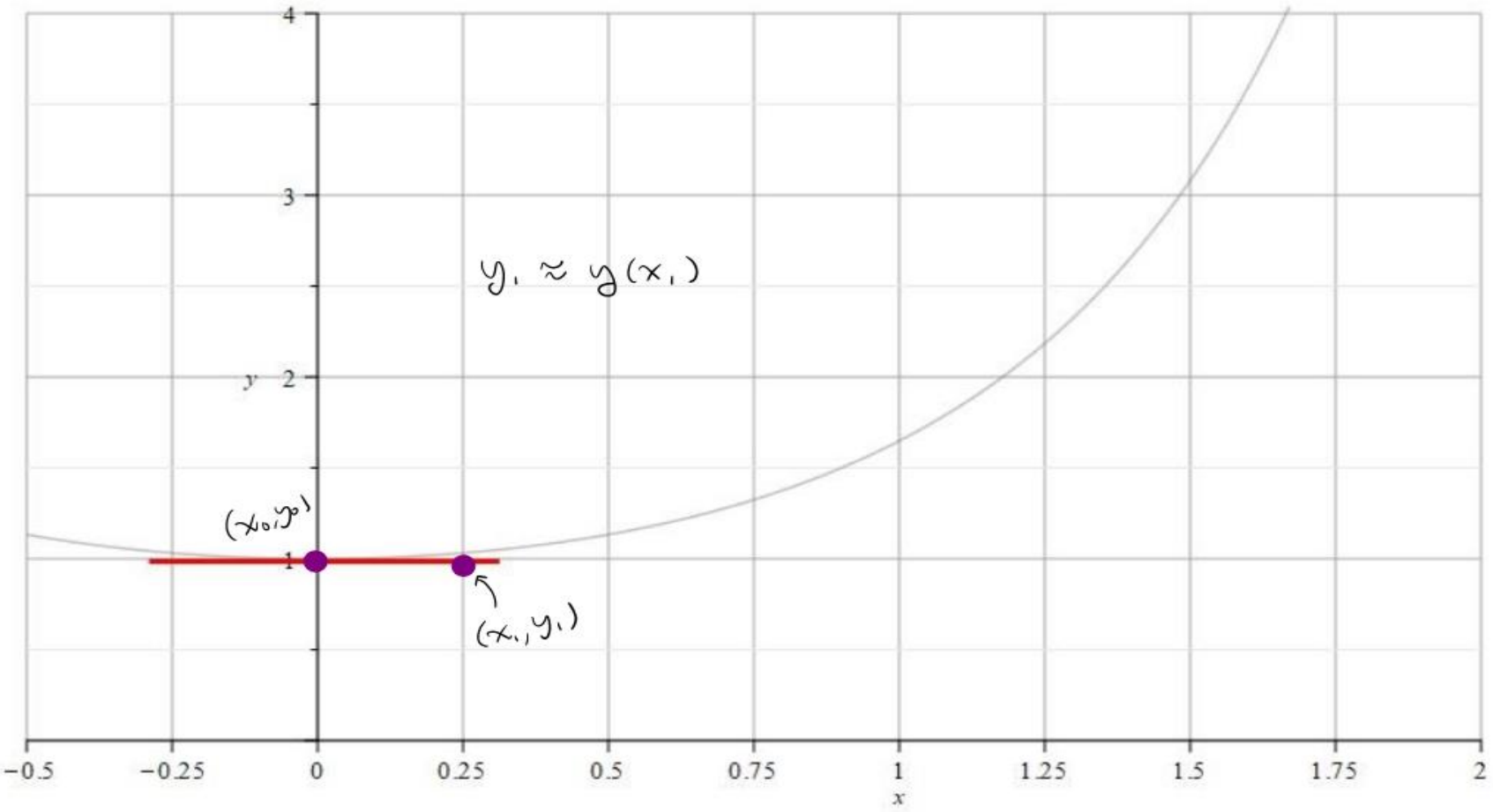


Figure: We take the approximation to the true function y at the point $x_1 = x_0 + h$ to be the point on the tangent line.

Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

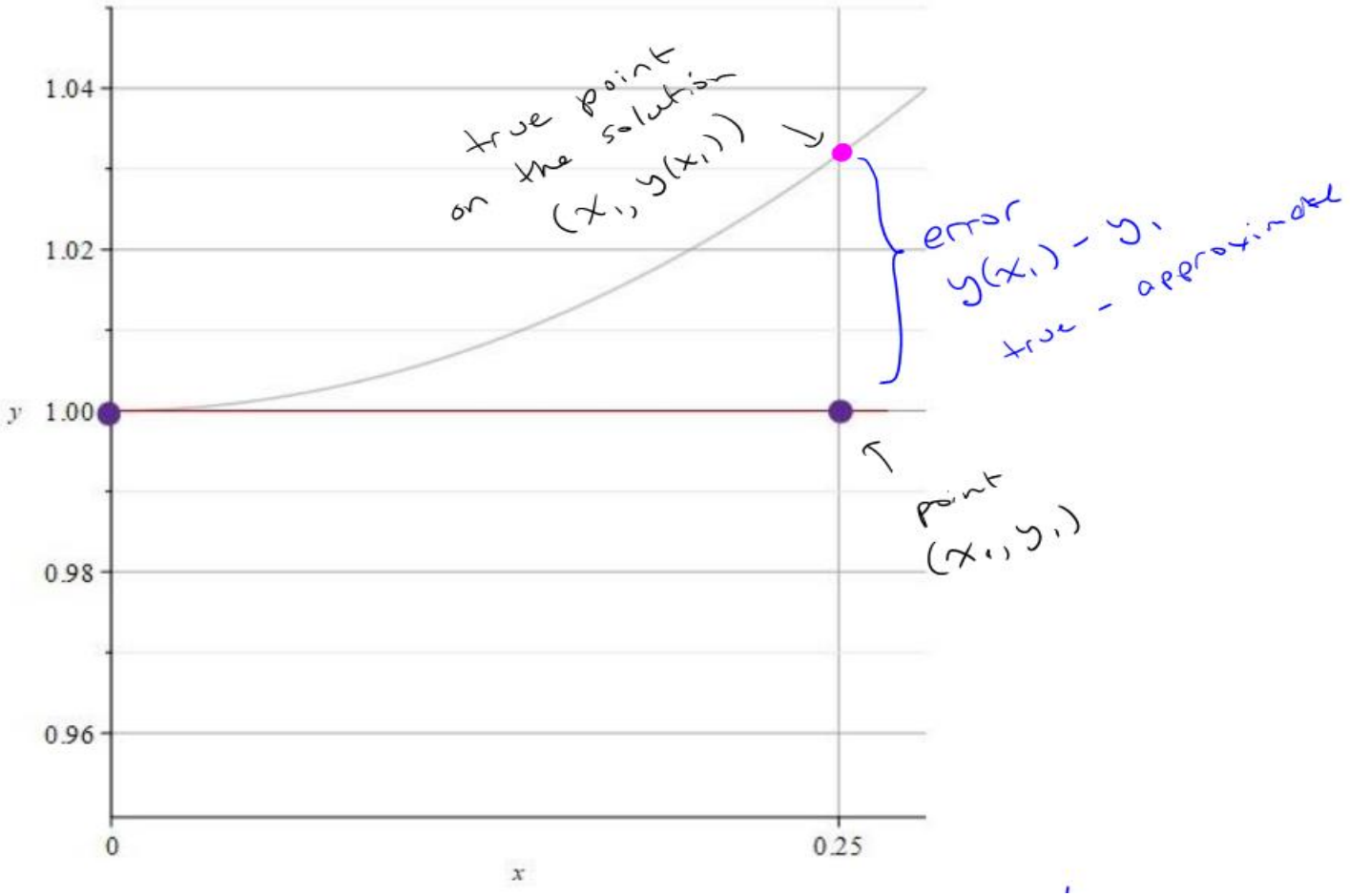


Figure: When h is very small, the true solution and the tangent line point will be close. Here, we've zoomed in to see that there is some error between the exact y value and the approximation from the tangent line.

Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

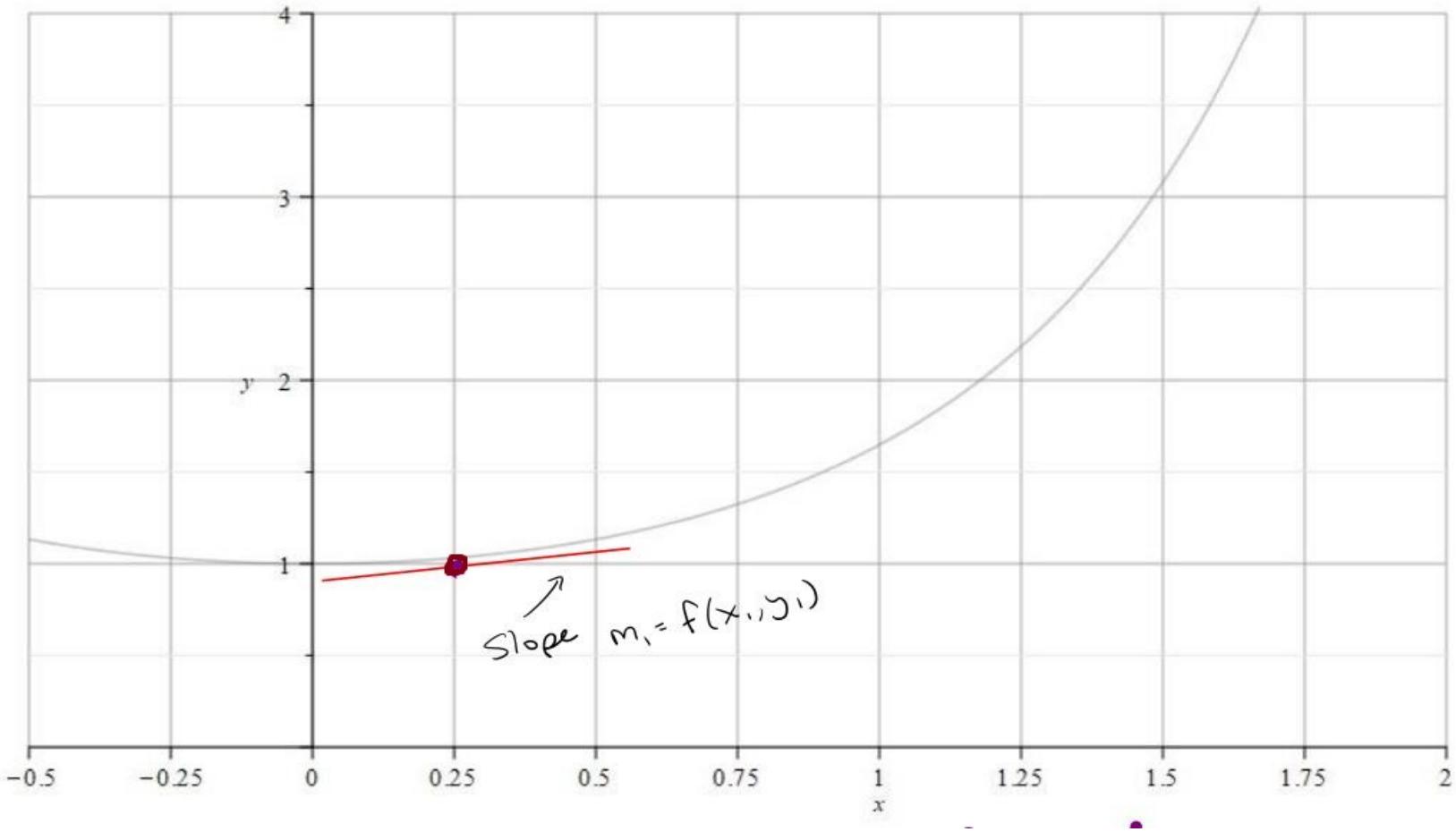


Figure: Now we start with the point (x_1, y_1) and repeat the process. We get the slope $m_1 = f(x_1, y_1)$ and draw a tangent line through (x_1, y_1) with slope m_1.

Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

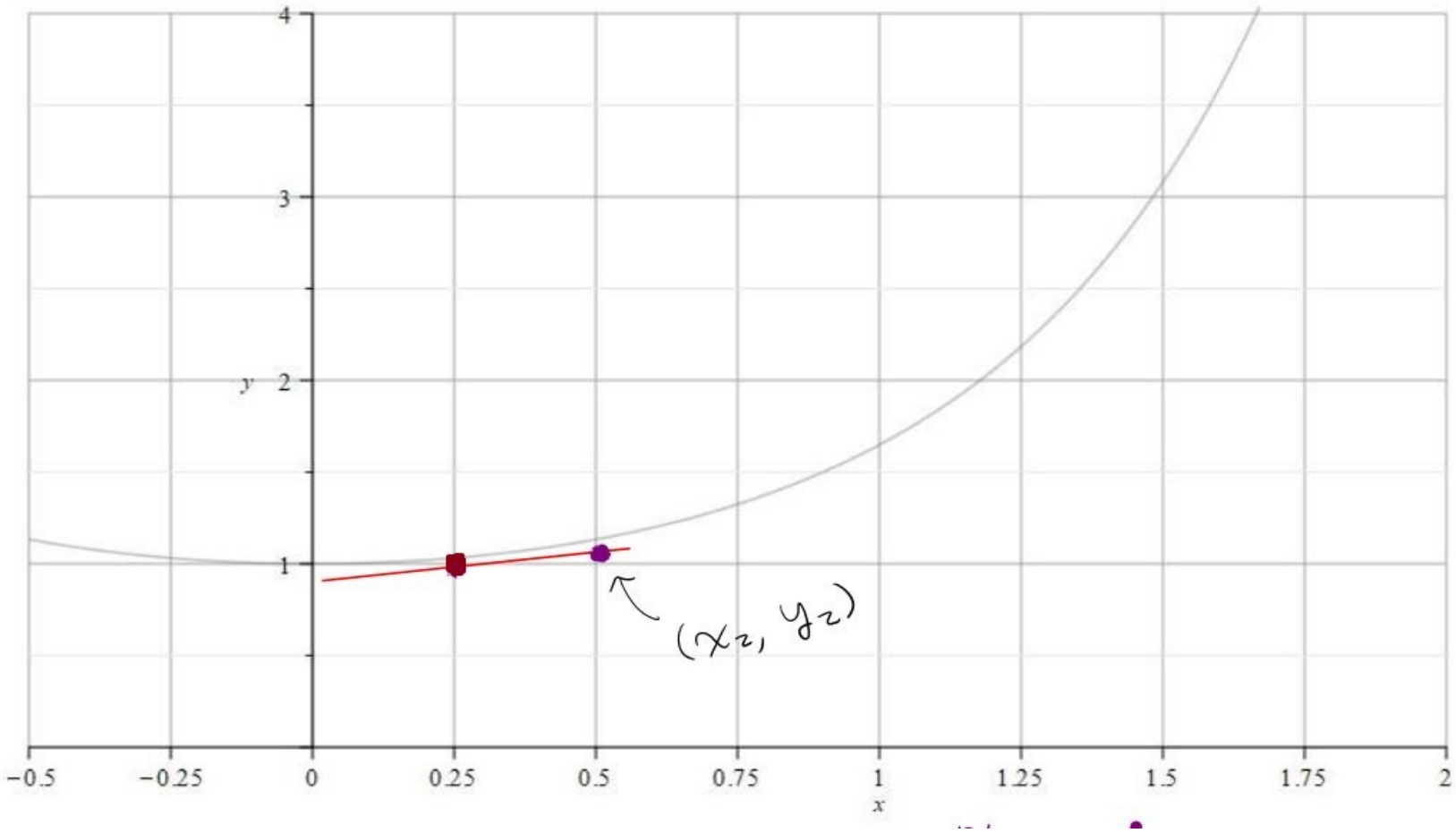


Figure: We go out h more units to $x_2 = x_1 + h$. Pick the point on the tangent line (x_2, y_2), and use this to approximate $y(x_2)$. So $y_2 \approx y(x_2)$

Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

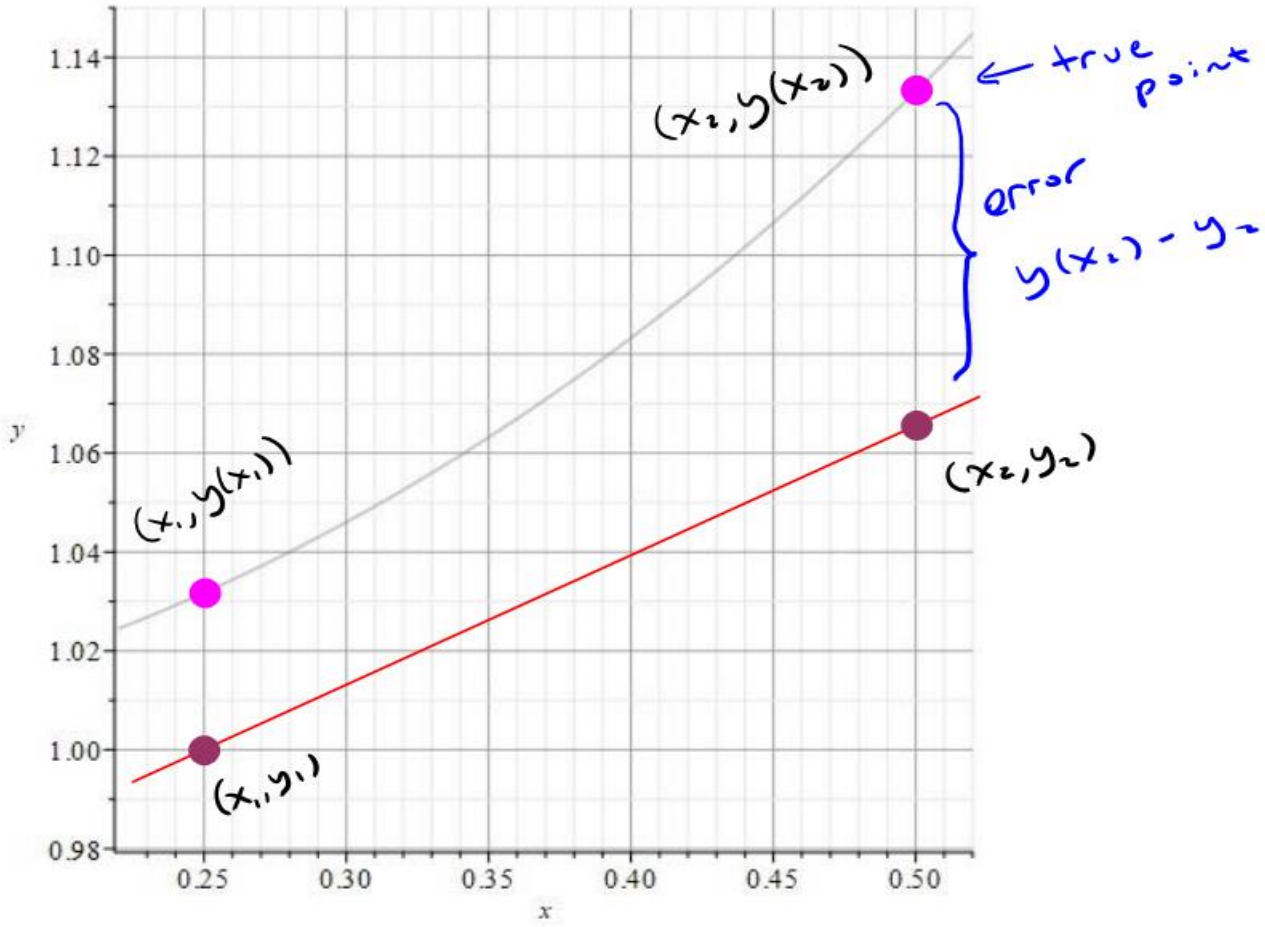


Figure: If we zoom in, we can see that there is some error. But as long as h is small, the point on the tangent line approximates the point on the actual solution curve.

Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

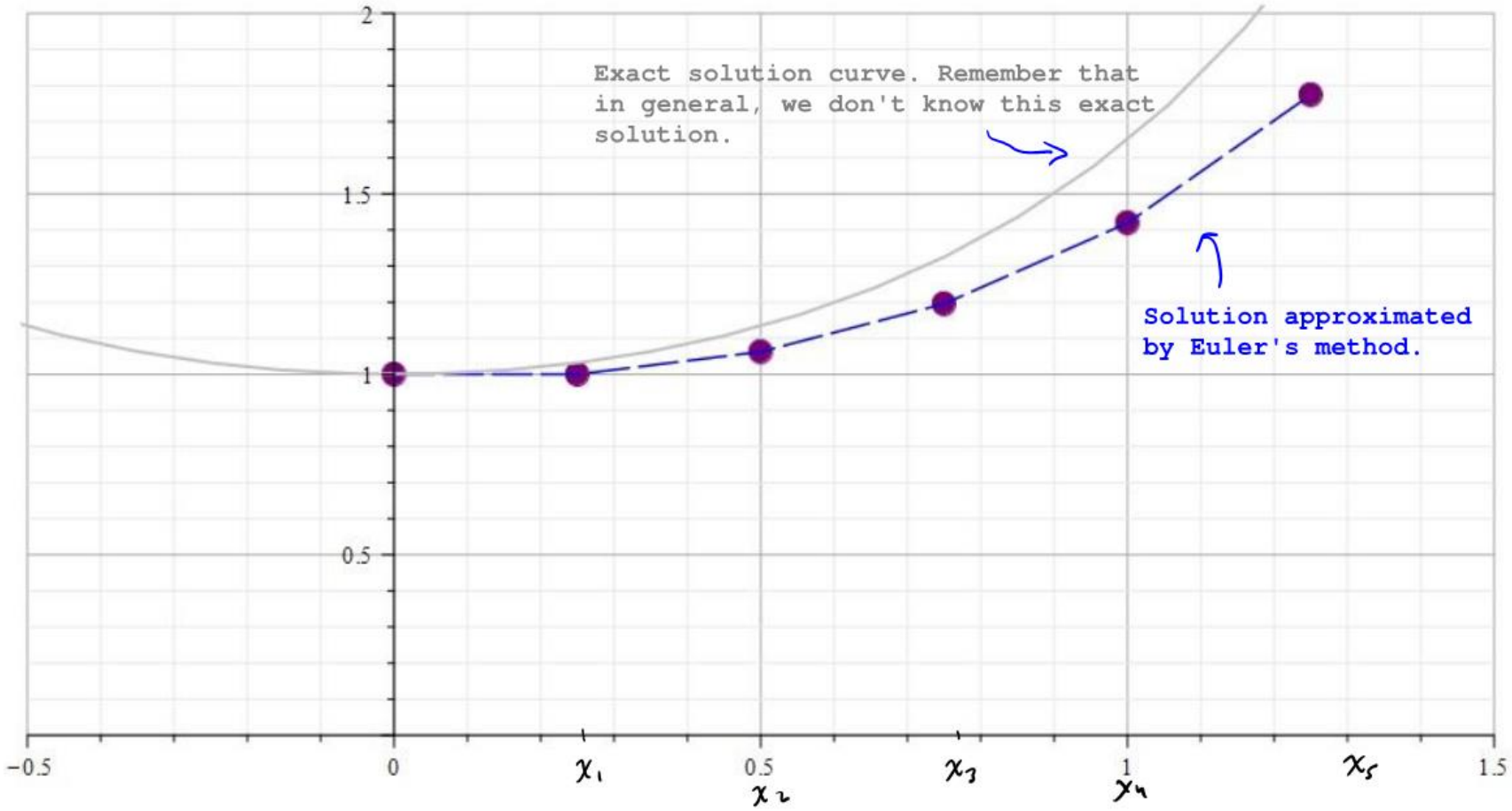


Figure: We can repeat this process at the new point to obtain the next point. We build an approximate solution by advancing the independent variable and connect the points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$.

Euler's Method: An Algorithm & Error

We start with the IVP

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

We build a sequence of points that approximates the true solution y

$$(x_0, y_0),\ (x_1, y_1),\ (x_2, y_2),\ \ldots, (x_N, y_N).$$

We'll take the x values to be equally spaced with a common difference of h. That is

$$\begin{aligned} x_1 &= x_0 + h \\ x_2 &= x_1 + h = x_0 + 2h \\ x_3 &= x_2 + h = x_0 + 3h \\ &\vdots \\ x_n &= x_0 + nh \end{aligned}$$

Euler's Method: An Algorithm

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

Notation:

- The approximate value of the solution will be denoted by y_n,
- and the exact values (that we don't expect to actually know) will be denoted $y(x_n)$.

To build a formula for the approximation y_1, let's approximate the derivative at (x_0, y_0).

$$f(x_0, y_0) = \left.\frac{dy}{dx}\right|_{(x_0,y_0)} \approx \frac{y_1 - y_0}{x_1 - x_0}$$

(Notice that's the standard formula for slope.)

Euler's Method: An Algorithm

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

We know x_0 and y_0, and we also know that $x_1 = x_0 + h$ so that $x_1 - x_0 = h$. Thus, we can solve for y_1.

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{y_1 - y_0}{h} = f(x_0, y_0)$$

$$\implies \quad y_1 - y_0 = hf(x_0, y_0)$$

$$\implies \quad y_1 = y_0 + hf(x_0, y_0)$$

The formula to approximate y_1 is therefore

$$y_1 = y_0 + hf(x_0, y_0)$$

Euler's Method: An Algorithm

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

We can continue this process. So we use

$$\frac{y_2 - y_1}{h} = f(x_1, y_1) \quad \Longrightarrow \quad y_2 = y_1 + hf(x_1, y_1)$$

and so forth. We have

Euler's Method Formula: The nth approximation y_n to the exact solution $y(x_n)$ is given by

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$$

with (x_0, y_0) given in the original IVP and h the choice of step size.

Euler's Method Example: $\frac{dy}{dx} = xy, \quad y(0) = 1$

Let's take $h = 0.25$ and find the first three iterates y_1, y_2, and y_3. We have $x_0 = 0$ and $y_0 = 1$. So

$$y_1 = y_0 + hf(x_0, y_0) = 1 + 0.25(0 \cdot 1) = 1$$

Now we repeat to find y_2. We have $x_1 = 0.25$ and $y_1 = 1$.

$$y_2 = y_1 + hf(x_1, y_1) = 1 + 0.25(0.25 \cdot 1) = 1.0625$$

Now we repeat to find y_3. We have $x_2 = 0.5$ and $y_2 = 1.0625$.

$$y_3 = y_2 + hf(x_2, y_2) = 1.0625 + 0.25(0.5 \cdot 1.0625) = 1.19531$$

Euler's Method Example: $\dfrac{dx}{dt} = \dfrac{x^2 - t^2}{xt}, \quad x(1) = 2$

Take $h = 0.2$ and use Euler's method to approximate $x(1.4)$.

We'll need to use two steps. With $h = 0.2$, we'll move from $t = 1$ to $t = 1.2$ and then to $t = 1.4$. First, let's determine the formula using Euler's method. We have $f(t, x) = \frac{x^2 - t^2}{xt}$, so the general formula will be

$$x_n = x_{n-1} + hf(t_{n-1}, x_{n-1}) = x_{n-1} + 0.2\left(\frac{x_{n-1}^2 - t_{n-1}^2}{x_{n-1}t_{n-1}}\right)$$

We also have

$$t_0 = 1 \quad \text{and} \quad x_0 = 2.$$

Euler's Method Example: $\dfrac{dx}{dt} = \dfrac{x^2 - t^2}{xt}, \quad x(1) = 2$

First, we approximate $x(1.2)$

$$x_1 = x_0 + 0.2\left(\frac{x_0^2 - t_0^2}{x_0 t_0}\right) = 2 + 0.2\left(\frac{2^2 - 1^2}{2 \cdot 1}\right) = 2.3$$

So we have the point $(1.2, 2.3)$.

Euler's Method Example: $\dfrac{dx}{dt} = \dfrac{x^2 - t^2}{xt}, \quad x(1) = 2$

Now we move on to the next point to approximate $x(1.4)$. From the last step, we have $t_1 = 1.2$ and $x_1 = 2.3$ so

$$x_2 = x_1 + 0.2\left(\frac{x_1^2 - t_1^2}{x_1 t_1}\right) = 2.3 + 0.2\left(\frac{2.3^2 - 1.2^2}{2.3 \cdot 1.2}\right) = 2.579$$

So we have the point $(1.4, 2.579)$.

The approximation

$$x(1.4) \approx 2.579.$$

It is possible to solve this IVP exactly to obtain the solution $x = \sqrt{4t^2 - 2t^2 \ln(t)}$. The true value $x(1.4) = 2.554$ to four decimal digits.

Euler's Method: Error

As the previous graph suggests, the approximate solution obtained using Euler's method has error. Moreover, the error can be expected to become more pronounced, the farther away from the intial condition we get.

First, let's define what we mean by the term *error*. There are a couple of types of error that we can talk about. These are[5]

$$\text{Absolute Error} = |\text{True Value} - \text{Approximate Value}|$$

and

$$\text{Relative Error} = \frac{\text{Absolute Error}}{|\text{True value}|}$$

[5] Some authors will define absolute error without use of absolute value bars so that absolute error need not be nonnegative.

Euler's Method: Error

We expect to get better results taking smaller steps. We can ask, how does the error depend on the step size? Let's look at some error for one of the examples.

$$\frac{dy}{dx} = xy, \quad y(0) = 1$$

I programed Euler's method into Matlab and used different h values to approximate $y(1)$. The number of iterations depends on the step size. For example, if $h = 0.2$, it takes five steps to get from $x_0 = 0$ to $x_5 = 1$. In general, the number of steps is $n = \frac{1}{h}$.

h	$y(1) - y_n$	$\frac{y(1)-y_n}{y(1)}$
0.2	0.1895	0.1149
0.1	0.1016	0.0616
0.05	0.0528	0.0320
0.025	0.0269	0.0163
0.0125	0.0136	0.0082

Euler's Method: Error

We notice from this example that cutting the step size in half, seems to cut the error and relative error in half. This suggests the following:

The absolute error in Euler's method is proportional to the step size.

There are two sources of error for Euler's method (not counting numerical errors due to machine rounding).

- The error in approximating the curve with a tangent line, and
- using the approximate value y_{n-1} to get the slope at the next step.

Euler's Method: Error

For numerical schemes of this sort, we often refer to the *order* of the scheme. If the error satisfies

$$\text{Absolute Error} = Ch^p$$

where C is some constant, then the order of the scheme is p.

Euler's method is an order 1 scheme.

Euler's Method

Euler's method is simple and intuitive. However, it is rarely used in practice because of its error. There are more widely used schemes. Other methods tend to use multiple tangent lines for each iteration and are sometimes referred as mutli-step methods. The two most common are

- Improved Euler[6] which is order 2, and
- Runge-Kutta[7] which is order 4.

[6] a.k.a. RK2
[7] a.k.a. RK4

Improved Euler's Method: $\dfrac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$

Euler's method approximated y_1 using the slope $m_0 = f(x_0, y_0)$ for the tangent line. An initial improvement on the method can be made by using this as an intermediate point to give a second approximation to the slope. That is, let

$$m_0 = f(x_0, y_0)$$

as before, and now let

$$\hat{m}_0 = f(x_1, y_0 + m_0 h).$$

Then we take y_1 to be the point on the line that has the average of these two slopes

$$y_1 = y_0 + \frac{1}{2}(m_0 + \hat{m}_0)h.$$

Other methods will use the weighted averages of 3, 4 or more tangent lines.

Existence and Uniqueness

Two important questions we can always pose (and sometimes answer) are

(1) Does an IVP have a solution? (existence) and

(2) If it does, is there just one? (uniqueness)

Hopefully it's obvious that we can't solve $\left(\frac{dy}{dx}\right)^2 + 1 = -y^2$.

(Not if we are only interested in real valued solutions.)

Uniqueness

Consider the IVP

$$\frac{dy}{dx} = x\sqrt{y} \quad y(0) = 0$$

Exercise: Verify that $y = \frac{x^4}{16}$ is a solution of the IVP.

Can you find a second solution of the IVP by inspection (i.e. clever guessing)?

Section 3: First Order Equations: Separation of Variables

The simplest type of equation we could encounter would be of the form

$$\frac{dy}{dx} = g(x).$$

If $G(x)$ is any antiderivative of $g(x)$, the solutions to this ODE would be

$$y = G(x) + c$$

obtained by simply integrating.

Separable Equations

Definition: The first order equation $y' = f(x, y)$ is said to be **separable** if the right side has the form

$$f(x, y) = g(x)h(y).$$

That is, a separable equation is one that has the form

$$\frac{dy}{dx} = g(x)h(y).$$

Separable -vs- Nonseparable

$$\frac{dy}{dx} = x^3 y$$

is separable as the right side is the product of $g(x) = x^3$ and $h(y) = y$.

$$\frac{dy}{dx} = 2x + y$$

is not separable. You can try, but it is not possible to write $2x + y$ as the **product** of a function of x alone and a function of y alone.

Solving Separable Equations

Let's assume that it's safe to divide by $h(y)$ and let's set $p(y) = 1/h(y)$. We solve (usually find an implicit solution) by **separating the variables**.

$$\frac{dy}{dx} = g(x)h(y)$$

Note that $\dfrac{1}{h(y)}\dfrac{dy}{dx} = g(x) \quad \Longrightarrow \quad p(y)\dfrac{dy}{dx}\,dx = g(x)\,dx$

Since $\frac{dy}{dx}\,dx = dy$, we integrate both sides

$$\int p(y)\,dy = \int g(x)\,dx \quad \Longrightarrow \quad P(y) = G(x) + c$$

where P and G are any antiderivatives of p and g, respectively. The expression

$$P(y) = G(x) + c$$

defines a one parameter family of solutions implicitly.

Caveat regarding division by $h(y)$.

Recall that the IVP

$$\frac{dy}{dx} = x\sqrt{y}, \quad y(0) = 0$$

has two solutions

$$y = \frac{x^2}{16} \quad \text{and} \quad y = 0.$$

Exercise: Solve this by separation of variables. Note that only one of these solutions is recoverable. Why is the second one lost?

Solutions Defined by Integrals

Recall (Fundamental Theorem of Calculus) for g continuous on an interval containing x_0 and x

$$\frac{d}{dx}\int_{x_0}^{x} g(t)\,dt = g(x) \quad \text{and} \quad \int_{x_0}^{x} \frac{dy}{dt}\,dt = y(x) - y(x_0).$$

Assuming g is continuous at x_0, we can use this to solve

$$\frac{dy}{dx} = g(x), \quad y(x_0) = y_0.$$

Expressing the solution in terms of an integral

$$y(x) = y_0 + \int_{x_0}^{x} g(t)\,dt.$$

Verify that this is a solution.

Section 4: First Order Equations: Linear & Special

A first order linear equation has the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

If $g(x) = 0$ the equation is called **homogeneous**. Otherwise it is called **nonhomogeneous**.

Provided $a_1(x) \neq 0$ on the interval I of definition of a solution, we can write the **standard form** of the equation

$$\frac{dy}{dx} + P(x)y = f(x).$$

We'll be interested in equations (and intervals I) for which P and f are continuous on I.

Solutions (the General Solution)

$$\frac{dy}{dx} + P(x)y = f(x).$$

It turns out the solution will always have a basic form of $y = y_c + y_p$ where

- y_c is called the **complementary** solution and would solve the problem
$$\frac{dy}{dx} + P(x)y = 0$$
(called the associated homogeneous equation), and
- y_p is called the **particular** solution, and is heavily influenced by the function $f(x)$.

The cool thing is that our solution method will get both parts in one process—we won't get this benefit with higher order equations!

Motivating Example

$$x^2\frac{dy}{dx}+2xy = e^x$$

Note that the left side is a product rule

$$\frac{d}{dx}\left[x^2y\right] = x^2\frac{dy}{dx} + 2xy.$$

Hence the equation reduces to

$$\frac{d}{dx}\left[x^2y\right] = e^x \quad \Longrightarrow \quad \int \frac{d}{dx}\left[x^2y\right]\,dx = \int e^x\,dx$$

Integrate and divide by x^2 to obtain

$$y = \frac{e^x}{x^2} + \frac{c}{x^2}$$

Derivation of Solution via Integrating Factor

We seek to solve the equation in standard form

$$\frac{dy}{dx} + P(x)y = f(x)$$

Based on the previous example, we seek a function $\mu(x)$ such that when we multiply the above equation by this new function, the left side collapses as a product rule. We wish to have

$$\mu\frac{dy}{dx} + \mu Py = \mu f \quad \Longrightarrow \quad \frac{d}{dx}[\mu y] = \mu f.$$

Matching the left sides

$$\mu\frac{dy}{dx} + \mu Py = \frac{d}{dx}[\mu y] = \mu\frac{dy}{dx} + \frac{d\mu}{dx}y$$

which requires

$$\frac{d\mu}{dx} = P\mu.$$

$$\frac{d\mu}{dx} = P\mu.$$

We've obtained a separable equation for the sought after function μ. Using separation of variables, we find that

$$\mu(x) = \exp\left(\int P(x)\,dx\right).$$

This function is called an *integrating factor*.

General Solution of First Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)f(x)\,dx = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx} f(x)\,dx + C\right)$$

Example: Find the general solution.

$$x\frac{dy}{dx} - y = 2x^2$$

In standard form the equation is

$$\frac{dy}{dx} - \frac{1}{x}y = 2x, \quad \text{so that} \quad P(x) = -\frac{1}{x}.$$

Then[8]

$$\mu(x) = \exp\left(\int -\frac{1}{x}\,dx\right) = \exp(-\ln|x|) = x^{-1}$$

The equation becomes

$$\frac{d}{dx}[x^{-1}y] = x^{-1}(2x) = 2$$

[8] We will take the constant of integration to be zero when finding μ.

$$\frac{d}{dx}[x^{-1}y] = x^{-1}(2x) = 2$$

Next we integrate both sides—μ makes this possible, hence the name *integrating factor*—and solve for our solution y.

$$\int \frac{d}{dx}[x^{-1}y]\,dx = \int 2\,dx \quad \Longrightarrow \quad x^{-1}y = 2x + C$$

and finally

$$y = 2x^2 + Cx.$$

Note that this solution has the form $y = y_p + y_c$ where $y_c = Cx$ and $y_p = 2x^2$. The complementary part comes from the constant of integration and is independent of the right side of the ODE $2x$. The particular part comes from the right hand side integration of $x^{-1}(2x)$.

Steady and Transient States

For some linear equations, the term y_c decays as x (or t) grows. For example

$$\frac{dy}{dx} + y = 3xe^{-x} \quad \text{has solution} \quad y = \frac{3}{2}x^2 + Ce^{-x}.$$

$$\text{Here,} \quad y_p = \frac{3}{2}x^2 \quad \text{and} \quad y_c = Ce^{-x}.$$

Such a decaying complementary solution is called a **transient state**.

The corresponding particular solution is called a **steady state**.

Bernoulli Equations

Suppose $P(x)$ and $f(x)$ are continuous on some interval (a, b) and n is a real number different from 0 or 1 (not necessarily an integer). An equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

is called a **Bernoulli** equation.

Observation: This equation has the flavor of a linear ODE, but since $n \neq 0, 1$ it is necessarily nonlinear. So our previous approach involving an integrating factor does not apply directly. Fortunately, we can use a change of variables to obtain a related linear equation.

Solving the Bernoulli Equation

$$\frac{dy}{dx} + P(x)y = f(x)y^n \tag{3}$$

Let $u = y^{1-n}$. Then

$$\frac{du}{dx} = (1-n)y^{-n}\frac{dy}{dx} \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{y^n}{1-n}\frac{du}{dx}.$$

Substituting into (3) and dividing through by $y^n/(1-n)$

$$\frac{y^n}{1-n}\frac{du}{dx} + P(x)y = f(x)y^n \quad \Longrightarrow \frac{du}{dx} + (1-n)P(x)y^{1-n} = (1-n)f(x)$$

Given our choice of u, this is the first order linear equation

$$\frac{du}{dx} + P_1(x)u = f_1(x), \quad \text{where} \quad P_1 = (1-n)P, \quad f_1 = (1-n)f.$$

Example

Solve the initial value problem $y' - y = -e^{2x}y^3$, subject to $y(0) = 1$.

Here, $n = 3$ so we set $u = y^{1-3} = y^{-2}$. Observe then that

$$\frac{du}{dx} = -2y^{-3}\frac{dy}{dx} \quad \text{so} \quad \frac{dy}{dx} = -\frac{1}{2}y^3\frac{du}{dx}.$$

Upon substitution

$$-\frac{1}{2}y^3\frac{du}{dx} - y = -e^{2x}y^3$$

Multiplying through by $-2y^{-3}$ gives

$$\frac{du}{dx} + 2y^{-2} = 2e^{2x}$$

As expected, the second term on the left is $(1-n)P(x)u$, here $2u$.

Example Continued

Now we solve the first order linear equation for u using an integrating factor. Omitting the details, we obtain

$$u(x) = \frac{1}{2}e^{2x} + Ce^{-2x}$$

Of course, we need to remember that our goal is to solve the original equation for y. But the relationship between y and u is known. From $u = y^{-2}$, we know that $y = u^{-1/2}$. Hence

$$y = \frac{1}{\sqrt{\frac{1}{2}e^{2x} + Ce^{-2x}}}$$

Applying $y(0) = 1$ we find that $C = 1/2$ for a solution to the IVP

$$y = \frac{\sqrt{2}}{\sqrt{e^{2x} + e^{-2x}}}$$

Exact Equations

We considered first order equations of the form

$$M(x, y)\, dx + N(x, y)\, dy = 0. \tag{4}$$

The left side is called a *differential form*. We will assume here that M and N are continuous on some (shared) region in the plane.

Definition: The equation (4) is called an **exact equation** on some rectangle R if there exists a function $F(x, y)$ such that

$$\frac{\partial F}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial F}{\partial y} = N(x, y)$$

for every (x, y) in R.

Exact Equation Solution

If $M(x, y)\,dx + N(x, y)\,dy = 0$ happens to be exact, then it is equivalent to

$$\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = 0$$

This implies that the function F is constant on R and solutions to the

DE are given by the relation

$$F(x, y) = C$$

Recognizing Exactness

There is a theorem from calculus that ensures that if a function F has first partials on a domain, and if those partials are continuous, then the second mixed partials are equal. That is,

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}.$$

If it is true that

$$\frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N$$

this provides a condition for exactness, namely

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Exact Equations

Theorem: Let M and N be continuous on some rectangle R in the plane. Then the equation

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Example

Show that the equation is exact and obtain a family of solutions.

$$(2xy - \sec^2 x)\,dx + (x^2 + 2y)\,dy = 0$$

First note that for $M = 2xy - \sec^2 x$ and $N = x^2 + 2y$ we have

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}.$$

Hence the equation is exact. We obtain the solutions (implicitly) by finding a function F such that $\partial F/\partial x = M$ and $\partial F/\partial y = N$. Using the first relation we get[9]

$$F(x, y) = \int M(x, y)\,dx = \int (2xy - \sec^2 x)\,dx = x^2 y - \tan x + g(y)$$

[9] Holding y constant while integrating with respect to x means that the *constant* of integration may well depend on y

Example Continued

We must find g to complete our solution. We know that

$$F(x, y) = x^2 y - \tan x + g(y) \quad \text{and} \quad \frac{\partial F}{\partial y} = N(x, y) = x^2 + 2y$$

Differentiating the expression on the left with respect to y and equating

$$\frac{\partial F}{\partial y} = x^2 + g'(y) = x^2 + 2y$$

from which if follows that $g'(y) = 2y$. An antiderivative is given by $g(y) = y^2$. Since our solutions are $F = C$, we arrive at the family of solutions

$$x^2 y - \tan x + y^2 = C$$

Special Integrating Factors

Suppose that the equation $M\,dx + N\,dy = 0$ is not exact. Clearly our approach to exact equations would be fruitless as there is no such function F to find. It may still be possible to solve the equation if we can find a way to morph it into an exact equation. As an example, consider the DE

$$(2y - 6x)\,dx + (3x - 4x^2y^{-1})\,dy = 0$$

Note that this equation is NOT exact. In particular

$$\frac{\partial M}{\partial y} = 2 \neq 3 - 8xy^{-1} = \frac{\partial N}{\partial x}.$$

Special Integrating Factors

But note what happens when we multiply our equation by the function $\mu(x, y) = xy^2$.

$$xy^2(2y - 6x)\,dx + xy^2(3x - 4x^2y^{-1})\,dy = 0, \quad \Longrightarrow$$

$$(2xy^3 - 6x^2y^2)\,dx + (3x^2y^2 - 4x^3y)\,dy = 0$$

Now we see that

$$\frac{\partial(\mu M)}{\partial y} = 6xy^2 - 12x^2y = \frac{\partial(\mu N)}{\partial x}$$

The new equation[10] IS exact!

Of course this raises the question: **How would we know to use $\mu = xy^2$?**

[10] The solutions sets for these equations are almost the same. However, it is possible to introduce or lose solutions employing this approach.We won't worry about this here.

Special Integrating Factors

The function μ is called a *special integrating factor*. Finding one (assuming one even exists) may require ingenuity and likely a bit of luck. However, there are certain cases we can look for and perhaps use them to solve the occasional equation. A useful method is to look for μ of a certain *form* (usually $\mu = x^n y^m$ for some powers n and m). We will restrict ourselves to two possible cases:

There is an integrating faction $\mu = \mu(x)$ depending only on x, or there is an integrating factor $\mu = \mu(y)$ depending only on y.

Special Integrating Factor $\mu = \mu(x)$

Suppose that

$$M\,dx + N\,dy = 0$$

is NOT exact, but that

$$\mu M\,dx + \mu N\,dy = 0$$

IS exact where $\mu = \mu(x)$ does not depend on y. Then

$$\frac{\partial(\mu(x)M)}{\partial y} = \frac{\partial(\mu(x)N)}{\partial x}.$$

Let's use the product rule in the right side.

Special Integrating Factor $\mu = \mu(x)$

Since μ is *constant* in y,

$$\mu \frac{\partial M}{\partial y} = \frac{d\mu}{dx} N + \mu \frac{\partial N}{\partial x} \tag{5}$$

Rearranging (5), we get both a condition for the existence of such a μ as well as an equation for it. The function μ must satisfy the separable equation

$$\frac{d\mu}{dx} = \mu \left(\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \right) \tag{6}$$

Note that this equation is solvable, insofar as μ depends only on x, only if

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

depends only on x!

Special Integrating Factor

When solvable, equation (6) has solution

$$\mu = \exp\left(\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\, dx\right)$$

A similar manipulation assuming a function $\mu = \mu(y)$ depending only on y leads to the existence condition requiring

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$$

depend only on y.

Special Integrating Factor

$$M\,dx + N\,dy = 0 \tag{7}$$

Theorem: If $(\partial M/\partial y - \partial N/\partial x)/N$ is continuous and depends only on x, then

$$\mu = \exp\left(\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\,dx\right)$$

is a special integrating factor for (7). If $(\partial N/\partial x - \partial M/\partial y)/M$ is

continuous and depends only on y, then

$$\mu = \exp\left(\int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\,dy\right)$$

is a special integrating factor for (7).

Example

Solve the equation $2xy\,dx + (y^2 - 3x^2)\,dy = 0$.

Note that $\partial M/\partial y = 2x$ and $\partial N/\partial x = -6x$. The equation is not exact. Looking to see if there may be a special integrating factor, note that

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{8x}{y^2 - 3x^2}$$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-8x}{2xy} = \frac{-4}{y}$$

The first does not depend on x alone. But the second does depend on y alone. So there is a special integrating factor

$$\mu = \exp\left(\int -\frac{4}{y}\,dy\right) = y^{-4}$$

Example Continued

The new equation obtained by multiplying through by μ is

$$2xy^{-3}\,dx + (y^{-2} - 3x^2y^{-4})\,dy = 0.$$

Note that

$$\frac{\partial}{\partial y}2xy^{-3} = -6xy^{-4} = \frac{\partial}{\partial x}(y^{-2} - 3x^2y^{-4})$$

so this new equation is exact. Solving for F

$$F(x,y) = \int 2xy^{-3}\,dx = x^2y^{-3} + g(y)$$

and $g'(y) = y^{-2}$ so that $g(y) = -y^{-1}$. The solutions are given by

$$\frac{x^2}{y^3} - \frac{1}{y} = C.$$

Section 5: First Order Equations: Models and Applications

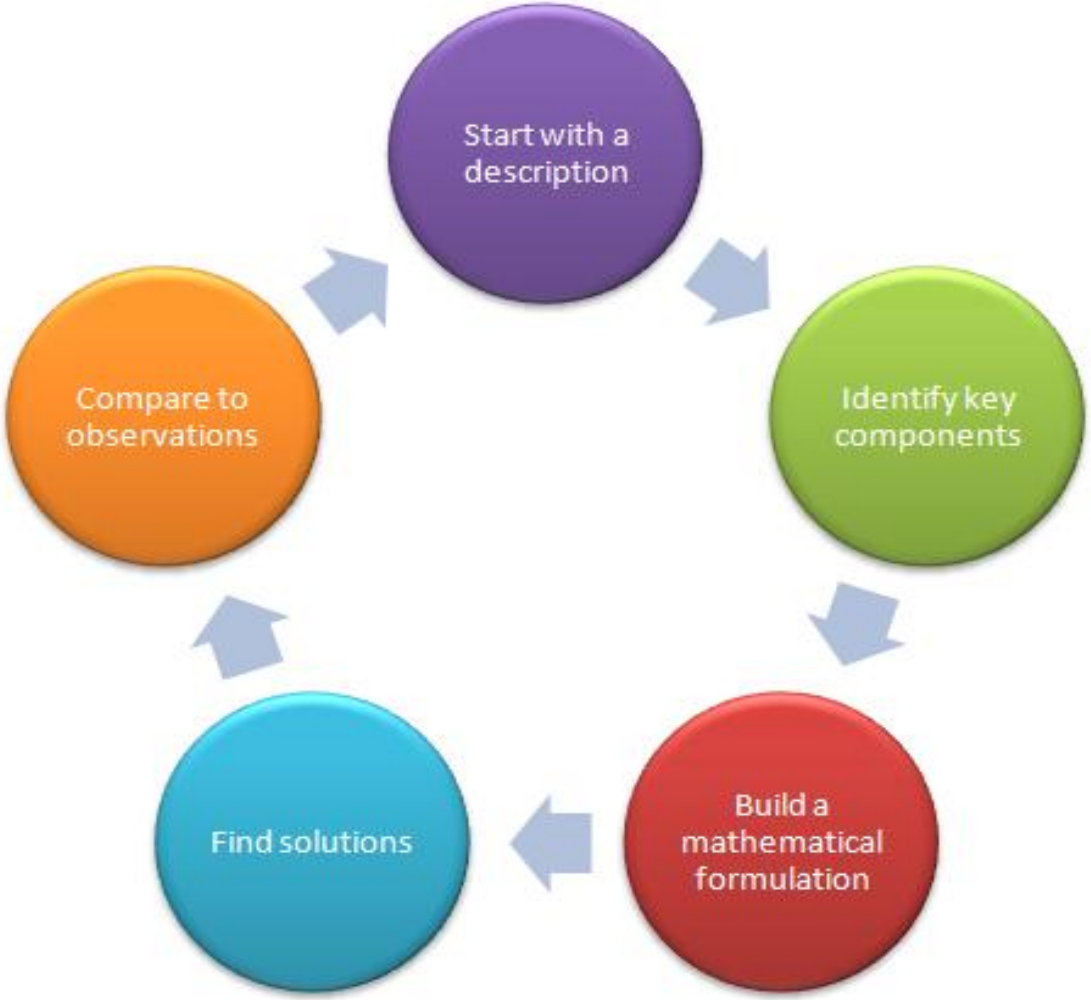


Figure: Mathematical Models give Rise to Differential Equations

Population Dynamics

A population of dwarf rabbits grows at a rate proportional to the current population. In 2011, there were 58 rabbits. In 2012, the population was up to 89 rabbits. Estimate the number of rabbits expected in the population in 2021.

We can translate this into a mathematical statement then hope to solve the problem and answer the question. Letting the population of rabbits (say population density) at time t be given by $P(t)$, to say that the rate of change is proportional to the population is to say

$$\frac{dP}{dt} = kP(t) \quad \text{for some constant } k.$$

This is a differential equation! To answer the question, we will require the value of k as well as some initial information—i.e. we will need an IVP.

Example Continued...[11]

We can choose units for time t. Based on the statement, taking t in years is well advised. Letting $t = 0$ in year 2011, the second and third sentences translate as

$$P(0) = 58, \quad \text{and} \quad P(1) = 89$$

Without knowing k, we can solve the IVP

$$\frac{dP}{dt} = kP, \quad P(0) = 58$$

by separation of variables to obtain

$$P(t) = 58e^{kt}.$$

[11] Taking P as the population density, i.e. number of rabbits per unit habitat, allows us to consider non-integer P values. Thus fractional and even irrational P values are reasonable and not necessarily gruesome.

Example Continued...

To evaluate the population function (for a real number), we still need to know k. We have the additional information $P(1) = 89$. Note that this gives

$$P(1) = 89 = 58e^{1k} \quad \Longrightarrow \quad k = \ln\left(\frac{89}{58}\right).$$

Hence the function

$$P(t) = 58e^{t\ln(89/58)}.$$

Finally, the population in 2021 is approximately

$$P(10) = 58e^{10\ln(89/58)} \approx 4200$$

Exponential Growth or Decay

If a quantity P changes continuously at a rate proportional to its current level, then it will be governed by a differential equation of the form

$$\frac{dP}{dt} = kP \quad \text{i.e.} \quad \frac{dP}{dt} - kP = 0.$$

Note that this equation is both separable and first order linear. If $k > 0$, P experiences **exponential growth**. If $k < 0$, then P experiences **exponential decay**.

Series Circuits: RC-circuit

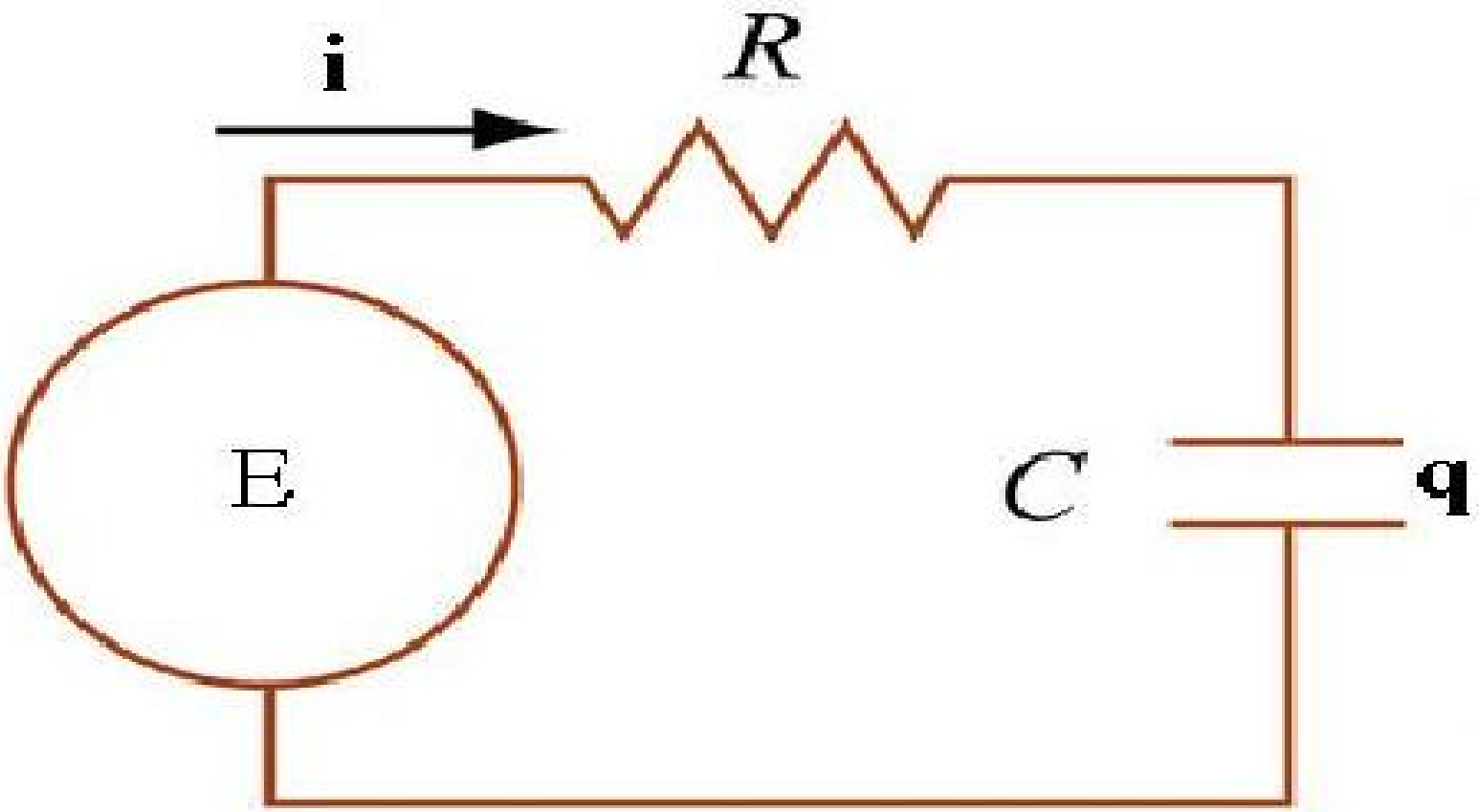


Figure: Series Circuit with Applied Electromotive force E, Resistance R, and Capcitance C. The charge of the capacitor is q and the current $i = \frac{dq}{dt}$.

Series Circuits: LR-circuit

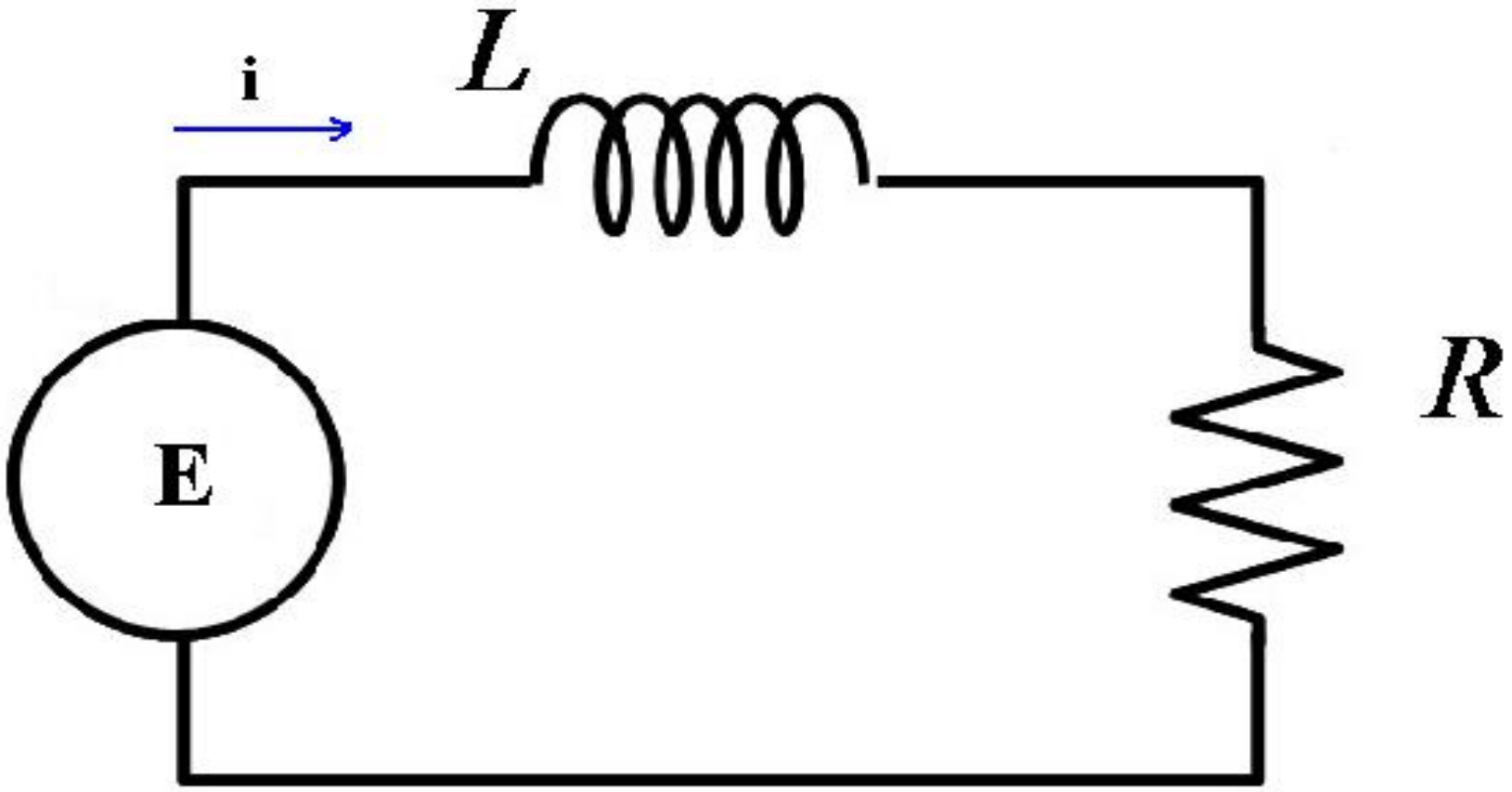


Figure: Series Circuit with Applied Electromotive force E, Inductance L, and Resistance R. The current is i.

Measurable Quantities:

Resistance R in ohms (Ω),
Inductance L in henries (h),
Capacitance C in farads (f),

Implied voltage E in volts (V),
Charge q in coulombs (C),
Current i in amperes (A)

Current is the rate of change of charge with respect to time: $i = \dfrac{dq}{dt}$.

Component	Potential Drop
Inductor	$L\frac{di}{dt}$
Resistor	Ri i.e. $R\frac{dq}{dt}$
Capacitor	$\frac{1}{C}q$

Kirchhoff's Law

The sum of the voltages around a closed circuit is zero.

In other words, the sum of potential drops across the passive components is equal to the applied electromotive force.

For an RC series circuit, this tells us that

$$\begin{array}{ccccc} \text{drop across resistor} & + & \text{drop across capacitor} & = & \text{applied force} \\ R\frac{dq}{dt} & + & \frac{1}{C}q & = & E(t) \end{array}$$

Series Circuit Equations

For an LR series circuit, we have

$$\begin{array}{ccccc} \text{drop across inductor} & + & \text{drop across resistor} & = & \text{applied force} \\ L\frac{di}{dt} & + & Ri & = & E(t) \end{array}$$

If the initial charge (RC) or initial current (LR) is known, we can solve the corresponding IVP.

(Note: We will consider LRC series circuits later as these give rise to second order ODEs.)

Example

A 200 volt battery is applied to an RC series circuit with resistance 1000Ω and capacitance 5×10^{-6} f. Find the charge $q(t)$ on the capacitor if $i(0) = 0.4A$. Determine the charge as $t \to \infty$.

Using the equation for an RC circuit $Rq' + (1/C)q = E$ we have

$$1000\frac{dq}{dt} + \frac{1}{5 \cdot 10^{-6}}q = 200, \quad q'(0) = 0.4$$

(Note that this is a slightly irregular IVP since the condition is given on $i = q'$.) In standard form the equation is $q' + 200q = 1/5$ with integrating factor $\mu = \exp(200t)$. The general solution is

$$q(t) = \frac{1}{1000} + Ke^{-200t}.$$

Example Continued...

Applying the initial condition we obtain the charge $q(t)$

$$q(t) = \frac{1}{1000} - \frac{e^{-200t}}{500}.$$

The long time charge on the capacitor is therefore

$$\lim_{t\to\infty} q(t) = \frac{1}{1000}.$$

A Classic Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

In order to answer such a question, we need to convert the problem statement into a mathematical one.

A Classic Mixing Problem

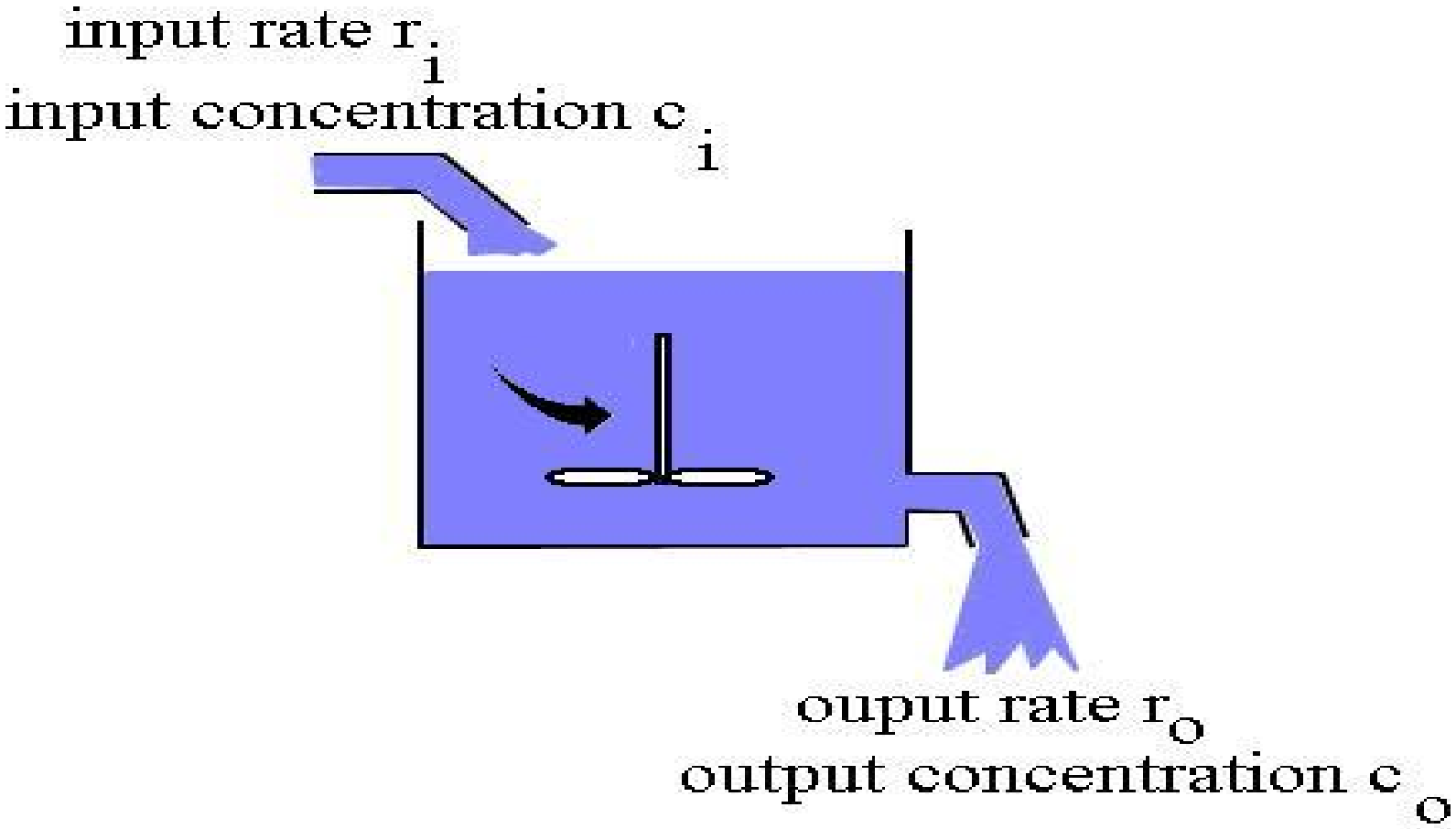


Figure: Spatially uniform composite fluids (e.g. salt & water, gas & ethanol) being mixed. Concentrations of substances change in time. The ”well mixed” condition ensures that concentrations do not change with space.

Building an Equation

The rate of change of the amount of salt

$$\frac{dA}{dt} = \left(\begin{array}{c} \textit{input rate} \\ \textit{of salt} \end{array} \right) - \left(\begin{array}{c} \textit{output rate} \\ \textit{of salt} \end{array} \right)$$

The input rate of salt is

$$\text{fluid rate in} \cdot \text{concentration of inflow} \quad = \quad r_i \cdot c_i.$$

The output rate of salt is

$$\text{fluid rate out} \cdot \text{concentration of outflow} \quad = \quad r_o \cdot c_o.$$

Building an Equation

The concentration of the outflowing fluid is

$$\frac{\text{total salt}}{\text{total volume}} = \frac{A(t)}{V(t)} = \frac{A(t)}{V(0) + (r_i - r_o)t}.$$

$$\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V}.$$

This equation is first order linear.

Note that the volume

$$V(t) = \textit{initial volume} + \textit{rate in} \times \textit{time} - \textit{rate out} \times \textit{time}.$$

If $r_i = r_o$, then $V(t) = V(0)$ a constant.

Solve the Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

We can take A in pounds, V in gallons, and t in minutes. Here, $V(0) = 500$ gal, $r_i = 5$ gal/min, $c_i = 2$ lb/gal, and $r_o = 5$ gal/min. Since the incoming and outgoing rates are the same, the volume $V(t) = 500$ gallons for all t. This gives an outgoing concentration of

$$c_o = \frac{A(t)}{V(t)} = \frac{A(t)}{500 + 5t - 5t} = \frac{A(t)}{500}.$$

Since the tank originally contains pure water (no salt), we have $A(0) = 0$.

Mixing Example

Our IVP is

$$\frac{dA}{dt} = 5\text{gal/min} \cdot 2\text{lb/gal} - 5\text{gal/min} \cdot \frac{A}{500}\text{lb/gal}, \quad A(0) = 0$$

$$\frac{dA}{dt} + \frac{1}{100}A = 10, \quad A(0) = 0.$$

The IVP has solution

$$A(t) = 1000\left(1 - e^{-t/100}\right).$$

The concentration c of salt in the tank after five minutes is therefore

$$c = \frac{A(5)\,\text{lb}}{V(5)\,\text{gal}} = \frac{1000(1 - e^{-5/100})}{500}\,\text{lb/gal} \approx 0.01\,\text{lb/gal}.$$

A Nonlinear Modeling Problem

A population $P(t)$ of tilapia changes at a rate jointly proportional to the current population and the difference between the constant carrying capacity[12] M of the environment and the current population. Determine the differential equation satsified by P.

To say that P has a rate of change jointly proportional to P and the difference between P and M is

$$\frac{dP}{dt} \propto P(M-P) \quad \text{i.e.} \quad \frac{dP}{dt} = kP(M-P)$$

for some constant of proportionality k.

[12] The carrying capacity is the maximum number of individuals that the environment can support due to limitation of space and resources.

Logistic Differential Equation

The equation

$$\frac{dP}{dt} = kP(M - P), \quad k, M > 0$$

is called a **logistic growth equation**.
Solve this equation[13] and show that for any $P(0) \neq 0$, $P \to M$ as $t \to \infty$.

The equation is separable. The general solution to the DE is

$$P(t) = \frac{MCe^{Mkt}}{1 + Ce^{Mkt}}.$$

[13] The partial fraction decomposition

$$\frac{1}{P(M-P)} = \frac{1}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right)$$

is useful.

Logistic Growth: $P'(t) = kP(M - P)$ $P(0) = P_0$

Applying the condition $P(0) = P_0$ to find the constant C, we obtain the solution to the IVP

$$P(t) = \frac{P_0 M e^{Mkt}}{M - P_0 + P_0 e^{Mkt}} = \frac{P_0 M}{(M - P_0)e^{-Mkt} + P_0}.$$

If $P_0 = 0$, then $P(t) = 0$ for all t. Otherwise, we can take the limit as $t \to \infty$ to obtain

$$\lim_{t\to\infty} P(t) = \frac{P_0 M}{0 + P_0} = M$$

as expected.

Section 6: Linear Equations: Theory and Terminology

Recall that an nth order linear IVP consists of an equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

to solve subject to conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}.$$

The problem is called **homogeneous** if $g(x) \equiv 0$. Otherwise it is called **nonhomogeneous**.

Theorem: Existence & Uniqueness

Theorem: If $a_0, \ldots, a_n$ and g are continuous on an interval I, $a_n(x) \neq 0$ for each x in I, and x_0 is any point in I, then for any choice of constants $y_0, \ldots, y_{n-1}$, the IVP has a unique solution $y(x)$ on I.

Put differently, we're guaranteed to have a solution exist, and it is the only one there is!

Example

Use only a little clever intuition to solve the IVP

$$y'' + 3y' - 2y = 0, \quad y(0) = 0, \quad y'(0) = 0$$

Exercise left to the reader. (Hint: Think *simple*. While we usually consider the initial conditions at the end, it may help to think about them first.)

A Second Order Linear Boundary Value Problem

consists of a problem

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \quad a < x < b$$

to solve subject to a pair of conditions[14]

$$y(a) = y_0, \quad y(b) = y_1.$$

However similar this is in appearance, the existence and uniqueness result **does not hold** for this BVP!

[14] Other conditions on y and/or y' can be imposed. The key characteristic is that conditions are imposed at both end points $x = a$ and $x = b$.

BVP Example

Consider the three similar BVPs:

$$(1) \quad y'' + 4y = 0, \quad 0 < x < \frac{\pi}{4} \quad y(0) = 0, \quad y\left(\frac{\pi}{4}\right) = 0.$$

$$(2) \quad y'' + 4y = 0, \quad 0 < x < \frac{\pi}{2} \quad y(0) = 0, \quad y\left(\frac{\pi}{2}\right) = 0.$$

$$(3) \quad y'' + 4y = 0, \quad 0 < x < \frac{\pi}{2} \quad y(0) = 0, \quad y\left(\frac{\pi}{2}\right) = 1.$$

BVP Examples

All solutions of the ODE $y'' + 4y = 0$ are of the form

$$y = c_1 \cos(2x) + c_2 \sin(2x).$$

It can readily be shown (do it!) that

- Problem (1) has exactly one solution $y = 0$.
- Problem (2) has infinitely many solutions $y = c_2 \sin(2x)$ where c_2 is any real number.
- And problem (3) has no solutions.

Homogeneous Equations

We'll consider the equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

and assume that each a_i is continuous and a_n is never zero on the interval of interest.

Theorem: If $y_1, y_2, \ldots, y_k$ are all solutions of this homogeneous equation on an interval I, then the *linear combination*

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x)$$

is also a solution on I for any choice of constants $c_1, \ldots, c_k$.

This is called the **principle of superposition**.

Corollaries

(i) If y_1 solves the homogeneous equation, the any constant multiple $y = cy_1$ is also a solution.

(ii) The solution $y = 0$ (called the trivial solution) is always a solution to a homogeneous equation.

Big Questions:

- Does an equation have any **nontrivial** solution(s), and
- since y_1 and cy_1 aren't truly *different* solutions, what criteria will be used to call solutions distinct?

Linear Dependence

Definition: A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ are said to be **linearly dependent** on an interval I if there exists a set of constants $c_1, c_2, \ldots, c_n$ with at least one of them being nonzero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all} \quad x \text{ in } I.$$

A set of functions that is not linearly dependent on I is said to be **linearly independent** on I.

Note: It is always possible to form the above linear combination to obtain 0 by simply taking all of the coefficients $c_i = 0$. The question here is whether it is possible to have at least one of the c's be nonzero. If so, the functions are linearly Dependent.

Example: A linearly Dependent Set

The functions $f_1(x) = \sin^2 x$, $f_2(x) = \cos^2 x$, and $f_3(x) = 1$ are linearly dependent on $I = (-\infty, \infty)$.

Making use of our knowledge of Pythagorean IDs, we can take $c_1 = c_2 = 1$ and $c_3 = -1$ (this isn't the only choice, but it will do the trick). Note that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = \sin^2 x + \cos^2 x - 1 = 0 \quad \text{for all real } x.$$

Example: A linearly Independent Set

The functions $f_1(x) = \sin x$ and $f_2(x) = \cos x$ are linearly independent on $I = (-\infty, \infty)$.

Suppose $c_1 f_1(x) + c_2 f_2(x) = 0$ for all real x. Then the equation must hold when $x = 0$, and it must hold when $x = \pi/2$. Consequently

$$c_1 \cos(0) + c_2 \sin(0) = 0 \quad \Longrightarrow \quad c_1 = 0 \quad \text{and}$$

$$0 \cdot \cos(\pi/2) + c_2 \sin(\pi/2) = 0 \quad \Longrightarrow \quad c_2 = 0.$$

We see that the only way for our linear combination to be zero is for both coefficients to be zero. Hence the functions are linearly independent.

Determine if the set is Linearly Dependent or Independent

$$f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2$$

Looking at the functions, we should suspect that they are linearly dependent. Why? Because f_3 is a linear combination of f_1 and f_2. In fact,

$$f_3(x) = \frac{1}{4}f_2(x) - f_1(x) \quad \text{i.e.} \quad f_1(x) - \frac{1}{4}f_2(x) + f_3(x) = 0$$

for all real x. The latter is our linear combination with $c_1 = c_3 = 1$ and $c_2 = -\frac{1}{4}$ (not all zero).

With only two or three functions, we may be able to intuit linear dependence/independence. What follows will provide an alternative method.

Definition of Wronskian

Let $f_1, f_2, \ldots, f_n$ posses at least $n-1$ continuous derivatives on an interval I. The **Wronskian** of this set of functions is the determinant

$$W(f_1, f_2, \ldots, f_n)(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

(Note that, in general, this Wronskian is a function of the independent variable x.)

Determine the Wronskian of the Functions

$$f_1(x) = \sin x, \quad f_2(x) = \cos x$$

$$W(f_1, f_2)(x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$$= \sin x(-\sin x) - \cos x(\cos x)$$

$$= -\sin^2 x - \cos^2 x = -1$$

Determine the Wronskian of the Functions

$$f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2$$

$$W(f_1, f_2, f_3)(x) = \begin{vmatrix} x^2 & 4x & x - x^2 \\ 2x & 4 & 1 - 2x \\ 2 & 0 & -2 \end{vmatrix}$$

$$2\begin{vmatrix} 4x & x - x^2 \\ 4 & 1 - 2x \end{vmatrix} + (-2)\begin{vmatrix} x^2 & 4x \\ 2x & 4 \end{vmatrix}$$

$$= 2(-4x^2) - 2(-4x^2) = 0$$

Theorem (a test for linear independence)

Let $f_1, f_2, \ldots, f_n$ be $n-1$ times continuously differentiable on an interval I. If there exists x_0 in I such that $W(f_1, f_2, \ldots, f_n)(x_0) \neq 0$, then the functions are **linearly independent** on I.

If $y_1, y_2, \ldots, y_n$ are n solutions of the linear homogeneous n^{th} order equation on an interval I, then the solutions are **linearly independent** on I if and only if $W(y_1, y_2, \ldots, y_n)(x) \neq 0$ for[15] each x in I.

[15] For solutions of one linear homogeneous ODE, the Wronskian is either always zero or is never zero.

Determine if the functions are linearly dependent or independent:

$$y_1 = e^x, \quad y_2 = e^{-2x} \quad I = (-\infty, \infty)$$

Exercise left to the reader. (Hint: Use the Wronskian.)

Fundamental Solution Set

We're still considering this equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

with the assumptions $a_n(x) \neq 0$ and $a_i(x)$ are continuous on I.

Definition: A set of functions $y_1, y_2, \ldots, y_n$ is a **fundamental solution set** of the nth order homogeneous equation provided they

(i) are solutions of the equation,

(ii) there are n of them, and

(iii) they are linearly independent.

Theorem: Under the assumed conditions, the equation has a fundamental solution set.

General Solution of nth order Linear Homogeneous Equation

Let $y_1, y_2, \ldots, y_n$ be a fundamental solution set of the nth order linear homogeneous equation. Then the **general solution** of the equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Example

Verify that $y_1 = e^x$ and $y_2 = e^{-x}$ form a fundamental solution set of the ODE

$$y'' - y = 0 \quad \text{on} \quad (-\infty, \infty),$$

and determine the general solution.

Note that

$$(i) \quad y_1'' - y_1 = e^x - e^x = 0 \quad \text{and} \quad y_2'' - y_2 = e^{-x} - e^{-x} = 0.$$

Also note that (ii) we have two solutions for this second order equation. And finally (iii)

$$W(y_1, y_2)(x) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2 \neq 0.$$

Hence the functions are linearly independent. We have a fundamental solution set, and the general solution is

$$y = c_1 e^x + c_2 e^{-x}.$$

Nonhomogeneous Equations

Now we will consider the equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

where g is not the zero function. We'll continue to assume that a_n doesn't vanish and that a_i and g are continuous.

The **associated homogeneous equation** is

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

Theorem: General Solution of Nonhomogeneous Equation

Let y_p be any solution of the nonhomogeneous equation, and let y_1, $y_2, \ldots, y_n$ be any fundamental solution set of the associated homogeneous equation.

Then the general solution of the nonhomogeneous equation is

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x)$$

where c_1, $c_2, \ldots, c_n$ are arbitrary constants.

Note the form of the solution $y_c + y_p$!
(complementary plus particular)

Another Superposition Principle (for nonhomogeneous eqns.)

Let $y_{p_1}, y_{p_2}, \ldots, y_{p_k}$ be k particular solutions to the nonhomogeneous linear equations

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g_i(x)$$

for $i = 1, \ldots, k$. Assume the domain of definition for all k equations is a common interval I.

Then

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_k}$$

is a particular solution of the nonhomogeneous equation

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_0(x)y = g_1(x) + g_2(x) + \cdots + g_k(x).$$

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(a) Verify that

$$y_{p_1} = 6 \quad \text{solves} \quad x^2y'' - 4xy' + 6y = 36.$$

Exercise left to the reader.

(b) Verify that

$$y_{p_2} = -7x \quad \text{solves} \quad x^2y'' - 4xy' + 6y = -14x.$$

Exercise left to the reader.

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

It can be readily verified that $y_1 = x^2$ and $y_2 = x^3$ is a fundamental solution set of

$$x^2y'' - 4xy' + 6y = 0.$$

Use this along with results (a) and (b) to write the general solution of $x^2y'' - 4xy' + 6y = 36 - 14x$.

By the definition of the general solution along with the principle of superposition, we have

$$y = c_1x^2 + c_2x^3 + 6 - 7x.$$

Section 7: Reduction of Order

We'll focus on second order, linear, homogeneous equations. Recall that such an equation has the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

Let us assume that $a_2(x) \neq 0$ on the interval of interest. We will write our equation in **standard form**

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

where $P = a_1/a_2$ and $Q = a_0/a_2$.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Recall that every fundmantal solution set will consist of two linearly independent solutions y_1 and y_2, and the general solution will have the form

$$y = c_1 y_1(x) + c_2 y_2(x).$$

Suppose we happen to know one solution $y_1(x)$. **Reduction of order** is a method for finding a second linearly independent solution $y_2(x)$ that starts with the assumption that

$$y_2(x) = u(x)y_1(x)$$

for some function $u(x)$. The method involves finding the function u.

Reduction of Order

Consider the equation **in standard form** with one known solution. Determine a second linearly independent solution.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \quad y_1(x) - -\text{is known.}$$

We begin by assuming that $y_2 = u(x)y_1(x)$ for some yet to be determined function $u(x)$. Note then that

$$y_2' = u'y_1 + uy_1', \quad \text{and} \quad y_2'' = u''y_1 + 2u'y_1' + uy_1''.$$

Since y_2 must solve the homogeneous equation, we can substitute the above into the equation to obtain a condition on u.

$$y_2'' + Py_2' + Qy_2 = u''y_1 + (2y_1' + Py_1)u' + (y_1'' + Py_1' + Qy_1)u = 0.$$

Reduction of Order

Since y_1 is a solution of the homogeneous equation, the last expression in parentheses is zero. So we obtain an equation for u

$$u''y_1 + (2y_1' + Py_1)u' = 0.$$

While this appears as a second order equation, the absences of u makes this equation first order in u' (hence the name *reduction of order*). If we let $w = u'$, we can express the equation in standard form (assuming y_1 doesn't vanish)

$$w' + (2y_1'/y_1 + P)w = 0.$$

This first order linear equation has a solution

$$w = \frac{\exp\left(-\int P(x)\,dx\right)}{y_1^2}.$$

Reduction of Order

With w determined, we integrate once to obtain u and conclude that a second linearly independent solution

$$y_2(x) = y_1(x) \int \frac{\exp\left(-\int P(x)\,dx\right)}{(y_1(x))^2}\,dx.$$

Reduction of Order Formula

For the second order, homogeneous equation **in standard form** with one known solution y_1, a second linearly independent solution y_2 is given by

$$y_2 = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx$$

Example

Find the general solution of the ODE given one known solution

$$x^2y'' - 3xy' + 4y = 0, \quad x > 0, \quad y_1 = x^2$$

In standard form, the equation is

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0 \quad \text{so that} \quad P(x) = -\frac{3}{x}.$$

Hence

$$-\int P(x)\,dx = -\int \left(-\frac{3}{x}\right)\,dx = 3\ln(x) = \ln x^3.$$

A second solution is therefore

$$y_2 = x^2 \int \frac{\exp(\ln x^3)}{(x^2)^2}\,dx = x^2 \int \frac{x^3}{x^4}\,dx = x^2 \int \frac{dx}{x} = x^2 \ln x.$$

Note that we can take the constant of integration here to be zero (why?). The general solution of the ODE is

$$y = c_1x^2 + c_2x^2 \ln x.$$

Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order, linear, homogeneous equation with constant coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

Question: What sort of function y could be expected to satisfy

$$y'' = \text{constant}\, y' + \text{constant}\, y?$$

We look for solutions of the form $y = e^{mx}$ with m constant.

If $y = e^{mx}$, then $y' = me^{mx}$ and $y'' = m^2 e^{mx}$. Upon substitution into the DE we get

$$\begin{aligned} 0 &= ay'' + by' + cy \\ &= am^2 e^{mx} + bme^{mx} + ce^{mx} \\ &= (am^2 + bm + c)e^{mx} \end{aligned}$$

Noting that the exponential is never zero, the truth of the above equation requires m to satisfy

$$am^2 + bm + c = 0.$$

This quadratic equation is called the *characteristic* or *auxiliary* equation. The polynomial on the left side is the *characteristic* or *auxiliary* polynomial.

Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

There are three cases:

I $b^2 - 4ac > 0$ and there are two distinct real roots $m_1 \neq m_2$

II $b^2 - 4ac = 0$ and there is one repeated real root $m_1 = m_2 = m$

III $b^2 - 4ac < 0$ and there are two roots that are complex conjugates $m_{1,2} = \alpha \pm i\beta$

Case I: Two distinct real roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac > 0$$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \quad \text{where} \quad m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Show that $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ are linearly independent.

Exercise left to the reader.

Case II: One repeated real root

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0$$

$$y = c_1 e^{mx} + c_2 x e^{mx} \quad \text{where} \quad m = \frac{-b}{2a}$$

Use reduction of order to show that if $y_1 = e^{\frac{-bx}{2a}}$, then $y_2 = xe^{\frac{-bx}{2a}}$.

Exercise left to the reader.

Case III: Complex conjugate roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac < 0$$

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)), \quad \text{where the roots}$$

$$m = \alpha \pm i\beta, \quad \alpha = \frac{-b}{2a} \quad \text{and} \quad \beta = \frac{\sqrt{4ac - b^2}}{2a}$$

The solutions can initially be written as

$$Y_1 = e^{(\alpha + i\beta)x} = e^{\alpha x} e^{i\beta x}, \quad \text{and} \quad Y_2 = e^{(\alpha - i\beta)x} = e^{\alpha x} e^{-i\beta x}.$$

Note: We are not interested in solutions expressed as complex valued functions. So we wish to rewrite the above in terms of real valued functions of our real variable.

Deriving the solutions Case III

Recall Euler's Formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

We can express Y_1 and Y_2 as

$$Y_1 = e^{\alpha x}e^{i\beta x} = e^{\alpha x}(\cos(\beta x) + i\sin(\beta x))$$

$$Y_2 = e^{\alpha x}e^{-i\beta x} = e^{\alpha x}(\cos(\beta x) - i\sin(\beta x))$$

Now apply the principle of superposition and set

$$y_1 = \frac{1}{2}(Y_1 + Y_2) = e^{\alpha x}\cos(\beta x), \quad \text{and}$$

$$y_2 = \frac{1}{2i}(Y_1 - Y_2) = e^{\alpha x}\sin(\beta x).$$

Examples

Solve the ODE

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 6x = 0$$

The characteristic equation is

$$m^2 + 4m + 6 = 0 \quad \text{with roots} \quad -2 \pm \sqrt{2}i.$$

This is the complex conjugate case with $\alpha = -2$ and $\beta = \sqrt{2}$.. The general solution is therefore

$$y = c_1 e^{-2x}\cos(\sqrt{2}x) + c_2 e^{-2x}\sin(\sqrt{2}x).$$

Examples

Solve the ODE

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} - 5x = 0$$

The characteristic equation is

$$m^2 + 4m - 5 = 0 \quad \text{with roots} \quad -5, 1.$$

This is the two distinct real roots case. Hence $y_1 = e^{-5x}$, $y_2 = e^x$, and the general solution is therefore

$$y = c_1 e^{-5x} + c_2 e^x.$$

Examples

Solve the ODE

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 0$$

The characteristic equation is

$$m^2 + 4m + 4 = 0 \quad \text{with root} \quad -2 \text{ (repeated)}$$

This is the one repeated real root case. Hence $y_1 = e^{-2x}$, $y_2 = xe^{-2x}$, and the general solution is therefore

$$y = c_1 e^{-2x} + c_2 x e^{-2x}.$$

Higer Order Linear Constant Coefficient ODEs

- The same approach applies. For an nth order equation, we obtain an nth degree polynomial.
- Complex roots must appear in conjugate pairs (due to real coefficients) giving a pair of solutions $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$.
- If a root m is repeated k times, we get k linearly independent solutions

$$e^{mx}, \quad xe^{mx}, \quad x^2e^{mx}, \quad \ldots, \quad x^{k-1}e^{mx}$$

  or in conjugate pairs cases $2k$ solutions

$$e^{\alpha x}\cos(\beta x),\ e^{\alpha x}\sin(\beta x), \quad xe^{\alpha x}\cos(\beta x),\ xe^{\alpha x}\sin(\beta x), \ldots,$$

$$x^{k-1}e^{\alpha x}\cos(\beta x),\ x^{k-1}e^{\alpha x}\sin(\beta x)$$

- It may require a computer algebra system to find the roots for a high degree polynomial.

Example

Solve the ODE

$y''' - 4y' = 0$

The characteristic equation is

$$m^3 - 4m = 0 \quad \text{with roots} \quad -2, 0, 2.$$

A fundamental solution set is $y_1 = e^{-2x}$, $y_2 = e^{0x} = 1$, and $y_3 = e^{2x}$. The general solution is therefore

$$y = c_1 e^{-2x} + c_2 + c_3 e^{2x}.$$

Note that as expected, this third order equation has a fundamental solution set consisting of three linearly independent functions.

Example

Solve the ODE

$$y''' - 3y'' + 3y' - y = 0$$

The characteristic equation is

$$m^3 - 3m^2 + 3m - 1 = 0 \quad \text{with root} \quad 1 \text{ (repeated)}.$$

Every fundamental solution set must contain three linearly independent functions. The method for repeated roots in the second order case extends nicely to higher order equations. A fundamental solution set is

$$y_1 = e^x, \quad y_2 = xe^x, \quad y_3 = x^2 e^x.$$

The general solution is

$$y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x.$$

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Motivating Example

Find a particular solution of the ODE

$$y'' - 4y' + 4y = 8x + 1$$

We may guess that since $g(x)$ is a first degree polynomial, perhaps our particuar solution y_p is also.[16]. That is

$$y_p = Ax + B$$

for some pair of constants A, B. We can substitute this into the DE. We'll use that

$$y_p' = A, \quad \text{and} \quad y_p'' = 0.$$

[16] Note that this is an educated guess. If it doesn't work out, we can sigh and try something else. If it does work, we owe no apologies for starting with a guess.

$$y'' - 4y' + 4y = 8x + 1$$

Plugging y_p into the ODE we get

$$\begin{aligned} 8x + 1 &= y_p'' - 4y_p' + 4y_p \\ &= 0 - 4(A) + 4(Ax + B) \\ &= 4Ax + (-4A + 4B) \end{aligned}$$

We have first degree polynomials on both sides of the equation. They are equal if and only if they have the same corresponding coefficients. Matching the coefficients of x and the constants on the left and right we get the pair of equations

$$\begin{array}{rcrcl} 4A & & & = & 8 \\ -4A & + & 4B & = & 1 \end{array}$$

This has solution $A = 2$, $B = 9/4$. We've found a particular solution

$$y_p = 2x + \frac{9}{4}.$$

The Method of Undetermined Coefficients

As the previous example suggests, this method entails guessing that the particular solution has the same basic *form* as the right hand side function $g(x)$. We must keep in mind that the idea of *form* should be considered in the most general context. The following *forms* arise

- n^{th} degree polynomial,
- exponentials e^{mx} for m constant,
- a linear combination of $\sin(mx)$ AND $\cos(mx)$ for some constant m,
- a product of any two or three of the above

When we assume a form for y_p we leave unspecified *coefficients* (hence the name of the method). The values of factors such as m inside an exponential, sine, or cosine are fixed by reference to the function g.

The Method of Undetermined Coefficients

The gist of the method is that we assume that y_p matches g in form and

- account for each type of term that appears in g or can arise through differentiation[17]
- avoid any unwarranted conditions imposed upon coefficients[18].

[17] e.g. sines and cosines give rise to one another when derivatives are taken. Hence they should be consdered together in linear combinations.

[18] Most notably we don't assume *a priori* values for coefficients in our linear combinations and don't force them to have fixed relationships to one another prior to fitting them to the ODE

Examples of Forms of y_p based on g (Trial Guesses)

(a) $g(x) = 1$ (or really any constant) This is a degree zero polynomial.

$y_p = A$

(b) $g(x) = x - 7$ This is a degree 1 polynomial.

$y_p = Ax + B$

(c) $g(x) = 5x$ This is a degree 1 polynomial.

$y_p = Ax + B$

(d) $g(x) = 3x^3 - 5$ This is a degree 3 polynomial.

$y_p = Ax^3 + Bx^2 + Cx + D$

(e) $g(x) = xe^{3x}$ A degree1 polynomial times an exponential e^{3x}.

$y_p = (Ax + B)e^{3x}$

(f) $g(x) = \cos(7x)$ A linear combination of $\cos(7x)$ and $\sin(7x)$.

$y_p = A\cos(7x) + B\sin(7x)$

(g) $g(x) = \sin(2x) - \cos(4x)$ A linear combination of $\cos(2x)$ and $\sin(2x)$ plus a linear combination of $\cos(4x)$ and $\sin(4x)$.

$y_p = A\cos(2x) + B\sin(2x) + C\cos(4x) + D\sin(4x)$

(h) $g(x) = x^2\sin(3x)$ A product of a second degree polynomial and a linear combination of sines and cosines of $3x$.

$y_p = (Ax^2 + Bx + C)\cos(3x) + (Dx^2 + Ex + F)\sin(3x)$

Still More Trial Guesses

(i) $g(x) = e^x \cos(2x)$ A product of an exponential e^x and a linear combination of sines and cosines of $2x$.

$y_p = Ae^x \cos(2x) + Be^x \sin(2x)$

(j) $g(x) = x^3 e^{8x}$ A degree3 polynomial times an exponential e^{8x}.

$y_p = (Ax^3 + Bx^2 + Cx + D)e^{8x}$

(k) $g(x) = xe^{-x} \sin(\pi x)$ A product of a degree 1 polynomial, an exponential e^{-x} and a linear combination of sines and cosines of πx.

$y_p = (Ax + B)e^{-x} \cos(\pi x) + (Cx + D)e^{-x} \sin(\pi x)$

The Superposition Principle

$$y'' - y' = 20\sin(2x) + 4e^{-5x}$$

Given the theorem in section 6 regarding superposition for nonhomogeneous equations, we can consider two subproblems

$$y'' - y' = 20\sin(2x), \quad \text{and} \quad y'' - y' = 4e^{-5x}.$$

Calling the particular solutions y_{p_1} and y_{p_2}, respectively, the correct forms to guess would be

$$y_{p_1} = A\cos(2x) + B\sin(2x), \quad \text{and} \quad y_{p_2} = Ce^{-5x}$$

It can be shown (details left to the reader) that $A = 2$, $B = -4$ and $C = 2/15$.

The Superposition Principle

$$y'' - y' = 20\sin(2x) + 4e^{-5x}$$

The particular solution is

$$y = y_{p_1} + y_{p_2} = 2\cos(2x) - 4\sin(2x) + \frac{2e^{-5x}}{15}.$$

The general solution to the ODE is

$$y = c_1 e^x + c_2 + 2\cos(2x) - 4\sin(2x) + \frac{2e^{-5x}}{15}.$$

A Glitch!

$$y'' - y' = 3e^x$$

Here we note that $g(x) = 3e^x$ is a constant times e^x. So we may guess that our particular solution

$$y_p = Ae^x.$$

When we attempt the substitution, we end up with an unsolvable problem. Note that $y_p' = Ae^x$ and $y_p'' = Ae^x$ giving upon substitution

$$\begin{aligned} 3e^x &= y_p'' - y_p' \\ &= Ae^x - Ae^x \\ &= 0 \end{aligned}$$

This requires $3 = 0$ which is always false (i.e. we can't find a value of A to make a true statement out of this result.)

A Glitch!

$$y'' - y' = 3e^x$$

The reason for our failure here comes to light by consideration of the associated homogenous equation

$$y'' - y' = 0$$

with fundamental solution set $y_1 = e^x$, $y_2 = 1$. Our initial guess of Ae^x is a solution to the associated homogeneous equation for every constant A. And for any nonzero A, we've only duplicated part of the complementary solution. Fortunately, there is a fix for this problem. Taking a hint from a previous observation involving reduction of order, we may modify our initial guess by including a factor of x. If we guess

$$y_p = Axe^x$$

we find that this actuall works. It can be shown that (details left to the reader) $A = 3$. So $y_p = 3xe^x$ is a particular solution.

We'll consider cases

Using superposition as needed, begin with the assumption:

$$y_p = y_{p_1} + \cdots + y_{p_k}$$

where y_{p_i} has the same **general form** as $g_i(x)$.

Case I: y_p as first written has no part that duplicates the complementary solution y_c. Then this first form will suffice.

Case II: y_p has a term y_{p_i} that duplicates a term in the complementary solution y_c. Multiply that term by x^n, where n is the smallest positive integer that eliminates the duplication.

Case II Example $y'' - 2y' + y = -4e^x$

The associated homogeneous equation $y'' - 2y' + y = 0$ has $y_1 = e^x$ and $y_2 = xe^x$ as a fundamental solution set. At first pass, we may assume that

$$y_p = Ae^x$$

based on the form of the right hand side. A quick look at y_1 and y_2 shows that this guess will not work. We might try modifying our guess as

$$y_p = Axe^x,$$

but again this just duplicates y_2. The correct modification is therefore

$$y_p = Ax^2e^x.$$

This does not duplicate the complementary solution and will work as the correct form (i.e. it is possible to find a value of A such that this function solves the nonhomogeneous ODE).

Find the form of the particular soluition

$$y'' - 4y' + 4y = \sin(4x) + xe^{2x}$$

We consider the subproblems

$$y'' - 4y' + 4y = \sin(4x) \quad \text{and} \quad y'' - 4y' + 4y = xe^{2x}$$

At first pass, we may guess particular solutions

$$y_{p_1} = A\sin(4x) + B\cos(4x), \quad \text{and} \quad y_{p_2} = (Cx + D)e^{2x}.$$

A fundamental solution set to the associated homogeneous equation is

$$y_1 = e^{2x}, \quad y_2 = xe^{2x}.$$

Comparing this set to the first part of the particular solution y_{p_1}, we see that there is no correlation between them. Hence y_{p_1} will suffice as written.

Find the form of the particular soluition

$$y'' - 4y' + 4y = \sin(4x) + xe^{2x}$$

Our guess at y_{p_2} however will fail since it contains at least one term (actually both are problematic) that solves the homogeneous equation. We may attempt a factor of x

$$y_{p_2} = x(Cx + D)e^{2x} = (Cx^2 + Dx)e^{2x}$$

to fix the problem. However, this still contains a term (Dxe^{2x}) that duplicates the fundamental solution set. Hence we introduce another factor of x putting

$$y_{p_2} = x^2(Cx + D)e^{2x} = (Cx^3 + Dx^2)e^{2x}.$$

Now we have a workable form.

Find the form of the particular soluition

$$y'' - 4y' + 4y = \sin(4x) + xe^{2x}$$

The particular solution for the whole ODE is of the form

$$y_p = A\sin(4x) + B\cos(4x) + (Cx^3 + Dx^2)e^{2x}.$$

It is important to note that modification of y_{p_2} does not impose any necessary modification of y_{p_1}. This is due to our principle of superposition.

Solve the IVP

$$y'' - y = 4e^{-x} \quad y(0) = -1, \quad y'(0) = 1$$

The associated homogeneous equation $y'' - y = 0$ has fundamental solution set $y_1 = e^x$, $y_2 = e^{-x}$. We may guess that our particular solution

$$y_p = Ae^{-x}$$

but seeing that this duplicates y_2 we will need to modify our guess as

$$y_p = Axe^{-x}.$$

Substitution into the ODE gives $A = -2$. So our particular solution is $y_p = -2xe^{-x}$ and the general solution of the ODE is

$$y = c_1 e^x + c_2 e^{-x} - 2xe^{-x}.$$

Solve the IVP

$$y'' - y = 4e^{-x} \quad y(0) = -1, \quad y'(0) = 1$$

We apply our initial conditions to the general solution. Note that

$$y = c_1 e^x + c_2 e^{-x} - 2xe^{-x} \quad \Longrightarrow \quad y' = c_1 e^x - c_2 e^{-x} - 2e^{-x} + 2xe^{-x}.$$

So

$$y(0) = c_1 + c_2 = -1 \quad \text{and} \quad y'(0) = c_1 - c_2 - 2 = 1.$$

Solving this system of equations for c_1 and c_2 we find $c_1 = 1$ and $c_2 = -2$. The solution to the IVP is

$$y = e^x - 2e^{-x} - 2xe^{-x}.$$

Section 10: Variation of Parameters

Suppose we wish to consider the nonhomogeneous equations

$$y'' + y = \tan x \quad \text{or} \quad x^2 y'' + xy' - 4y = e^x?$$

Neither of these equations lend themselves to the method of undetermined coefficients for identification of a particular solution.

- The first one fails because $g(x) = \tan x$ does not fall into any of the classes of functions required for the method.
- The second one fails because the left hand side is not a constant coefficient equation.

We need another method!

For the equation in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = g(x),$$

suppose $\{y_1(x),\ y_2(x)\}$ is a fundamental solution set for the associated homogeneous equation. We seek a particular solution of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where u_1 and u_2 are functions[19] we will determine (in terms of y_1, y_2 and g).

This method is called **variation of parameters**.

[19] Note the similarity to $y_c = c_1y_1 + c_2y_2$. The coefficients u_1 and u_2 are varying, hence the name *variation of parameters*.

Variation of Parameters: Derivation of y_p

$$y'' + P(x)y' + Q(x)y = g(x)$$

Set $\quad y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$

Note that we have two unknowns u_1, u_2 but only one equation (the ODE). Hence we will introduce a second equation. We'll do this with some degree of freedom but in a way that makes life a little bit easier. We wish to substitute our form of y_p into the ODE. Note that

$$y_p' = u_1 y_1' + u_2 y_2' + u_1' y_1 + u_2' y_2.$$

Here is that second condition: Let us assume that

$$u_1' y_1 + u_2' y_2 = 0.$$

Variation of Parameters: Derivation of y_p

$$y'' + P(x)y' + Q(x)y = g(x)$$

Given our assumption, we have

$$y_p'' = u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2''.$$

Substituting into the ODE we obtain

$$\begin{aligned} g(x) &= y_p'' + Py_p' + Qy_p \\ &= u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2'' + P(u_1y_1' + u_2y_2') \\ &\qquad + Q(u_1y_1 + u_2y_2) \\ &= u_1'y_1' + u_2'y_2' + (y_1'' + Py_1' + Qy_1)u_1 + (y_2'' + Py_2' + Qy_2)u_2 \end{aligned}$$

Remember that $\quad y_i'' + P(x)y_i' + Q(x)y_i = 0, \quad$ for $i = 1, 2$

Variation of Parameters: Derivation of y_p

$$y'' + P(x)y' + Q(x)y = g(x)$$

Since the last terms cancel, we obtain a second equation for our u's: $u_1'y_1' + u_2'y_2' = g$. We have a system

$$\begin{array}{ccccc} u_1'y_1 & + & u_2'y_2 & = & 0 \\ u_1'y_1' & + & u_2'y_2' & = & g \end{array}$$

We may express this using a convenient matrix formalism as

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}$$

The coefficient matrix on the left should be familiar!

Variation of Parameters: Derivation of y_p

$$y'' + P(x)y' + Q(x)y = g(x)$$

Finally, we may solve this using Crammer's Rule to obtain

$$u_1' = \frac{W_1}{W} = \frac{-y_2 g}{W} \quad u_2' = \frac{W_2}{W} = \frac{y_1 g}{W}$$

where

$$W_1 = \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix}$$

and W is the Wronskian of y_1 and y_2. We simply integrate to obtain u_1 and u_2.

Example:

Solve the ODE $y'' + y = \tan x$.

The associated homogeneous equation has fundamental solution[20] set $y_1 = \cos x$ and $y_2 = \sin x$. The Wronskian $W = 1$. The equation is already in standard form, so our $g(x) = \tan x$. We have

$$u_1 = \int \frac{-y_2 g}{W}\,dx = \int -\frac{\sin x \tan x}{1}\,dx = \sin x - \ln|\sec x + \tan x|$$

$$u_2 = \int \frac{y_1 g}{W}\,dx = \int \frac{\cos x \tan x}{1}\,dx = -\cos x$$

[20] How we number the functions in the fundamental solution set is completely arbitrary. However, the designations are important for finding our u's and constructing our y_p. So we pick an ordering at the beginning and stick with it.

Example Continued...

Solve the ODE $y'' + y = \tan x$.

The particular solution is

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 \\ &= (\sin x - \ln|\sec x + \tan x|)\cos x - \cos x \sin x \\ &= -\cos x \ln|\sec x + \tan x|. \end{aligned}$$

And the general solution is therefore

$$y = c_1 \cos x + c_2 \sin x - \cos x \ln|\sec x + \tan x|.$$

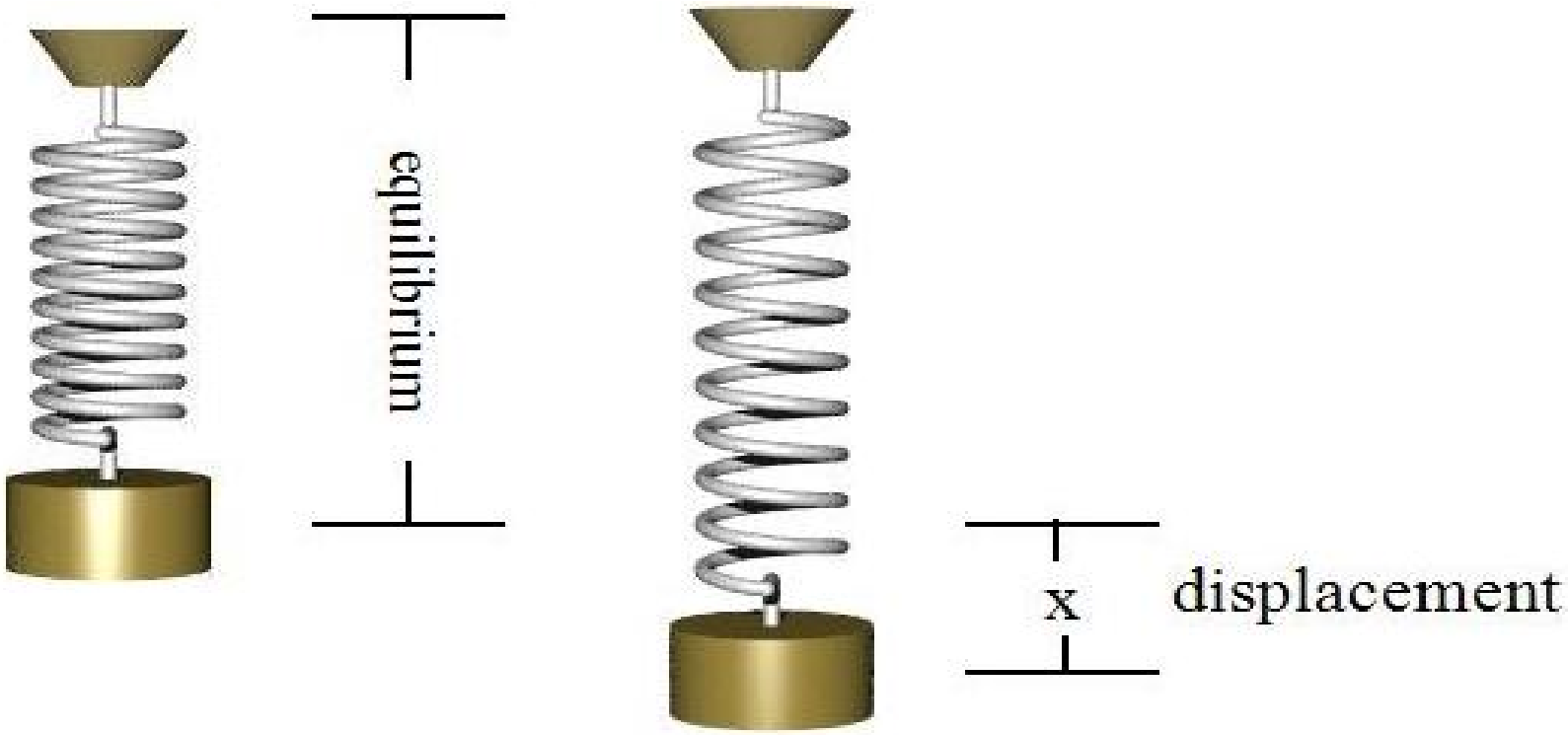
Section 11: Linear Mechanical Equations

Simple Harmonic Motion

We consider a flexible spring from which a mass is suspended. In the absence of any damping forces (e.g. friction, a dash pot, etc.), and free of any external driving forces, any initial displacement or velocity imparted will result in **free, undamped motion**–a.k.a. **simple harmonic motion**.

Harmonic Motion gif

Building an Equation: Hooke's Law



At equilibrium, displacement $x(t) = 0$.

Hooke's Law: $F_{spring} = k\,x$

Figure: In the absence of any displacement, the system is at equilibrium. Displacement $x(t)$ is measured from equilibrium $x = 0$.

Building an Equation: Hooke's Law

Newton's Second Law: $F = ma$ Force = mass times acceleration

$$a = \frac{d^2x}{dt^2} \quad \Longrightarrow \quad F = m\frac{d^2x}{dt^2}$$

Hooke's Law: $F = kx$ Force exerted by the spring is proportional to displacement
The force imparted by the spring opposes the direction of motion.

$$m\frac{d^2x}{dt^2} = -kx \quad \Longrightarrow \quad x'' + \omega^2 x = 0 \quad \text{where} \quad \omega = \sqrt{\frac{k}{m}}$$

Convention We'll Use: Up will be positive ($x > 0$), and down will be negative ($x < 0$). This orientation is arbitrary and follows the convention in Trench.

Obtaining the Spring Constant (US Customary Units)

If an object with weight W pounds stretches a spring δx feet from it's length with no mass attached, the by Hooke's law we compute the spring constant via the equation

$$W = k\delta x \quad \Longrightarrow \quad k = \frac{W}{\delta x}.$$

The units for k in this system of measure are lb/ft.

Note also that Weight = mass $\times$ acceleration due to gravity. Hence if we know the weight of an object, we can obtain the mass via

$$W = mg \quad \Longrightarrow \quad m = \frac{W}{g}.$$

We typically take the approximation $g = 32$ ft/sec^2. The units for mass are lb sec^2/ft which are called slugs.

Obtaining the Spring Constant (SI Units)

In SI units, the weight would be expressed in Newtons (N). The appropriate units for displacement would be meters (m). In these units, the spring constant would have units of N/m.

It is customary to describe an object by its mass in kilograms. When we encounter such a description, we deduce the weight in Newtons

$$W = mg \quad \text{taking the approximation} \quad g = 9.8\,\mathrm{m}/\mathrm{sec}^2.$$

Obtaining the Spring Constant: *Displacment in Equilibrium*

If an object stretches a spring δx units from it's length (with no object attached), we may say that it stretches the spring δx units *in equilibrium*. Applying Hooke's law with the weight as force, we have

$$mg = k\delta x.$$

We observe that the value ω can be deduced from δx by

$$\omega^2 = \frac{k}{m} = \frac{g}{\delta x}.$$

Provided that values for δx and g are used in appropriate units, ω is in units of per second.

Simple Harmonic Motion

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1 \tag{8}$$

Here, x_0 and x_1 are the initial position (relative to equilibrium) and velocity, respectively. The solution is

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t)$$

called the **equation of motion**.

Caution: The phrase *equation of motion* is used differently by different authors. Some, including Trench, use this phrase to refer the ODE of which (8) would be the example here. Others use it to refer to the **solution** to the associated IVP.

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t)$$

Characteristics of the system include

- the period $T = \frac{2\pi}{\omega}$,
- the frequency $f = \frac{1}{T} = \frac{\omega}{2\pi}$[21]
- the circular (or angular) frequency ω, and
- the amplitude or maximum displacement $A = \sqrt{x_0^2 + (x_1/\omega)^2}$

[21] Various authors call f the natural frequency and others use this term for ω.

Amplitude and Phase Shift

We can formulate the solution in terms of a single sine (or cosine) function. Letting

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) = A \sin(\omega t + \phi)$$

requires

$$A = \sqrt{x_0^2 + (x_1/\omega)^2},$$

and the **phase shift** ϕ must be defined by

$$\sin \phi = \frac{x_0}{A}, \quad \text{with} \quad \cos \phi = \frac{x_1}{\omega A}.$$

(Alternatively, we can let $x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) = A \cos(\omega t - \hat{\phi})$ in which case $\hat{\phi}$ is defined by

$$\cos \hat{\phi} = \frac{x_0}{A}, \quad \text{with} \quad \sin \hat{\phi} = \frac{x_1}{\omega A}.$$

The phase shift defined above $\phi = \frac{\pi}{2} - \hat{\phi}$.)

Example

An object stretches a spring 6 inches in equilibrium. Assuming no driving force and no damping, set up the differential equation describing this system.

Letting the displacement at time t be $x(t)$ feet, we have

$$mx'' + kx = 0 \quad \Longrightarrow \quad x'' + \omega^2 x = 0$$

where $\omega = \sqrt{k/m}$. We seek the value of ω, but we do not have the mass of the object to calculate the weight. Since the displacement is described as displacment in equilibrium, we can calculate

$$\omega = \sqrt{g/\delta x}.$$

Example Continued...

The streching is given in inches which we convert to feet. Using the appropriate value for g we have

$$\omega = \sqrt{\frac{32\,\text{ft/sec}^2}{\frac{1}{2}\,\text{ft}}} = 8\frac{1}{\text{sec}}.$$

The differential equation is therefore

$$x'' + 64x = 0.$$

Example

A 4 pound weight stretches a spring 6 inches. The mass is released from a position 4 feet above equilibrium with an initial downward velocity of 24 ft/sec. Find the equation of motion, the period, amplitude, phase shift, and frequency of the motion. (Take $g = 32$ ft/sec^2.)

We can calculate the spring constant and the mass from the given information. Converting inches to feet, we have

$$4\,\text{lb} = \frac{1}{2}\,\text{ft}k \quad \Longrightarrow \quad k = 8\text{lb/ft} \quad \text{and}$$

$$m = \frac{4\,\text{lb}}{32\,\text{ft/sec}^2} = \frac{1}{8}\,\text{slugs}.$$

The value of ω is therefore

$$\omega = \sqrt{\frac{k}{m}} = 8\,\frac{1}{\text{sec}}.$$

Example Continued...

Along with the initial conditions, we have the IVP

$$x'' + 64x = 0 \quad x(0) = 4, \quad x'(0) = -24.$$

The equation of motion is therefore

$$x(t) = 4\cos(8t) - 3\sin(8t).$$

The period and frequency are

$$T = \frac{2\pi}{8} = \frac{\pi}{4}\,\text{sec} \quad \text{and} \quad f = \frac{1}{T} = \frac{4}{\pi}\,\frac{1}{\text{sec}}.$$

The amplitude

$$A = \sqrt{4^2 + (-3)^2} = 5\,\text{ft}.$$

Example Continued...

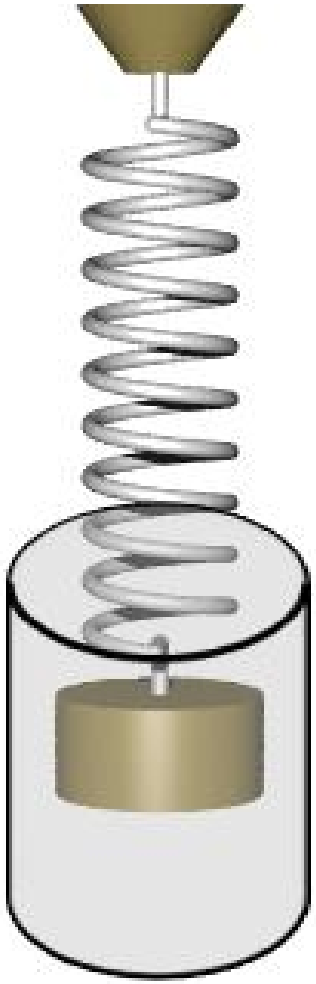
The phase shift ϕ satisfies the equations

$$\sin\phi = \frac{4}{5} \quad \text{and} \quad \cos\phi = -\frac{3}{5}.$$

We note that $\sin\phi > 0$ and $\cos\phi < 0$ indicating that ϕ is a quadrant II angle (in standard position). Taking the smallest possible positive value, we have

$$\phi \approx 2.21 \quad (\text{roughly } 127^\circ).$$

Free Damped Motion



fluid resists motion

$$F_{\text{damping}} = \beta \frac{dx}{dt}$$

$\beta > 0$ (by conservation of energy)

Figure: If a damping force is added, we'll assume that this force is proportional to the instantaneous velocity.

Free Damped Motion

Now we wish to consider an added force corresponding to damping—friction, a dashpot, air resistance.

Total Force = Force of spring + Force of damping

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx \quad \Longrightarrow \quad \frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

where

$$2\lambda = \frac{\beta}{m} \quad \text{and} \quad \omega = \sqrt{\frac{k}{m}}.$$

Three qualitatively different solutions can occur depending on the nature of the roots of the characteristic equation

$$r^2 + 2\lambda r + \omega^2 = 0 \quad \text{with roots} \quad r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}.$$

Case 1: $\lambda^2 > \omega^2$ Overdamped

$$x(t) = e^{-\lambda t}\left(c_1 e^{t\sqrt{\lambda^2-\omega^2}} + c_2 e^{-t\sqrt{\lambda^2-\omega^2}}\right)$$

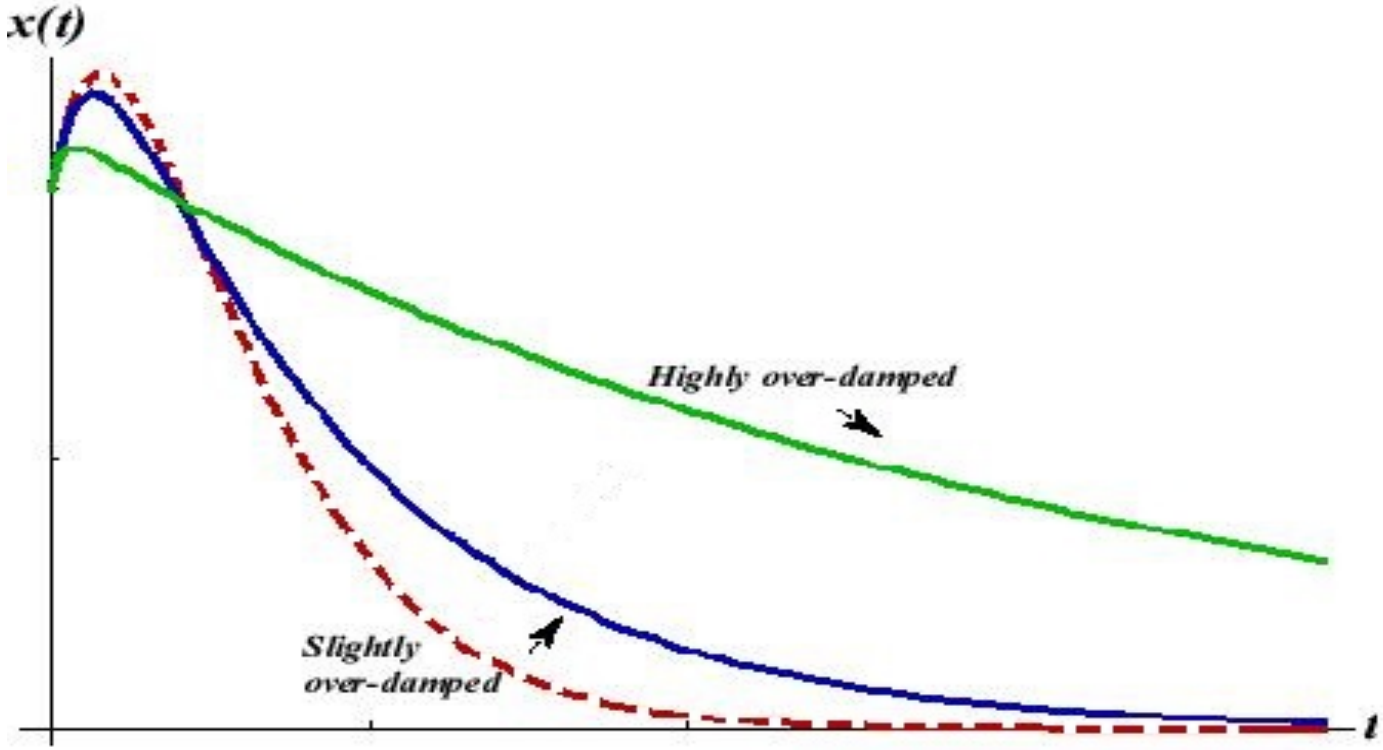


Figure: Two distinct real roots. No oscillations. Approach to equilibrium may be slow.

Case 2: $\lambda^2 = \omega^2$ Critically Damped

$$x(t) = e^{-\lambda t}(c_1 + c_2 t)$$

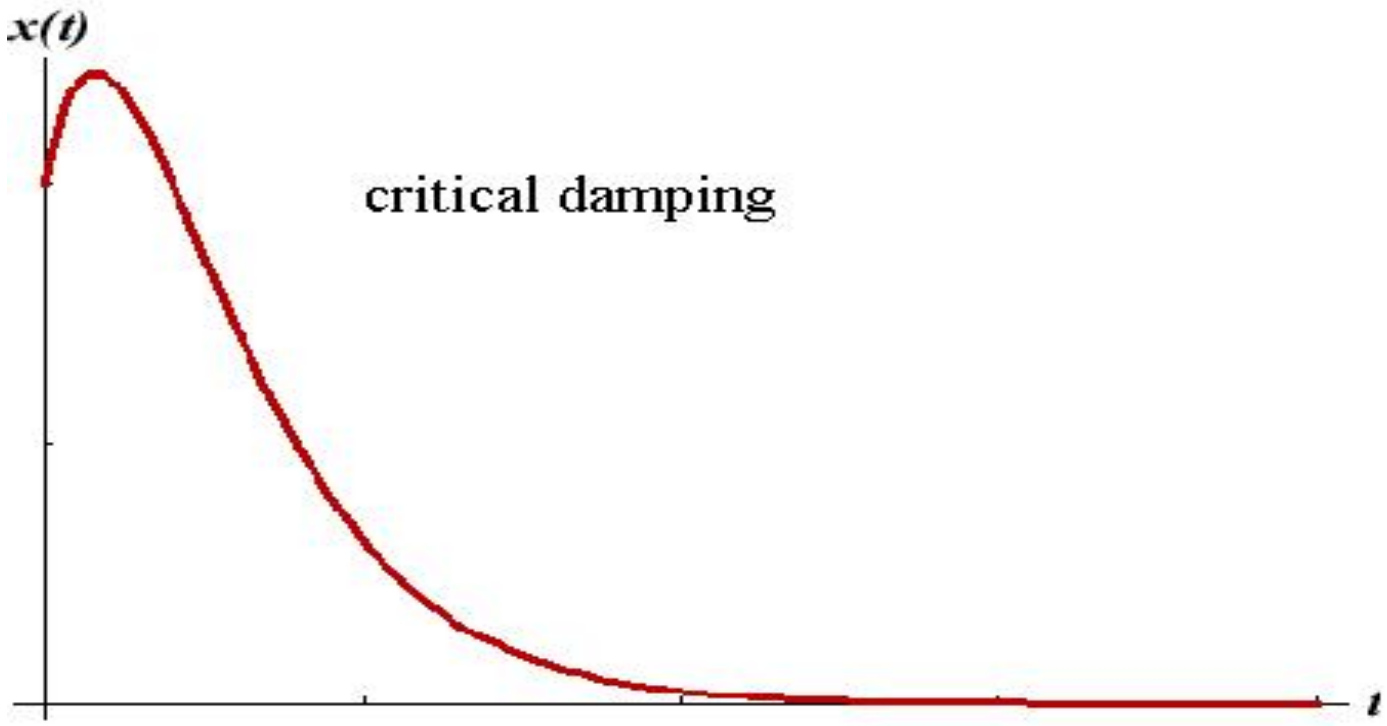


Figure: One real root. No oscillations. Fastest approach to equilibrium.

Case 3: $\lambda^2 < \omega^2$ Underdamped

$$x(t) = e^{-\lambda t}\left(c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t)\right), \quad \omega_1 = \sqrt{\omega^2 - \lambda^2}$$

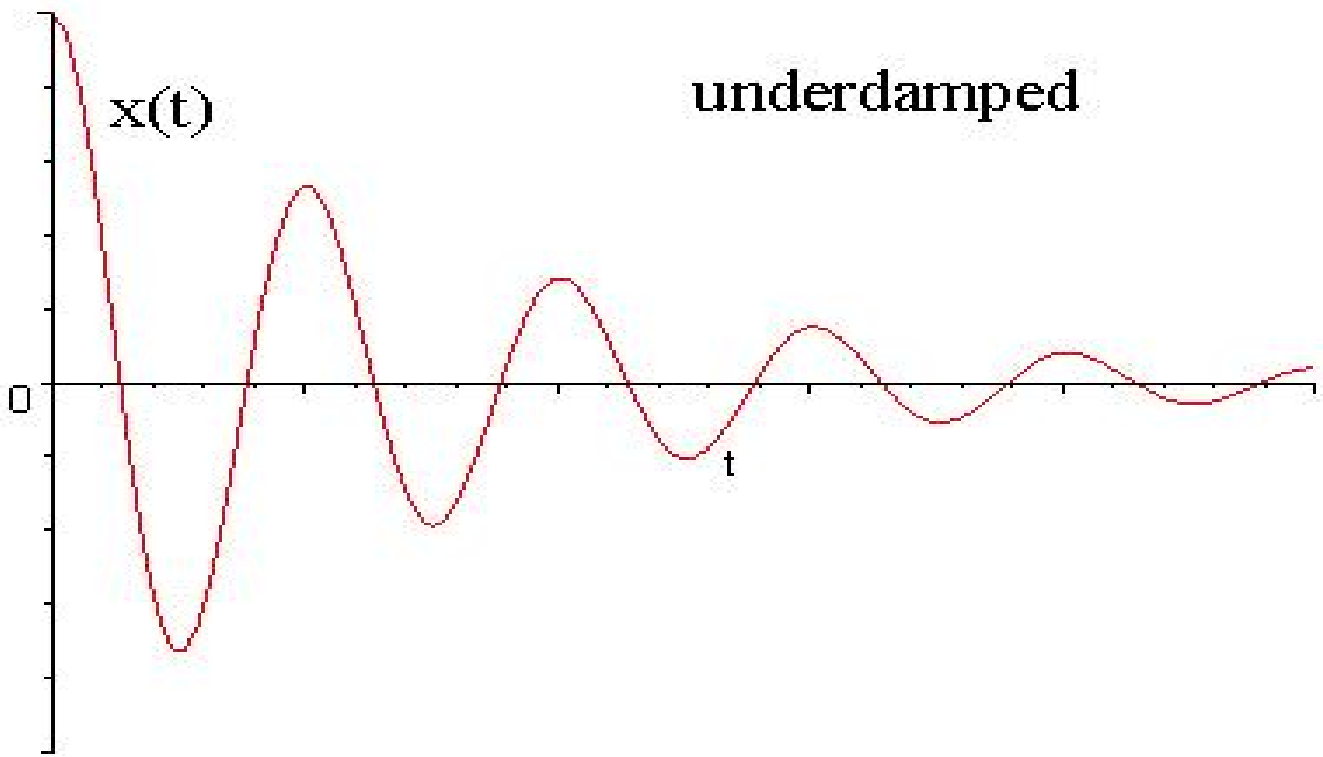


Figure: Complex conjugate roots. Oscillations occur as the system approaches (resting) equilibrium.

Damping Ratio

Engineers may refer to the *damping ratio* when determining which of the three types of damping a system exhibits. Simply put, the damping ratio is the ratio of the system damping to the critical damping for the given mass and spring constant. Calling this damping ratio ζ,

$$\zeta = \frac{\text{damping coefficient}}{\text{critical damping}} = \frac{\beta}{2\sqrt{mk}} = \frac{\lambda}{\omega}$$

Relative to this ratio, the damping cases are given by

$\zeta < 1$	under damped
$\zeta = 1$	critically damped
$\zeta > 1$	over damped

This criterion is identical. That is, if $\zeta < 1$, the characteristic equation has complex roots; if $\zeta = 1$ it has one real root, and if $\zeta > 1$ is has two real roots.

Comparison of Damping

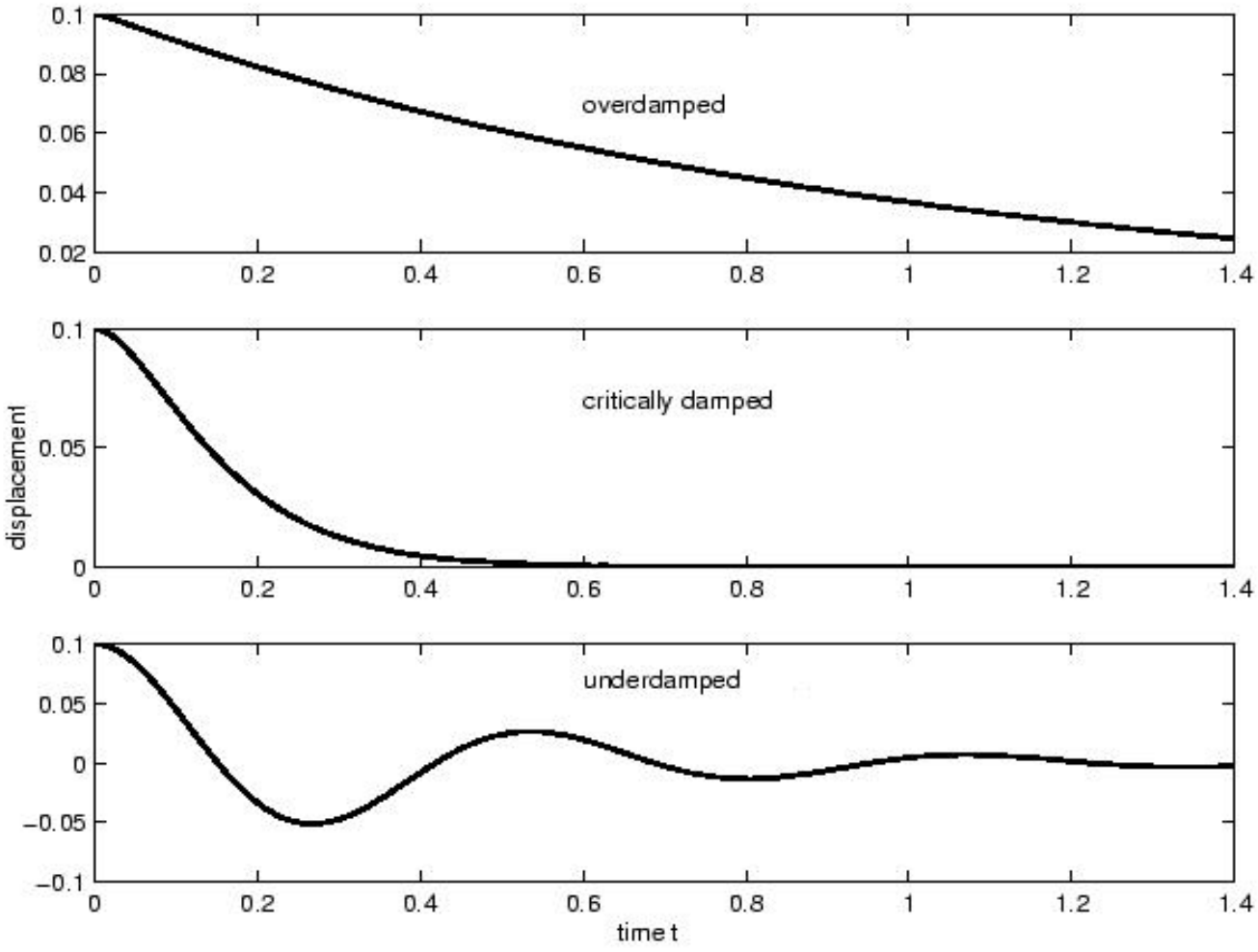


Figure: Comparison of motion for the three damping types.

Example

A 2 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 10 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped.

Our DE is

$$2x'' + 10x' + 12x = 0 \quad \Longrightarrow \quad x'' + 5x' + 6x = 0.$$

Hence

$$\lambda = \frac{5}{2} \quad \text{and} \quad \omega^2 = 6.$$

Note that

$$\lambda^2 - \omega^2 = \frac{25}{4} - 6 = \frac{1}{4} > 0.$$

This system is overdamped.

Example

A 3 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 12 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped. If the mass is released from the equilibrium position with an upward velocity of 1 m/sec, solve the resulting initial value problem.

From the description, the IVP is

$$3x'' + 12x' + 12x = 0, \quad x(0) = 0, \quad x'(0) = 1.$$

The equation of motion (solution of the IVP) is found to be

$$x(t) = te^{-2t}.$$

Driven Motion

We can consider the application of an external driving force (with or without damping). Assume a time dependent force $f(t)$ is applied to the system. The ODE governing displacement becomes

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx + f(t), \quad \beta \geq 0.$$

Divide out m and let $F(t) = f(t)/m$ to obtain the nonhomogeneous equation

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)$$

Forced Undamped Motion and Resonance

Consider the case $F(t) = F_0 \cos(\gamma t)$ or $F(t) = F_0 \sin(\gamma t)$, and $\lambda = 0$. Two cases arise

$$(1) \quad \gamma \neq \omega, \quad \text{and } (2) \quad \gamma = \omega.$$

Taking the sine case, the DE is

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

with complementary solution

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

Note that

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Using the method of undetermined coefficients, the **first guess** to the particular solution is

$$x_p = A\cos(\gamma t) + B\sin(\gamma t)$$

If $\omega \neq \gamma$, then this form does not duplicate the solution to the associated homogeneous equation. Hence it is the correct form for the particular solution. An equation of motion may consist of a sum of sines and cosines of ωt and γt.

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

Note that

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Using the method of undetermined coefficients, the **first guess** to the particular solution is

$$x_p = A\cos(\gamma t) + B\sin(\gamma t)$$

If $\omega = \gamma$, then this form DOES duplicate the solution to the associated homogeneous equation. In this case, the correct form for x_p is

$$x_p = At\cos(\omega t) + Bt\sin(\omega t).$$

Note that terms of this sort will produce an amplitude of motion that grows linearly in t.

Forced Undamped Motion and Resonance

For $F(t) = F_0 \sin(\gamma t)$ starting from rest at equilibrium:

$$\text{Case (1):} \quad x'' + \omega^2 x = F_0 \sin(\gamma t), \quad x(0) = 0, \quad x'(0) = 0$$

$$x(t) = \frac{F_0}{\omega^2 - \gamma^2} \left(\sin(\gamma t) - \frac{\gamma}{\omega} \sin(\omega t) \right)$$

If $\gamma \approx \omega$, the amplitude of motion could be rather large!

Pure Resonance

Case (2): $x'' + \omega^2 x = F_0 \sin(\omega t), \quad x(0) = 0, \quad x'(0) = 0$

$$x(t) = \frac{F_0}{2\omega^2}\sin(\omega t) - \frac{F_0}{2\omega} t\cos(\omega t)$$

Note that the amplitude, α, of the second term is a function of t:

$$\alpha(t) = \frac{F_0 t}{2\omega}$$

which grows without bound!

Forced Motion and Resonance Applet

Choose "Elongation diagram" to see a plot of displacement. Try exciter frequencies close to ω.

Section 12: LRC Series Circuits

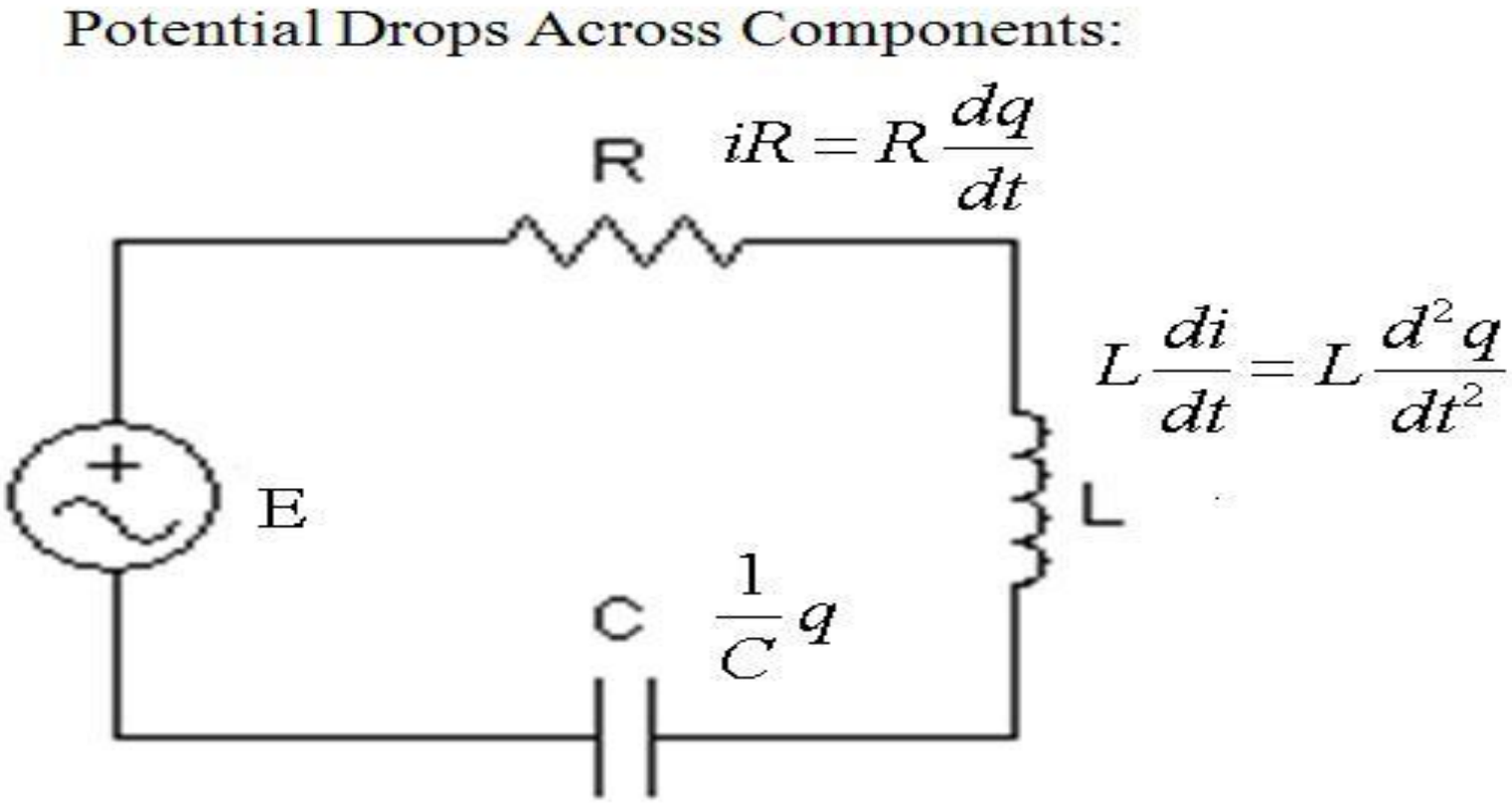


Figure: Kirchhoff's Law: The charge q on the capacitor satisfies $Lq'' + Rq' + \frac{1}{C}q = E(t)$.

LRC Series Circuit (Free Electrical Vibrations)

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$

If the applied force $E(t) = 0$, then the **electrical vibrations** of the circuit are said to be **free**. These are categorized as

overdamped if	$R^2 - 4L/C > 0,$
critically damped if	$R^2 - 4L/C = 0,$
underdamped if	$R^2 - 4L/C < 0.$

Steady and Transient States

Given a nonzero applied voltage $E(t)$, we obtain an IVP with nonhomogeneous ODE for the charge q

$$Lq'' + Rq' + \frac{1}{C}q = E(t), \quad q(0) = q_0, \quad q'(0) = i_0.$$

From our basic theory of linear equations we know that the solution will take the form

$$q(t) = q_c(t) + q_p(t).$$

The function of q_c is influenced by the initial state (q_0 and i_0) and will decay exponentially as $t \to \infty$. Hence q_c is called the **transient state charge** of the system.

The function q_p is independent of the initial state but depends on the characteristics of the circuit (L, R, and C) and the applied voltage E. q_p is called the **steady state charge** of the system.

Example

An LRC series circuit has inductance 0.5 h, resistance 10 ohms, and capacitance $4 \cdot 10^{-3}$ f. Find the steady state current of the system if the applied force is $E(t) = 5\cos(10t)$.

The ODE for the charge is

$$0.5q'' + 10q' + \frac{1}{4 \cdot 10^{-3}} q = 5\cos(10t) \implies q'' + 20q' + 500q = 10\cos(10t).$$

The characteristic equation $r^2 + 20r + 500 = 0$ has roots $r = -10 \pm 20i$. To determine q_p we can assume

$$q_p = A\cos(10t) + B\sin(10t)$$

which does not duplicate solutions of the homogeneous equation (such duplication would only occur if the roots above were $r = \pm 10i$).

Example Continued...

Working through the details, we find that $A = 1/50$ and $B = 1/100$. The steady state charge is therefore

$$q_p = \frac{1}{50}\cos(10t) + \frac{1}{100}\sin(10t).$$

The steady state current

$$i_p = \frac{dq_p}{dt} = -\frac{1}{5}\sin(10t) + \frac{1}{10}\cos(10t).$$

Section 13: The Laplace Transform

If $f = f(s, t)$ is a function of two variables s and t, and we compute a definite integral **with respect to** t,

$$\int_a^b f(s, t)\, dt$$

we are left with a function of s alone.

Example: The integral[22]

$$\int_0^4 (2st + s^2 - t)\, dt = st^2 + s^2 t - \frac{t^2}{2}\bigg|_0^4 = 16s + 4s^2 - 8$$

is a function of the variable s.

[22] The variable s is treated like a constant when integrating with respect to t—and visa versa.

Integral Transform

An **integral transform** is a mapping that assigns to a function $f(t)$ another function $F(s)$ via an integral of the form

$$\int_a^b K(s,t)f(t)\,dt.$$

- The function K is called the **kernel** of the transformation.
- The limits a and b may be finite or infinite.
- The integral may be improper so that convergence/divergence must be considered.
- This transform is **linear** in the sense that

$$\int_a^b K(s,t)(\alpha f(t)+\beta g(t))\,dt = \alpha\int_a^b K(s,t)f(t)\,dt + \beta\int_a^b K(s,t)g(t)\,dt.$$

The Laplace Transform

Definition: Let $f(t)$ be defined on $[0, \infty)$. The Laplace transform of f is denoted and defined by

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s).$$

The domain of the transformation $F(s)$ is the set of all s such that the integral is convergent.

Note: The kernel for the Laplace transform is $K(s, t) = e^{-st}$.

Find the Laplace transform of $f(t) = 1$

It is readily seen that if $s = 0$, the integral $\int_0^\infty e^{-st}\,dt$ is divergent. Otherwise[23]

$$\mathscr{L}\{1\} = \int_0^\infty e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^\infty$$

Convergence in the limit $t \to \infty$ requires $s > 0$. In this case, we have

$$\mathscr{L}\{1\} = -\frac{1}{s}(0-1) = \frac{1}{s}.$$

So we have the transform along with its domain

$$\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0.$$

[23] The integral is improper. We are in reality evaluating an integral of the form $\int_0^b e^{-st} f(t)\,dt$ and then taking the limit $b \to \infty$. We suppress some of the notation here with the understanding that this process is implied.

A piecewise defined function

Find the Laplace transform of f defined by

$$f(t) = \begin{cases} 2t, & 0 \le t < 10 \\ 0, & t \ge 10 \end{cases}$$

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \int_0^{10} 2te^{-st}\, dt + \int_{10}^\infty 0 \cdot e^{-st}\, dt$$

For $s \neq 0$, integration by parts gives

$$\mathscr{L}\{f(t)\} = \frac{2}{s^2} - \frac{2e^{-10s}}{s^2} - \frac{20e^{-10s}}{s}.$$

When $s = 0$, the value $\mathscr{L}\{f(t)\}|_{s=10} = 100$ can be computed by evaluating the integral or by taking the limit of the above as $s \to 0$.

The Laplace Transform is a Linear Transformation

Some basic results include:

- $\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$
- $\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0$
- $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$ for $n = 1, 2, \ldots$
- $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$
- $\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s > 0$
- $\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \quad s > 0$

Examples: Evaluate the Laplace Transform of

(a) $f(t) = (2-t)^2$

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{4 - 4t + t^2\} = 4\mathscr{L}\{1\} - 4\mathscr{L}\{t\} + \mathscr{L}\{t^2\}$$
$$= \frac{4}{s} - \frac{4}{s^2} + \frac{2}{s^3}$$

(b) $f(t) = \sin^2 5t$

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{\frac{1}{2} - \frac{1}{2}\cos(10t)\} = \frac{1}{2}\mathscr{L}\{1\} - \frac{1}{2}\mathscr{L}\{\cos(10t)\}$$
$$= \frac{1}{2s} - \frac{\frac{1}{2}s}{s^2 + 100}$$

Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

Definition: Let $c > 0$. A function f defined on $[0, \infty)$ is said to be of *exponential order c* provided there exists positive constants M and T such that $|f(t)| < Me^{ct}$ for all $t > T$.

Definition: A function f is said to be *piecewise continuous* on an interval $[a, b]$ if f has at most finitely many jump discontinuities on $[a, b]$ and is continuous between each such jump.

Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

Theorem: If f is piecewise continuous on $[0, \infty)$ and of exponential order c for some $c > 0$, then f has a Laplace transform for $s > c$.

Section 14: Inverse Laplace Transforms

Now we wish to go *backwards*: Given $F(s)$ can we find a function $f(t)$ such that $\mathscr{L}\{f(t)\} = F(s)$?

If so, we'll use the following notation

$$\mathscr{L}^{-1}\{F(s)\} = f(t) \quad \text{provided} \quad \mathscr{L}\{f(t)\} = F(s).$$

We'll call $f(t)$ an inverse Laplace transform of $F(s)$.

A Table of Inverse Laplace Transforms

- $\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$
- $\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$, for $n = 1, 2, \ldots$
- $\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
- $\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$
- $\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$

The inverse Laplace transform is also linear so that

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha f(t) + \beta g(t)$$

Find the Inverse Laplace Transform

When using the table, we have to match the expression inside the brackets $\{\}$ **EXACTLY**! Algebra, including partial fraction decomposition, is often needed.

(a) $\mathscr{L}^{-1}\left\{\dfrac{1}{s^7}\right\}$

$$\mathscr{L}^{-1}\left\{\frac{1}{s^7}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{6!}\frac{6!}{s^7}\right\}$$

$$= \frac{1}{6!}\mathscr{L}^{-1}\left\{\frac{6!}{s^7}\right\} = \frac{t^6}{6!}$$

Example: Evaluate

(b) $\mathscr{L}^{-1}\left\{\dfrac{s+1}{s^2+9}\right\}$

$$\mathscr{L}^{-1}\left\{\frac{s+1}{s^2+9}\right\} = \mathscr{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \frac{1}{3}\mathscr{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$= \cos(3t) + \frac{1}{3}\sin(3t)$$

Example: Evaluate

(c) $\mathscr{L}^{-1}\left\{\dfrac{s-8}{s^2-2s}\right\}$

First we perform a partial fraction decomposition on the argument to find that

$$\frac{s-8}{s(s-2)} = \frac{4}{s} - \frac{3}{s-2}.$$

Now

$$\mathscr{L}^{-1}\left\{\frac{s-8}{s^2-2s}\right\} = \mathscr{L}^{-1}\left\{\frac{4}{s} - \frac{3}{s-2}\right\}$$

$$= 4\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - 3\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= 4 - 3e^{2t}$$

Section 15: Shift Theorems

Suppose we wish to evaluate $\mathscr{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\}$. Does it help to know that $\mathscr{L}\left\{t^2\right\} = \frac{2}{s^3}$?

Note that by definition

$$\begin{aligned}\mathscr{L}\left\{e^t t^2\right\} &= \int_0^\infty e^{-st} e^t t^2 \, dt \\ &= \int_0^\infty e^{-(s-1)t} t^2\end{aligned}$$

Observe that this is simply the Laplace transform of $f(t) = t^2$ evaluated at $s-1$. Letting $F(s) = \mathscr{L}\left\{t^2\right\}$, we have

$$F(s-1) = \frac{2}{(s-1)^3}.$$

Theorem (translation in s)

Suppose $\mathscr{L}\{f(t)\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at} f(t)\right\} = F(s-a).$$

For example,

$$\mathscr{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}} \quad \Longrightarrow \quad \mathscr{L}\left\{e^{at} t^n\right\} = \frac{n!}{(s-a)^{n+1}}.$$

$$\mathscr{L}\left\{\cos(kt)\right\} = \frac{s}{s^2+k^2} \quad \Longrightarrow \quad \mathscr{L}\left\{e^{at}\cos(kt)\right\} = \frac{s-a}{(s-a)^2+k^2}.$$

Inverse Laplace Transforms (completing the square)

(a) $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+2s+2}\right\}$

Note that $s^2+2s+2=(s+1)^2+1$ and $s=s+1-1$. Hence

$$\mathscr{L}^{-1}\left\{\frac{s}{s^2+2s+2}\right\}=\mathscr{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\}-\mathscr{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$=e^{-t}\cos t-e^{-t}\sin t.$$

Inverse Laplace Transforms (repeat linear factors)

(b) $\mathscr{L}^{-1}\left\{\dfrac{1+3s-s^2}{s(s-1)^2}\right\}$

Doing a partial fraction decomposition, we find that

$$\frac{1+3s-s^2}{s(s-1)^2} = \frac{1}{s} - \frac{2}{s-1} + \frac{3}{(s-1)^2}.$$

So

$$\mathscr{L}^{-1}\left\{\frac{1+3s-s^2}{s(s-1)^2}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s} - \frac{2}{s-1} + \frac{3}{(s-1)^2}\right\}$$

$$= \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} + 3\mathscr{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

$$= 1 - 2e^t + 3te^t$$

The Unit Step Function

Let $a > 0$. The unit step function $\mathscr{U}(t-a)$ is defined by

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a \end{cases}$$

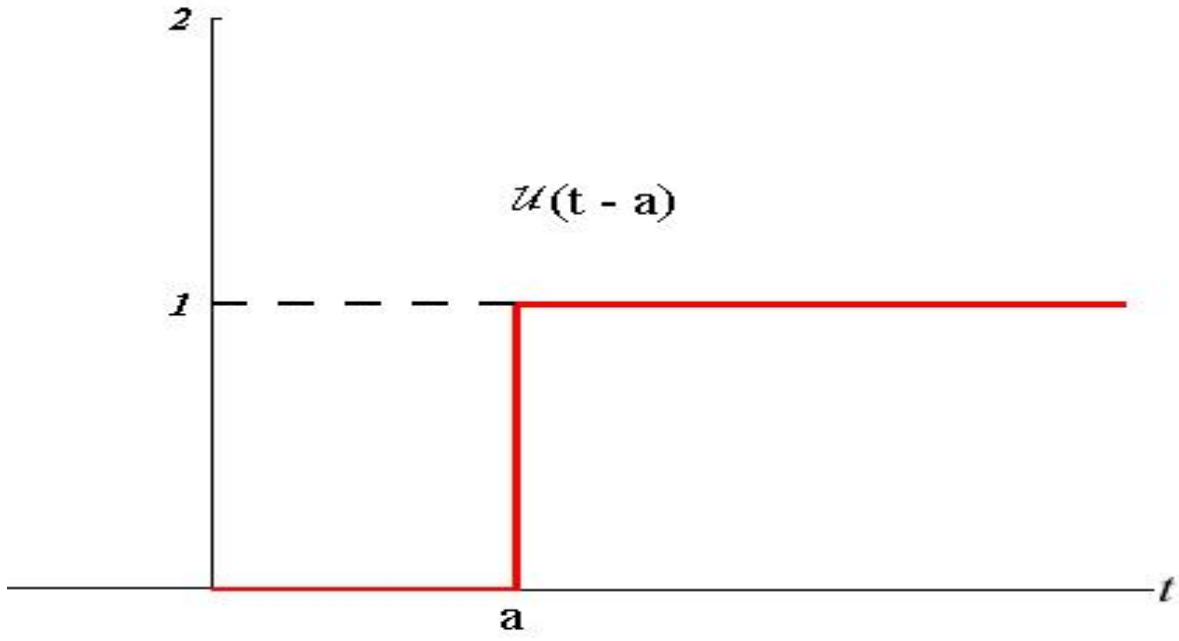


Figure: We can use the unit step function to provide convenient expressions for piecewise defined functions.

Piecewise Defined Functions

Verify that

$$f(t) = \begin{cases} g(t), & 0 \leq t < a \\ h(t), & t \geq a \end{cases} \quad = \; g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a)$$

Exercise left to the reader. (Hint: Consider the two intervals $0 \leq t < a$ and $t \geq a$.)

Translation in t

Given a function $f(t)$ for $t \geq 0$, and a number $a > 0$

$$f(t-a)\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ f(t-a), & t \geq a \end{cases} .$$

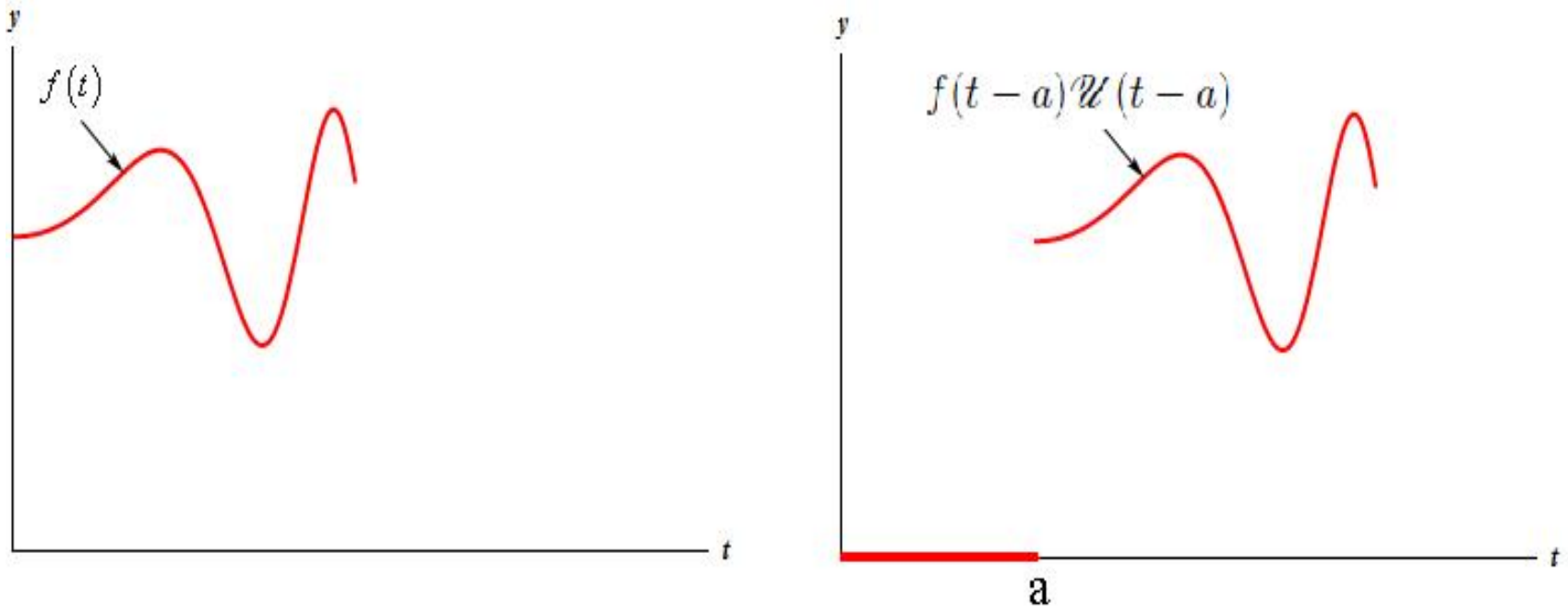


Figure: The function $f(t-a)\mathscr{U}(t-a)$ has the graph of f shifted a units to the right with value of zero for t to the left of a.

Theorem (translation in t)

If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s).$$

In particular,

$$\mathscr{L}\{\mathscr{U}(t-a)\} = \frac{e^{-as}}{s}.$$

As another example,

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad \Longrightarrow \quad \mathscr{L}\{(t-a)^n\mathscr{U}(t-a)\} = \frac{n!e^{-as}}{s^{n+1}}.$$

Example

Find the Laplace transform $\mathscr{L}\{f(t)\}$ where

$$f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ t, & t \geq 1 \end{cases}$$

Noting that $f(t) = 1 + (t-1)\mathscr{U}(t-1)$, we have

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{1\} + \mathscr{L}\{(t-1)\mathscr{U}(t-1)\}$$

$$= \frac{1}{s} + \frac{e^{-s}}{s^2}.$$

A Couple of Useful Results

Another formulation of this translation theorem is

(1) $\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}.$

For example (making use of a sum of angles formula)

$$\mathscr{L}\{\cos t\,\mathscr{U}\left(t-\frac{\pi}{2}\right)\} = e^{-\pi s/2}\mathscr{L}\{\cos\left(t+\frac{\pi}{2}\right)\}$$

$$= e^{-\pi s/2}\mathscr{L}\{-\sin t\} = -e^{-\pi s/2}\frac{1}{s^2+1}.$$

A Couple of Useful Results

The inverse form of this translation theorem is

(2) $\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a).$

For example, using a partial fraction decomposition as necessary

$$\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s(s+1)}\right\} = \mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s} - \frac{e^{-2s}}{s+1}\right\}$$

$$= \mathscr{U}(t-2) - e^{-(t-2)}\mathscr{U}(t-2).$$

Section 16: Laplace Transforms of Derivatives and IVPs

Suppose f has a Laplace transform and that f is differentiable on $[0, \infty)$. Obtain an expression for the Laplace tranform of $f'(t)$.

By definition $\quad \mathscr{L}\left\{f'(t)\right\} = \int_0^\infty e^{-st} f'(t)\, dt$

Let us assume that f is of exponential order c for some real c and take $s > c$. Integrate by parts to obtain

$$\begin{aligned}
\mathscr{L}\left\{f'(t)\right\} &= \int_0^\infty e^{-st} f'(t)\, dt = f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\, dt \\
&= 0 - f(0) + s\mathscr{L}\left\{f(t)\right\} \\
&= s\mathscr{L}\left\{f(t)\right\} - f(0) \qquad (9)
\end{aligned}$$

Transforms of Derivatives

If $\mathscr{L}\{f(t)\} = F(s)$, we have $\mathscr{L}\{f'(t)\} = sF(s) - f(0)$. We can use this relationship recursively to obtain Laplace transforms for higher derivatives of f.

For example

$$\begin{aligned}
\mathscr{L}\left\{f''(t)\right\} &= s\mathscr{L}\left\{f'(t)\right\} - f'(0) \\
&= s\left(sF(s) - f(0)\right) - f'(0) \\
&= s^2F(s) - sf(0) - f'(0)
\end{aligned}$$

Transforms of Derivatives

For $y = y(t)$ defined on $[0, \infty)$ having derivatives y', y'' and so forth, if

$$\mathscr{L}\{y(t)\} = Y(s),$$

then

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0),$$

$$\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0),$$

$$\vdots$$

$$\mathscr{L}\left\{\frac{d^ny}{dt^n}\right\} = s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0).$$

Differential Equation

For constants a, b, and c, take the Laplace transform of both sides of the equation

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1$$

Letting $\mathscr{L}\{y(t)\} = Y(s)$ and $\mathscr{L}\{g(t)\} = G(s)$, we take the Laplace transform of both sides of the ODE to obtain

$$\mathscr{L}\left\{ay'' + by' + cy\right\} = \mathscr{L}\{g(t)\} \quad \Longrightarrow$$

$$a\mathscr{L}\{y''\} + b\mathscr{L}\{y'\} + c\mathscr{L}\{y\} = G(s) \quad \Longrightarrow$$

$$a(s^2Y(s) - sy(0) - y'(0)) + b(sY(s) - y(0)) + cY(s) = G(s).$$

Differential Equation

For constants a, b, and c, take the Laplace transform of both sides of the equation

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1$$

Applying the initial conditions and solving for $Y(s)$

$$(as^2 + bs + c)Y(s) - asy_0 - ay_1 - by_0 = G(s) \quad \Longrightarrow$$

$$Y(s) = \frac{ay_0 s + ay_1 + by_0}{as^2 + bs + c} + \frac{G(s)}{as^2 + bs + c}.$$

The solution $y(t)$ of the IVP can be found by applying the inverse transform

$$y(t) = \mathscr{L}^{-1}\{Y(s)\}.$$

Solving IVPs

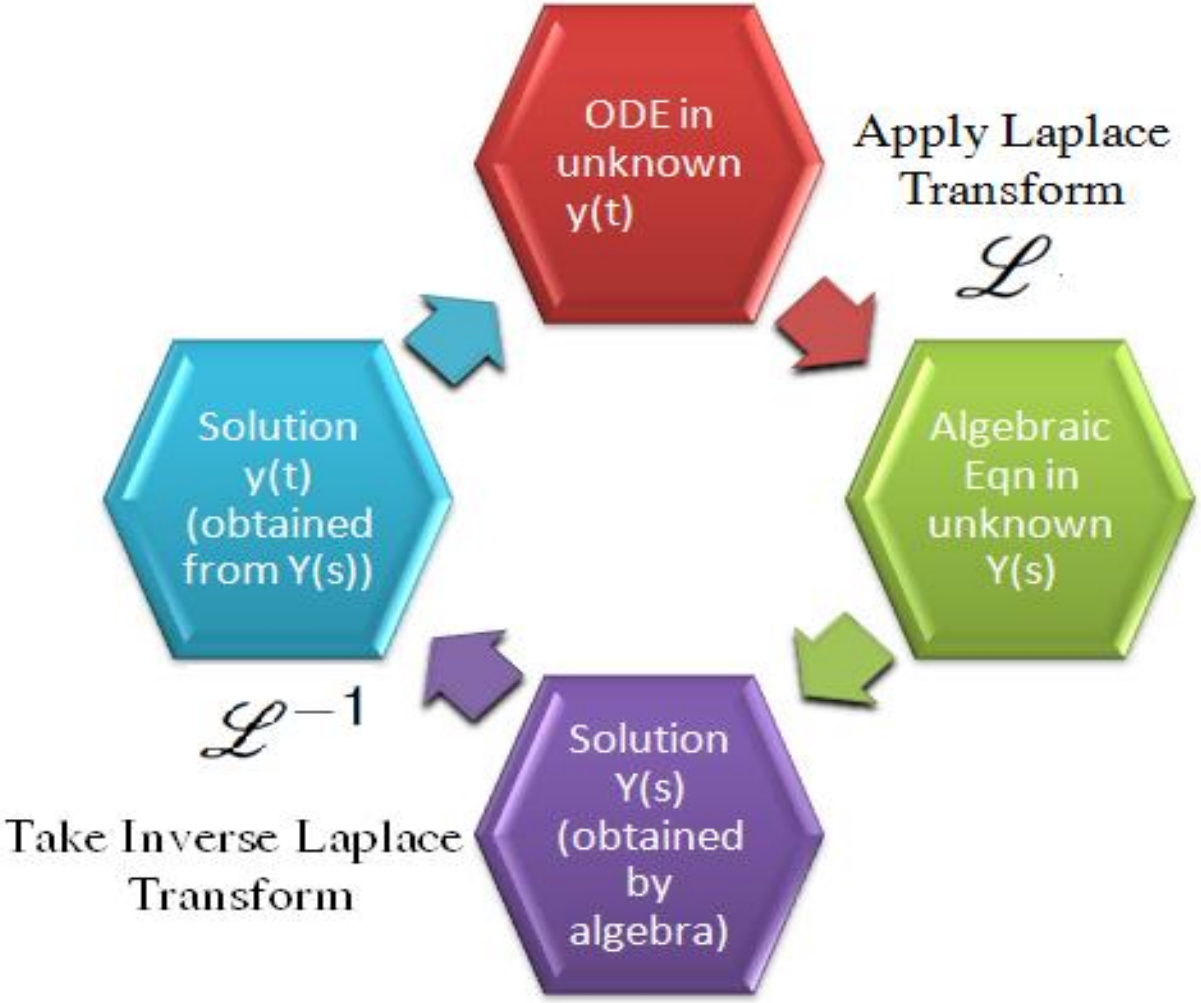


Figure: We use the Laplace transform to turn our DE into an algebraic equation. Solve this transformed equation, and then transform back.

General Form

We get

$$Y(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)}$$

where Q is a polynomial with coefficients determined by the initial conditions, G is the Laplace transform of $g(t)$ and P is the **characteristic polynomial** of the original equation.

$$\mathscr{L}^{-1}\left\{\frac{Q(s)}{P(s)}\right\} \quad \text{is called the } \textbf{zero input response},$$

and

$$\mathscr{L}^{-1}\left\{\frac{G(s)}{P(s)}\right\} \quad \text{is called the } \textbf{zero state response}.$$

Solve the IVP using the Laplace Transform

(a) $\dfrac{dy}{dt}+3y = 2t \quad y(0) = 2$

Apply the Laplace transform and use the initial condition. Let $Y(s) = \mathscr{L}\{y\}$

$$\mathscr{L}\{y' + 3y\} = \mathscr{L}\{2t\}$$

$$sY(s) - y(0) + 3Y(s) = \frac{2}{s^2}$$

$$(s+3)Y(s) - 2 = \frac{2}{s^2}$$

$$Y(s) = \frac{2}{s^2(s+3)} + \frac{2}{s+3} = \frac{2s^2+2}{s^2(s+3)}$$

Example Continued...

We use a partial fraction decomposition to facilitate taking the inverse transform.

$$Y(s) = \frac{-\frac{2}{9}}{s} + \frac{\frac{2}{3}}{s^2} + \frac{\frac{20}{9}}{s+3}$$

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = -\frac{2}{9} + \frac{2}{3}t + \frac{20}{9}e^{-3t}.$$

Solve the IVP using the Laplace Transform

$$y''+4y'+4y = te^{-2t} \quad y(0) = 1, y'(0) = 0$$

Again, let $Y(s) = \mathscr{L}\{y(t)\}$.

$$\mathscr{L}\{y'' + 4y' + 4y\} = \mathscr{L}\{te^{-2t}\}$$

$$s^2 Y(s) - sy(0) - y'(0) + 4sY(s) - 4y(0) + 4Y(s) = \frac{1}{(s+2)^2}$$

$$(s^2 + 4s + 4)Y(s) - s - 4 = \frac{1}{(s+2)^2}$$

$$Y(s) = \frac{1}{(s+2)^4} + \frac{s+4}{(s+2)^2}.$$

Example Continued...

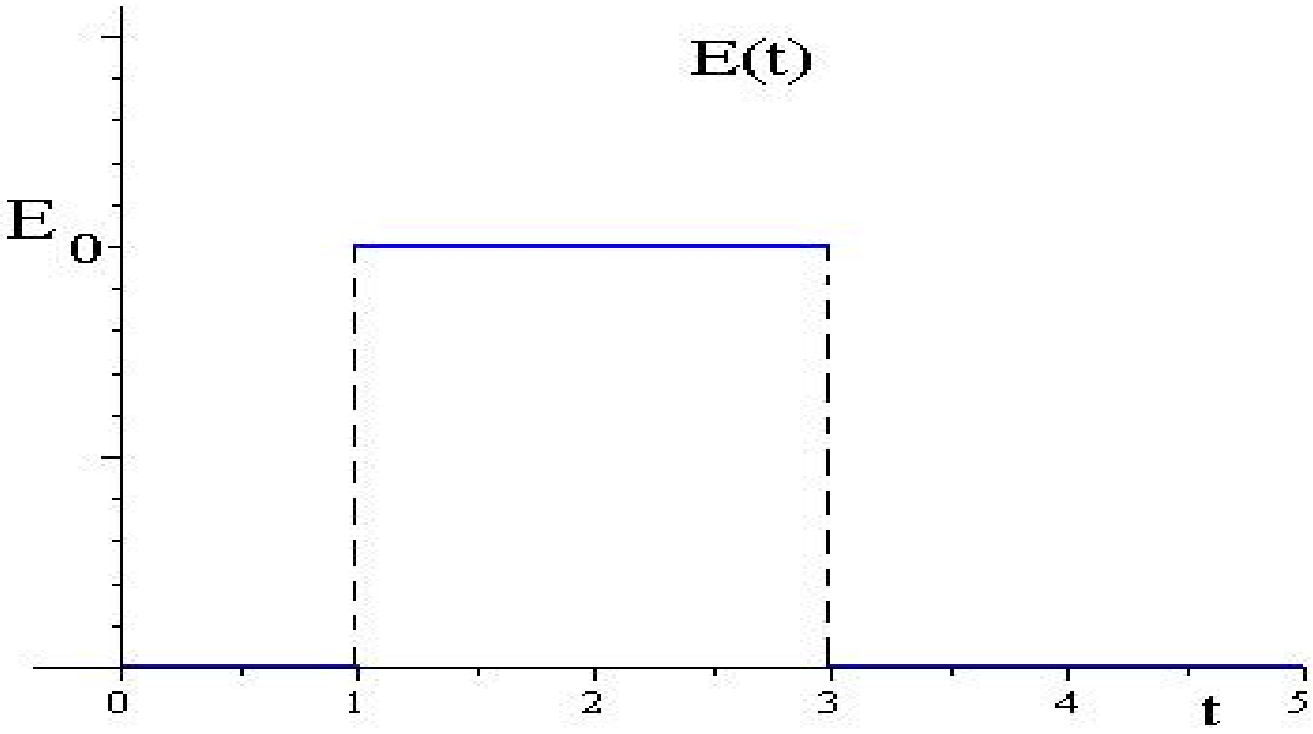
Perform a partial fraction decomposition on Y, and take the inverse transform to find the solution y.

$$Y(s) = \frac{1}{(s+2)^4} + \frac{1}{s+2} + \frac{2}{(s+2)^2}$$

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = \frac{1}{3!}t^3e^{-2t} + e^{-2t} + 2te^{-2t}.$$

Solve the IVP

An LR-series circuit has inductance $L = 1$h, resistance $R = 10\Omega$, and applied force $E(t)$ whose graph is given below. If the initial current $i(0) = 0$, find the current $i(t)$ in the circuit.



LR Circuit Example

The IVP can be stated as

$$\frac{di}{dt} + 10i = E_0\mathscr{U}(t-1) - E_0\mathscr{U}(t-3), \quad i(0) = 0.$$

Letting $I(s) = \mathscr{L}\{i(t)\}$, we apply the Laplace transform to obtain

$$\mathscr{L}\{i' + 10i\} = \mathscr{L}\{E_0\mathscr{U}(t-1) - E_0\mathscr{U}(t-3)\}$$

$$sI(s) - i(0) + 10I(s) = \frac{E_0 e^{-s}}{s} - \frac{E_0 e^{-3s}}{s}$$

$$I(s) = \frac{E_0}{s(s+10)}\left(e^{-s} - e^{-3s}\right).$$

Example Continued...

We can perform a partial fraction decomposition on the rational factor and recover the current i.

$$\begin{aligned} I(s) &= \left[\frac{\frac{E_0}{10}}{s} - \frac{\frac{E_0}{10}}{s+10}\right](e^{-s} - e^{-3s}) \\ &= \frac{E_0}{10}\frac{e^{-s}}{s} - \frac{E_0}{10}\frac{e^{-s}}{s+10} - \frac{E_0}{10}\frac{e^{-3s}}{s} + \frac{E_0}{10}\frac{e^{-3s}}{s+10} \end{aligned}$$

And finally

$$\begin{aligned} i(t) &= \mathscr{L}^{-1}\{I(s)\} \\ &= \frac{E_0}{10}\left(1 - e^{-10(t-1)}\right)\mathscr{U}(t-1) - \frac{E_0}{10}\left(1 - e^{-10(t-3)}\right)\mathscr{U}(t-3). \end{aligned}$$

Solving a System

We can solve a system of ODEs using Laplace transforms. Here, we'll consider systems that are

- linear,
- having initial conditions at $t = 0$, and
- constant coefficient.

Let's see it in action (i.e. with a couple of examples).

Example

Solve the system of equations

$$\begin{aligned} \frac{dx}{dt} &= -2x - 2y + 60, \quad x(0) = 0 \\ \frac{dy}{dt} &= -2x - 5y + 60, \quad y(0) = 0 \end{aligned}$$

We'll use the Laplace transforms. Sticking with the usual uppercase-lowercase convention, let's set

$$X(s) = \mathscr{L}\{x(t)\}, \quad \text{and} \quad Y(s) = \mathscr{L}\{y(t)\}.$$

Example Continued...

$$\begin{aligned}\frac{dx}{dt} &= -2x - 2y + 60, \quad x(0) = 0\\ \frac{dy}{dt} &= -2x - 5y + 60, \quad y(0) = 0\end{aligned}$$

Applying the transform to both sides of both equations

$$\begin{aligned}sX(s) - x(0) &= -2X(s) - 2Y(s) + \frac{60}{s}\\ sY(s) - y(0) &= -2X(s) - 5Y(s) + \frac{60}{s}\end{aligned}$$

We can subsitute in the initial conditions, and rearrange the equations to get an algebraic system

$$\begin{aligned}(s+2)X(s) + 2Y(s) &= \frac{60}{s}\\ 2X(s) + (s+5)Y(s) &= \frac{60}{s}\end{aligned}$$

Example Continued...

$$\begin{aligned}(s+2)X(s)+2Y(s) &= \frac{60}{s}\\ 2X(s)+(s+5)Y(s) &= \frac{60}{s}\end{aligned}$$

We can solve this system in any number of ways. For those familiar with it, Crammer's Rule is probably the easiest approach. Elimination will work just as well. We find

$$\begin{aligned}X(s) &= \frac{60(s+3)}{s(s+1)(s+6)}\\ Y(s) &= \frac{60}{(s+1)(s+6)}\end{aligned}$$

As is usually the case, a partial fraction decomposition will give us a form from which we can take the inverse transform using the table.

Example Continued...

Upon the decomposition, we have

$$\begin{aligned} X(s) &= \frac{30}{s} - \frac{24}{s+1} - \frac{6}{s+6} \\ Y(s) &= \frac{12}{s+1} - \frac{12}{s+6} \end{aligned}$$

Finally, we take the inverse transform to obtain the solution to the system.

$$\begin{aligned} x(t) &= \mathscr{L}^{-1}\{X(s)\} = 30 - 24e^{-t} - 6e^{-6t} \\ y(t) &= \mathscr{L}^{-1}\{Y(s)\} = 12e^{-t} - 12e^{-6t} \end{aligned}$$

Example

Use the Laplace transform to solve the system of equations

$$\begin{aligned} x''(t) &= y, \quad x(0) = 1, \quad x'(0) = 0 \\ y'(t) &= x, \quad y(0) = 1 \end{aligned}$$

This system is second order. Again, using the upper-lowercase convention, we take the Laplace transform of both equations to obtain

$$\begin{aligned} s^2X(s) - sx(0) - x'(0) &= Y(s) \\ sY(s) - y(0) &= X(s) \end{aligned}$$

As before, we substitute in the given initial conditions and rearrange the equations.

$$\begin{aligned} s^2X(s) - Y(s) &= s \\ -X(s) + sY(s) &= 1 \end{aligned}$$

Example Continued...

Using some method to solve for X and Y, we obtain

$$\begin{aligned} X(s) &= \frac{s^2+1}{s^3-1} = \frac{s^2+1}{(s-1)(s^2+s+1)} \\ Y(s) &= \frac{s^2+s}{s^3-1} = \frac{s(s+1)}{(s-1)(s^2+s+1)} \end{aligned}$$

The right most expressions come from factoring the difference of cubes. The decomposition is a bit more tedious. It will be useful to complete the square on the factor in the denominator.

$$s^2+s+1 = \left(s+\frac{1}{2}\right)^2 + \frac{3}{4}.$$

Example Continued...

With a little effort, we obtain the decomposition

$$\begin{aligned} X(s) &= \frac{2/3}{s-1} + \frac{1/3(s-1)}{s^2+s+1} \\ Y(s) &= \frac{2/3}{s-1} + \frac{1/3(s+2)}{s^2+s+1} \end{aligned}$$

Using the completed square along with $s-1 = s + \frac{1}{2} - \frac{3}{2}$ and $s+2 = s + \frac{1}{2} + \frac{3}{2}$. We see that the shift in s result is going to be used. It's also useful to note that $3/4 = \left(\sqrt{3}/2\right)^2$.

Example Continued...

$$
\begin{aligned}
X(s) &= \frac{2/3}{s-1} + \frac{1/3\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} - \frac{1/2}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} \\
Y(s) &= \frac{2/3}{s-1} + \frac{1/3\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} + \frac{1/2}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}
\end{aligned}
$$

We finally take the inverse Laplace transform using the table, $x(t) = \mathscr{L}^{-1}\{X(s)\}$ and $y(t) = \mathscr{L}^{-1}\{Y(s)\}$, and obtain the solution

$$
\begin{aligned}
x(t) &= \frac{2}{3}e^{t} + \frac{1}{3}e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{\sqrt{3}}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right) \\
y(t) &= \frac{2}{3}e^{t} + \frac{1}{3}e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{1}{\sqrt{3}}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)
\end{aligned}
$$

Section 17: Fourier Series: Trigonometric Series

Consider the following problem:

An undamped spring mass system has a mass of 2 kg attached to a spring with spring constant 128 N/m. The mass is driven by an external force $f(t) = 2t$ for $-1 < t < 1$ that is 2-periodic so that $f(t+2) = f(t)$ for all $t > 0$.

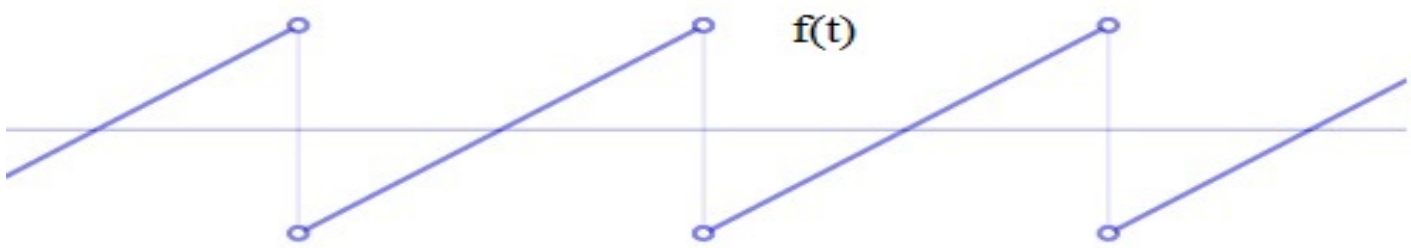


Figure: $2\dfrac{d^2x}{dt^2} + 128x = f(t)$

Question: How can we solve a problem like this with a right side with inifinitely many pieces?

Motivation for Fourier Series

Various applications in science and engineering involve periodic forcing or complex signals that can be considered sums of more elementary parts (e.g. harmonics).

- Signal processing (decomposing/reconstructing sound waves or voltage inputs)
- Control theory (qualitative assessment/control of dynamics)
- Approximation of forces or solutions of differential equations

A variety of interesting waveforms (periodic curves) arise in applications and can be expressed by series representations.

Common Models of Periodic Sources (e.g. Voltage)

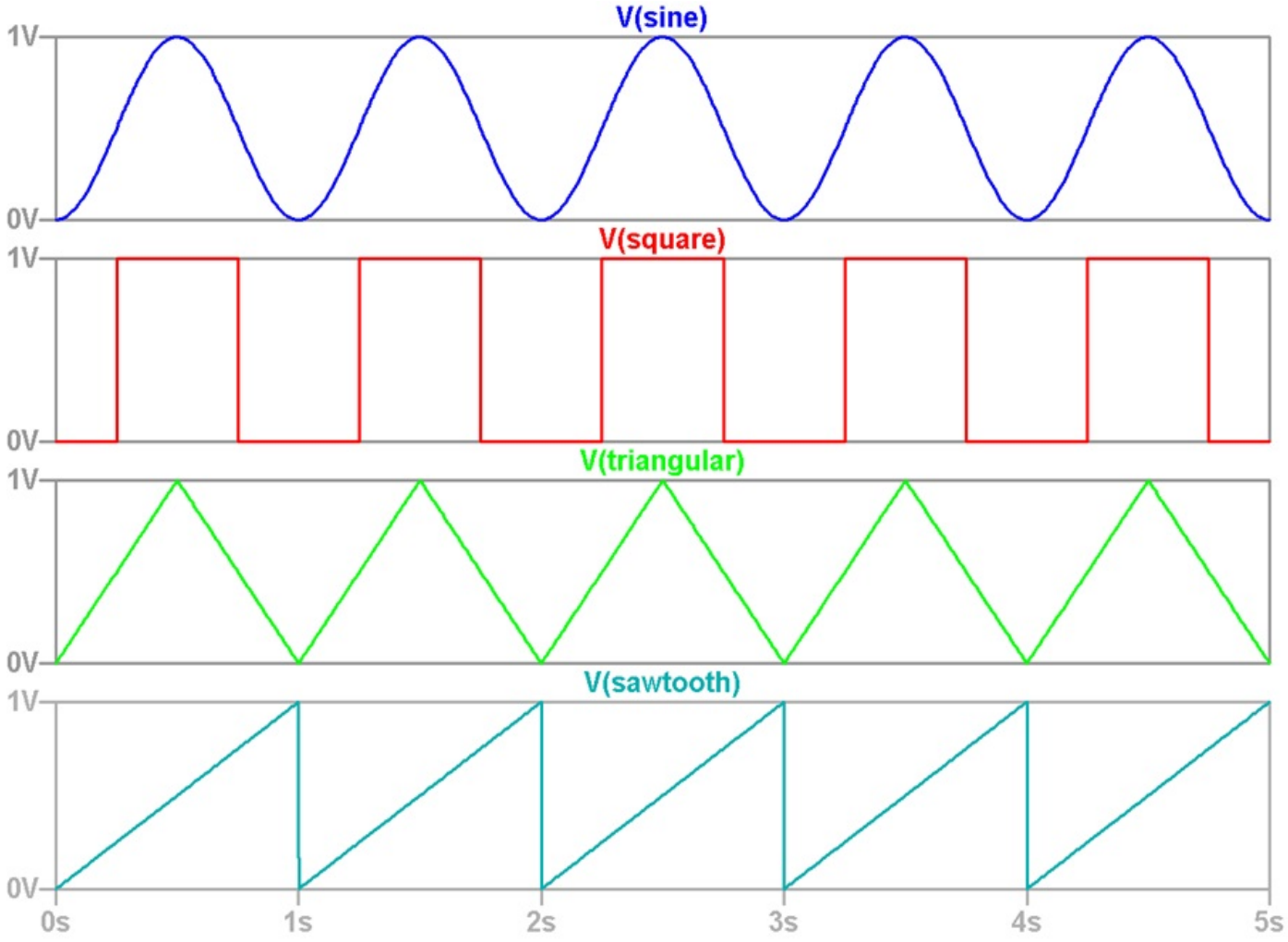


Figure: We'd like to solve, or at least approximate solutions, to ODEs and PDEs with periodic *right hand sides*.

Series Representations for Functions

The goal is to represent a function by a series

$$f(x) = \sum_{n=1}^{\infty} (\text{some simple functions})$$

In calculus, you saw power series $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ where the simple functions were powers $(x-c)^n$.

Here, you will see how some functions can be written as series of trigonometric functions

$$f(x) = \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)$$

We'll move the $n = 0$ to the front before the rest of the sum.

Some Preliminary Concepts

Suppose two functions f and g are integrable on the interval $[a, b]$. We define the **inner product** of f and g on $[a, b]$ as

$$\langle f, g \rangle = \int_a^b f(x)g(x)\,dx.$$

We say that f and g are **orthogonal** on $[a, b]$ if

$$\langle f, g \rangle = 0.$$

The product depends on the interval, so the orthogonality of two functions depends on the interval.

Properties of an Inner Product

Let f, g, and h be integrable functions on the appropriate interval and let c be any real number. The following hold

(i) $\langle f, g\rangle = \langle g, f\rangle$

(ii) $\langle f, g + h\rangle = \langle f, g\rangle + \langle f, h\rangle$

(iii) $\langle cf, g\rangle = c\langle f, g\rangle$

(iv) $\langle f, f\rangle \geq 0$ and $\langle f, f\rangle = 0$ if and only if $f = 0$

Orthogonal Set

A set of functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ is said to be **orthogonal** on an interval $[a, b]$ if

$$\langle \phi_m, \phi_n \rangle = \int_a^b \phi_m(x)\phi_n(x)\,dx = 0 \quad \text{whenever} \quad m \neq n.$$

Note that any function $\phi(x)$ that is not identically zero will satisfy

$$\langle \phi, \phi \rangle = \int_a^b \phi^2(x)\,dx > 0.$$

Hence we define the **square norm** of ϕ (on $[a, b]$) to be

$$\|\phi\| = \sqrt{\int_a^b \phi^2(x)\,dx}.$$

An Orthogonal Set of Functions

Consider the set of functions

$$\{1, \cos x, \cos 2x, \cos 3x, \ldots, \sin x, \sin 2x, \sin 3x, \ldots\} \quad \text{on} \quad [-\pi, \pi].$$

It can easily be verified that

$$\int_{-\pi}^{\pi} \cos nx \, dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \sin mx \, dx = 0 \quad \text{for all} \quad n, m \geq 1,$$

$$\int_{-\pi}^{\pi} \cos nx \, \sin mx \, dx = 0 \quad \text{for all} \quad m, n \geq 1, \quad \text{and}$$

$$\int_{-\pi}^{\pi} \cos nx \, \cos mx \, dx = \int_{-\pi}^{\pi} \sin nx \, \sin mx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & n = m \end{cases},$$

An Orthogonal Set of Functions

These integral values indicated that the set of functions

$$\{1, \cos x, \cos 2x, \cos 3x, \ldots, \sin x, \sin 2x, \sin 3x, \ldots\}$$

is an orthogonal set on the interval $[-\pi, \pi]$.

This set can be generalized by using a simple change of variables $t = \frac{\pi x}{p}$ to obtain the orthogonal set on $[-p, p]$

$$\left\{1, \cos \frac{n\pi x}{p}, \sin \frac{m\pi x}{p} \mid n, m \in \mathbb{N}\right\}$$

There are many interesting and useful orthogonal sets of functions (on appropriate intervals). What follows is readily extended to other such (infinite) sets.

Fourier Series

Suppose $f(x)$ is defined for $-\pi < x < \pi$. We would like to know how to write f as a series **in terms of sines and cosines**.

Task: Find coefficients (numbers) a_0, $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ such that[24]

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right).$$

[24] We'll write $\frac{a_0}{2}$ as opposed to a_0 purely for convenience.

Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right).$$

The question of convergence naturally arises when we wish to work with infinite series. To highlight convergence considerations, some authors prefer not to use the equal sign when expressing a Fourier series and instead write

$$f(x) \sim \frac{a_0}{2} + \cdots$$

Herein, we'll use the equal sign with the understanding that equality may not hold at each point.

Convergence will be address later.

Finding an Example Coefficient

For a known function f defined on $(-\pi, \pi)$, assume the series holds. We'll find the coefficient b_4. Multiply both sides by $\sin 4x$

$$f(x)\sin 4x = \frac{a_0}{2}\sin 4x + \sum_{n=1}^{\infty}\left(a_n \cos nx \sin 4x + b_n \sin nx \sin 4x\right).$$

Now integrate both sides with respect to x from $-\pi$ to π (assume it is valid to integrate first and sum later).

$$\int_{-\pi}^{\pi} f(x)\sin 4x\, dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\sin 4x\, dx \quad +$$

$$\sum_{n=1}^{\infty}\left(\int_{-\pi}^{\pi} a_n \cos nx \sin 4x\, dx + \int_{-\pi}^{\pi} b_n \sin nx \sin 4x\, dx\right).$$

$$\int_{-\pi}^{\pi} f(x)\sin 4x\, dx = \frac{a_0}{2}\int_{-\pi}^{\pi} \sin 4x\, dx \quad +$$

$$\sum_{n=1}^{\infty}\left(a_n \int_{-\pi}^{\pi} \cos nx \sin 4x\, dx + b_n \int_{-\pi}^{\pi} \sin nx \sin 4x\, dx\right).$$

Now we use the known orthogonality property. Recall that $\int_{-\pi}^{\pi} \sin 4x\, dx = 0$, and that for every $n = 1, 2, \ldots$

$$\int_{-\pi}^{\pi} \cos nx \sin 4x\, dx = 0$$

So the constant and all cosine terms are gone leaving

$$\int_{-\pi}^{\pi} f(x)\sin 4x\, dx = \sum_{n=1}^{\infty}\left(b_n \int_{-\pi}^{\pi} \sin nx \sin 4x\, dx\right).$$

But we also know that

$$\int_{-\pi}^{\pi} \sin nx \sin 4x\, dx = 0, \quad \text{for } n \neq 4, \text{ and} \quad \int_{-\pi}^{\pi} \sin 4x \sin 4x\, dx = \pi.$$

Hence the sum reduces to the single term

$$\int_{-\pi}^{\pi} f(x)\sin 4x\, dx = \pi b_4$$

from which we determine

$$b_4 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin 4x\, dx.$$

Note that there was nothing special about seeking the 4th sine coefficient b_4. We could have just as easily sought b_m for any positive integer m. We would simply start by introducing the factor $\sin(mx)$.

Moreover, using the same orthogonality property, we could pick on the a's by starting with the factor $\cos(mx)$—including the constant term since $\cos(0 \cdot x) = 1$. The only minor difference we want to be aware of is that

$$\int_{-\pi}^{\pi} \cos^2(mx)\, dx = \begin{cases} 2\pi, & m = 0 \\ \pi, & m \geq 1 \end{cases}$$

Careful consideration of this sheds light on why it is conventional to take the constant to be $\frac{a_0}{2}$ as opposed to just a_0.

The Fourier Series of $f(x)$ on $(-\pi, \pi)$

The **Fourier series** of the function f defined on $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\,.$$

Where

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx, \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx, \quad \text{and} \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx \end{aligned}$$

Example

Find the Fourier series of the piecewise defined function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 \le x < \pi \end{cases}$$

We can find the coefficients by using the integral formulas given.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\,dx = \frac{\pi}{2}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos(nx)\,dx = \frac{\cos(n\pi)-1}{\pi n^2}.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\sin(nx)\,dx = -\frac{\cos(n\pi)}{n}.$$

Example Continued...

It is convenient to use the relation $\cos(n\pi) = (-1)^n$—this comes up frequently in computing Fourier series. Using the determined coefficients we have

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \\ &= \frac{\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{(-1)^n - 1}{\pi n^2} \cos nx + \frac{(-1)^{n+1}}{n} \sin nx \right) \end{aligned}$$

Fourier Series on an interval $(-p, p)$

The set of functions $\{1, \cos\left(\frac{n\pi x}{p}\right), \sin\left(\frac{m\pi x}{p}\right) | n, m \geq 1\}$ is orthogonal on $[-p, p]$. Moreover, we have the properties

$$\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right) dx = 0 \quad \text{and} \quad \int_{-p}^{p} \sin\left(\frac{m\pi x}{p}\right) dx = 0 \ \text{ for all } \ n, m \geq 1,$$

$$\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right) \sin\left(\frac{m\pi x}{p}\right) dx = 0 \quad \text{for all} \quad m, n \geq 1,$$

$$\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right) \cos\left(\frac{m\pi x}{p}\right) dx = \begin{cases} 0, & m \neq n \\ p, & n = m \end{cases},$$

$$\int_{-p}^{p} \sin\left(\frac{n\pi x}{p}\right) \sin\left(\frac{m\pi x}{p}\right) dx = \begin{cases} 0, & m \neq n \\ p, & n = m \end{cases}.$$

Fourier Series on an interval $(-p, p)$

The orthogonality relations provide for an expansion of a function f defined on $(-p, p)$ as

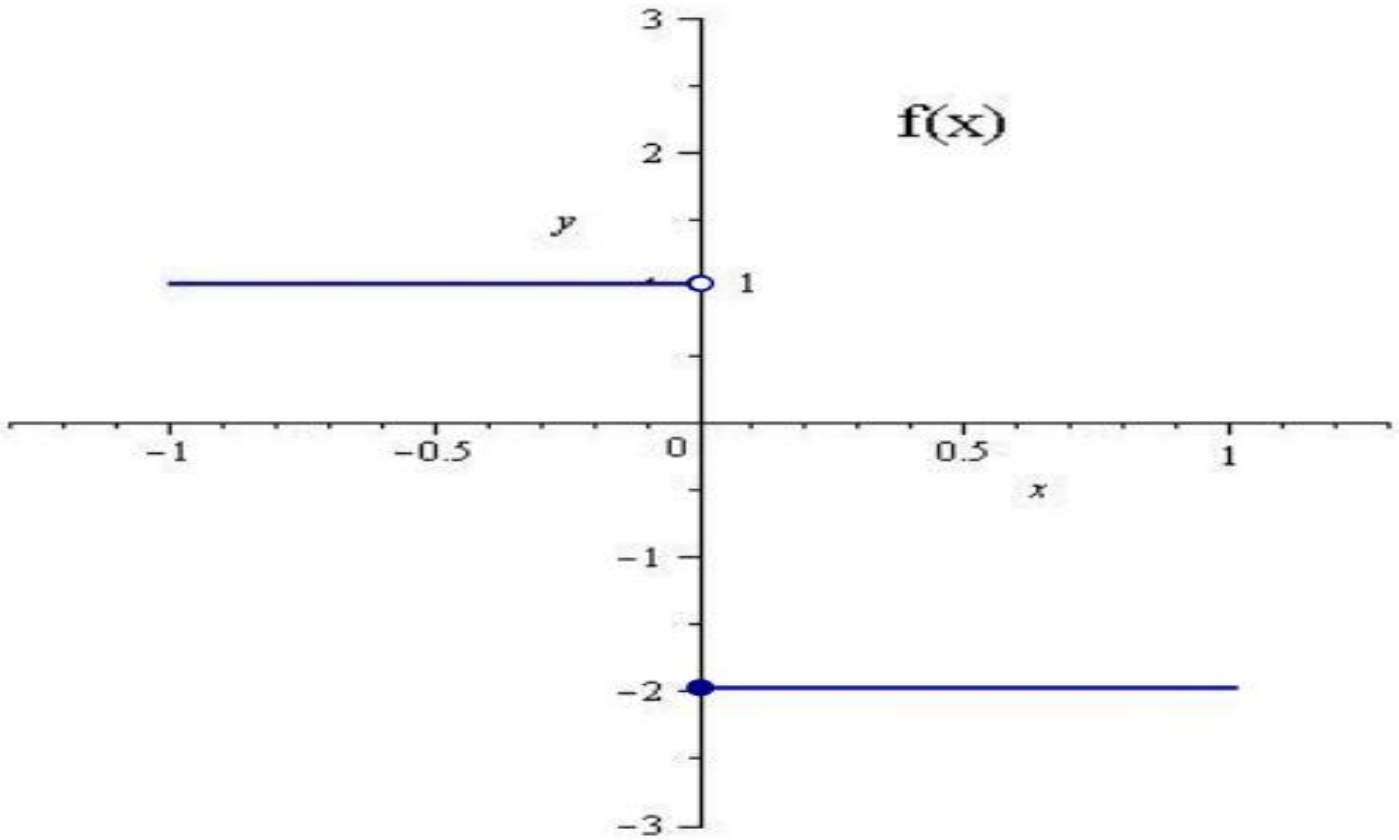
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right) \right)$$

where

$$\begin{aligned} a_0 &= \frac{1}{p} \int_{-p}^{p} f(x)\, dx, \\ a_n &= \frac{1}{p} \int_{-p}^{p} f(x) \cos\left(\frac{n\pi x}{p}\right)\, dx, \quad \text{and} \\ b_n &= \frac{1}{p} \int_{-p}^{p} f(x) \sin\left(\frac{n\pi x}{p}\right)\, dx \end{aligned}$$

Find the Fourier series of f

$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ -2, & 0 \leq x < 1 \end{cases}$$



Example

We apply the given formulas to find the coefficients. Noting that f is defined on the interval $(-1, 1)$ we have $p = 1$.

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} dx + \int_{0}^{1}(-2)\,dx = -1$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos\left(\frac{n\pi x}{1}\right)\,dx =$$

$$= \int_{-1}^{0}\cos(n\pi x)\,dx + \int_{0}^{1}(-2)\cos(n\pi x)\,dx = 0$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin\left(\frac{n\pi x}{1}\right)\,dx =$$

$$= \int_{-1}^{0}\sin(n\pi x)\,dx + \int_{0}^{1}(-2)\sin(n\pi x)\,dx = \frac{3((-1)^n - 1)}{n\pi}$$

Example Continued...

Putting the coefficients into the expansion, we get

$$f(x) = -\frac{1}{2} + \sum_{n=1}^{\infty} \frac{3((-1)^n - 1)}{n\pi} \sin(n\pi x).$$

This example raises an interesting question: The function f is not continuous on the interval $(-1, 1)$. However, each term in the Fourier series, and any partial sum thereof, is obviously continuous. This raises questions about properties (e.g. continuity) of the series. More to the point, we may ask: *what is the connection between f and its Fourier series at the point of discontinuity?*

This is the convergence issue mentioned earlier.

Convergence of the Series

Theorem: If f is continuous at x_0 in $(-p, p)$, then the series converges to $f(x_0)$ at that point. If f has a jump discontinuity at the point x_0 in $(-p, p)$, then the series **converges in the mean** to the average value

$$\frac{f(x_0-) + f(x_0+)}{2} \stackrel{\text{def}}{=} \frac{1}{2}\left(\lim_{x \to x_0^-} f(x) + \lim_{x \to x_0^+} f(x)\right)$$

at that point.

Periodic Extension:
The series is also defined for x outside of the original domain $(-p, p)$. The extension to all real numbers is $2p$-periodic.

Convergence of the Series

$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ -2, & 0 \le x < 1 \end{cases}, \quad f(x) = -\frac{1}{2} + \sum_{n=1}^{\infty} \frac{3((-1)^n - 1)}{n\pi} \sin(n\pi x).$$

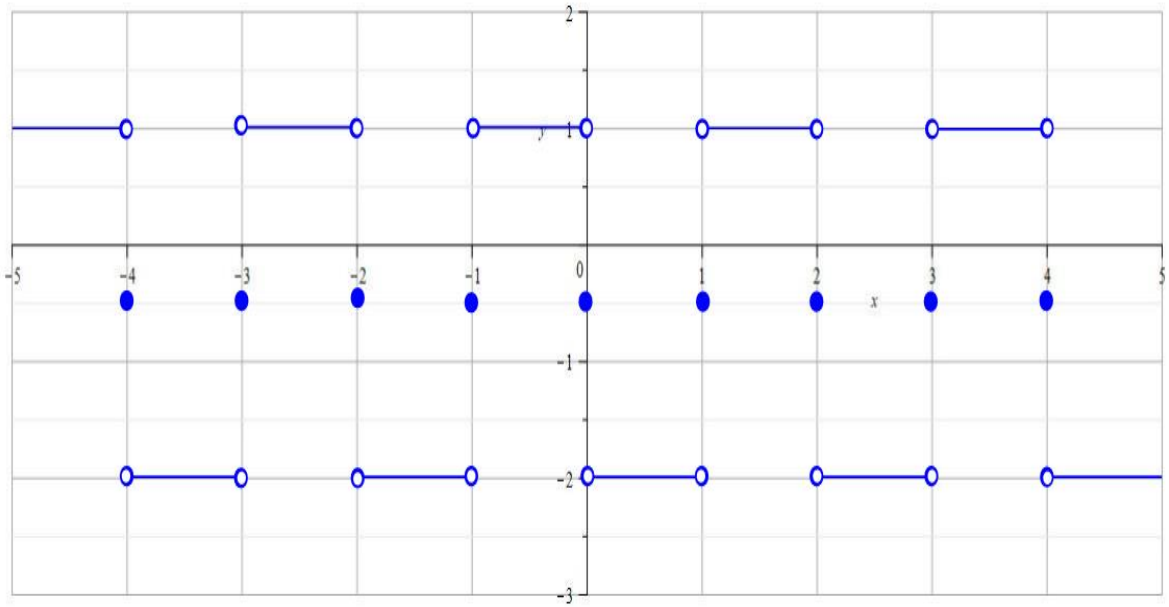


Figure: Plot of the infinite sum, the limit for the Fourier series of f. Note that the basic plot repeats every 2 units, and converges in the mean at each jump.

Find the Fourier Series for $f(x) = x$, $-1 < x < 1$

Again the value of $p = 1$. So the coefficients are

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} x\,dx = 0$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos\left(\frac{n\pi x}{1}\right)\,dx =$$

$$= \int_{-1}^{1} x\cos(n\pi x)\,dx = 0$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin\left(\frac{n\pi x}{1}\right)\,dx =$$

$$= \int_{-1}^{1} x\sin(n\pi x)\,dx = \frac{2(-1)^{n+1}}{n\pi}$$

Example Continued...

Having determined the coefficients, we have the Fourier series

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi x)$$

Observation: f is an odd function. It is not surprising then that there are no nonzero constant or cosine terms (which have even symmetry) in the Fourier series for f.

The following plots show f, f plotted along with some partial sums of the series, and f along with a partial sum of its series extended outside of the original domain $(-1, 1)$.

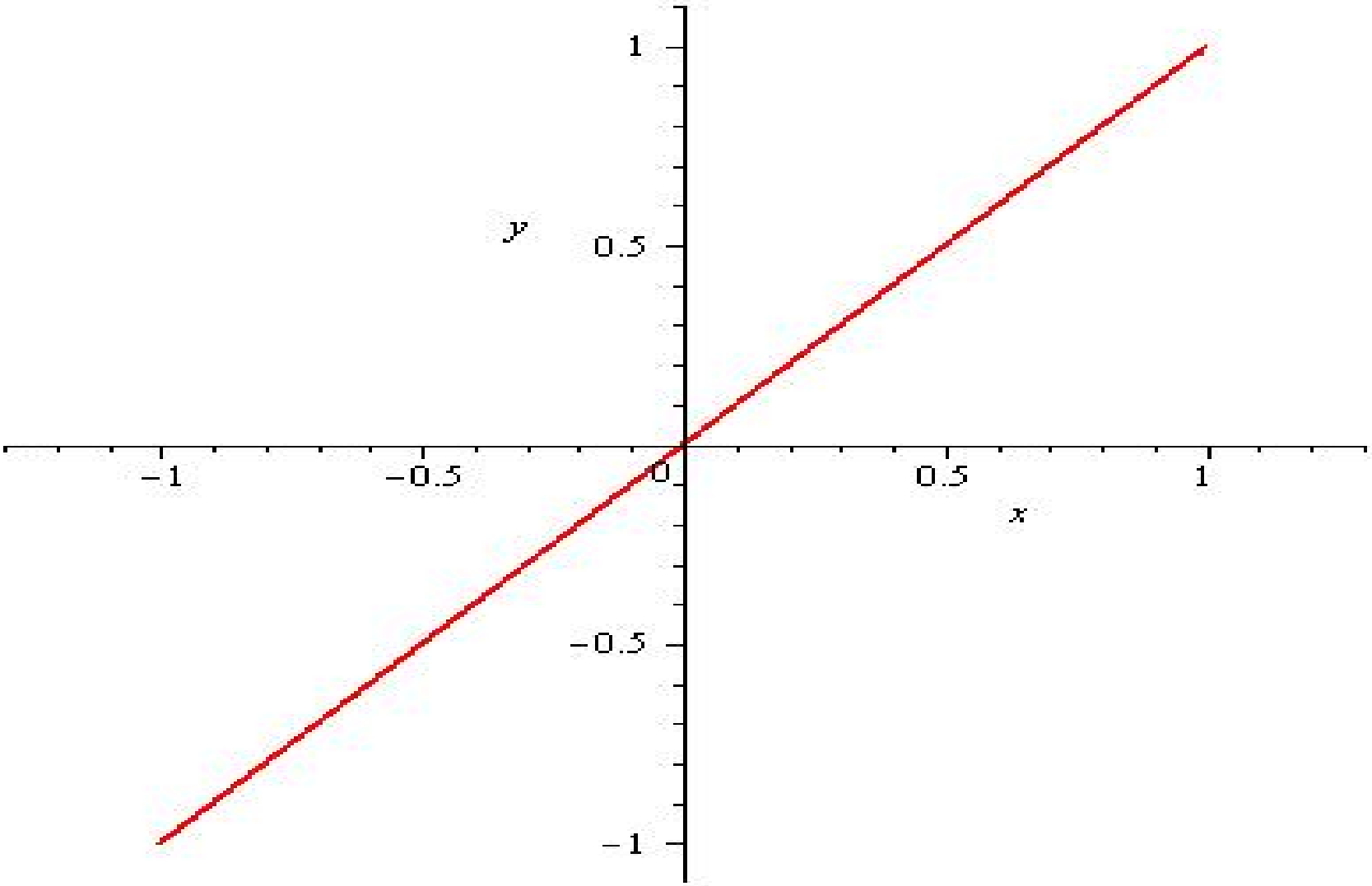


Figure: Plot of $f(x) = x$ for $-1 < x < 1$

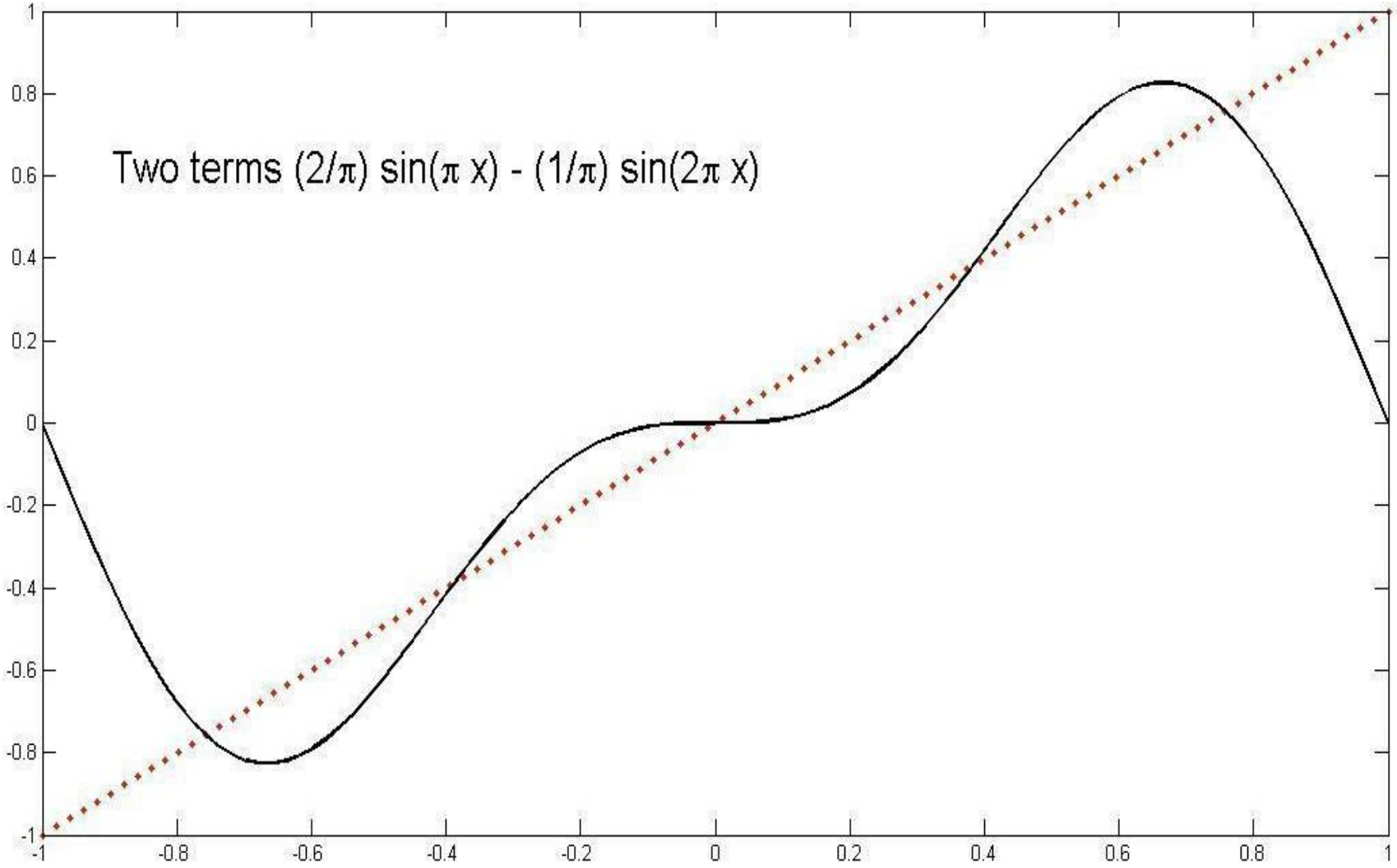


Figure: Plot of $f(x) = x$ for $-1 < x < 1$ with two terms of the Fourier series.

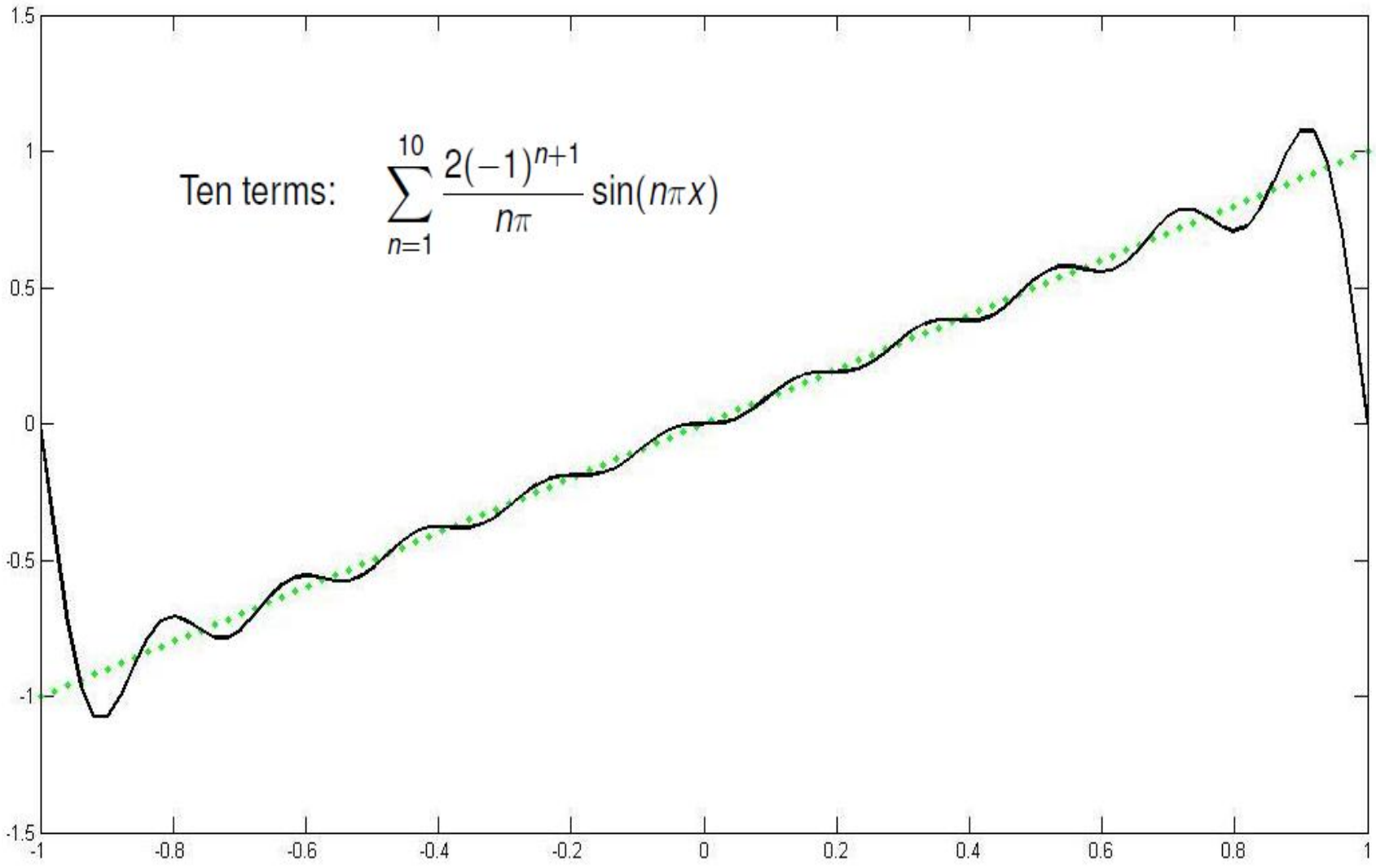


Figure: Plot of $f(x) = x$ for $-1 < x < 1$ with 10 terms of the Fourier series

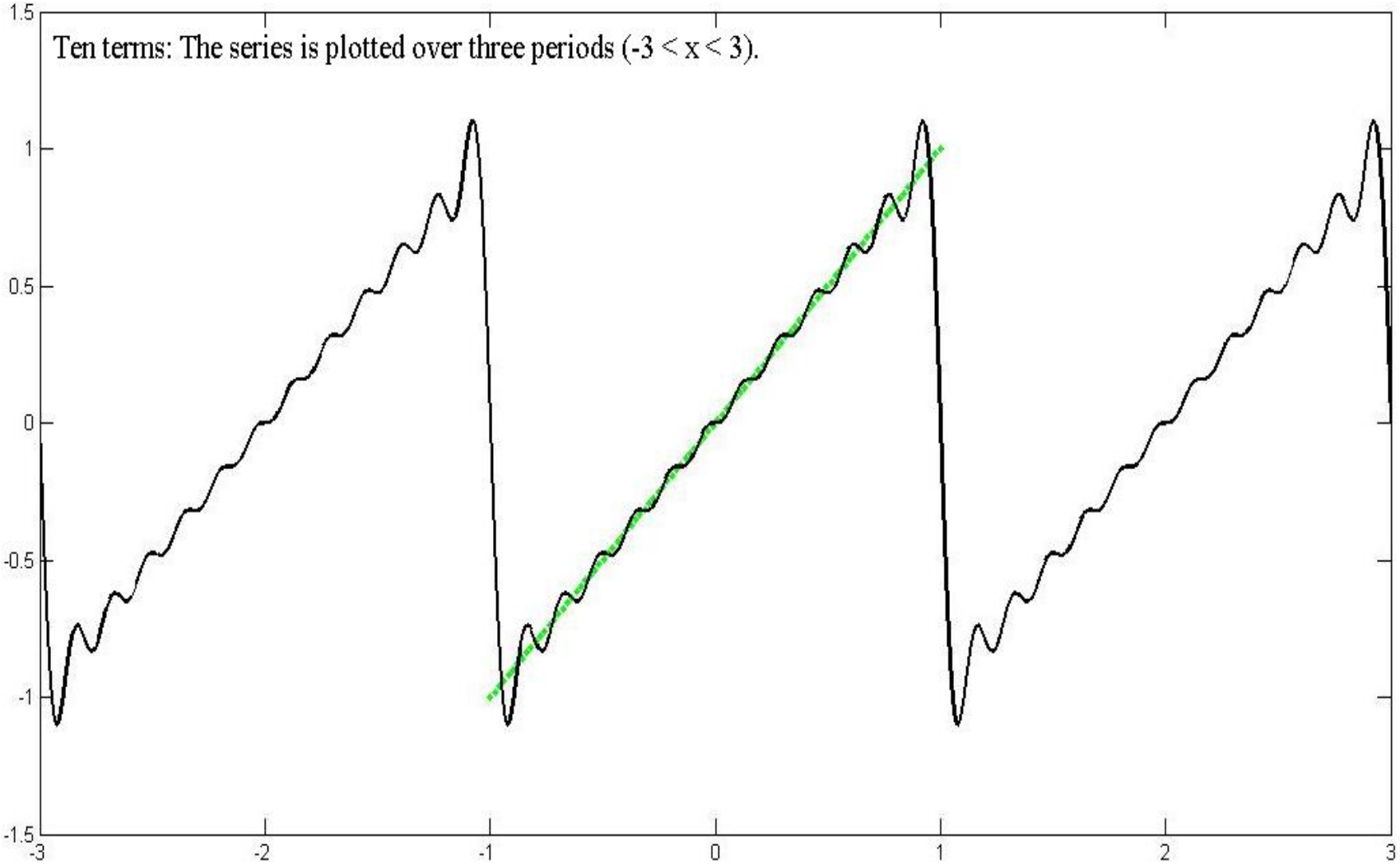
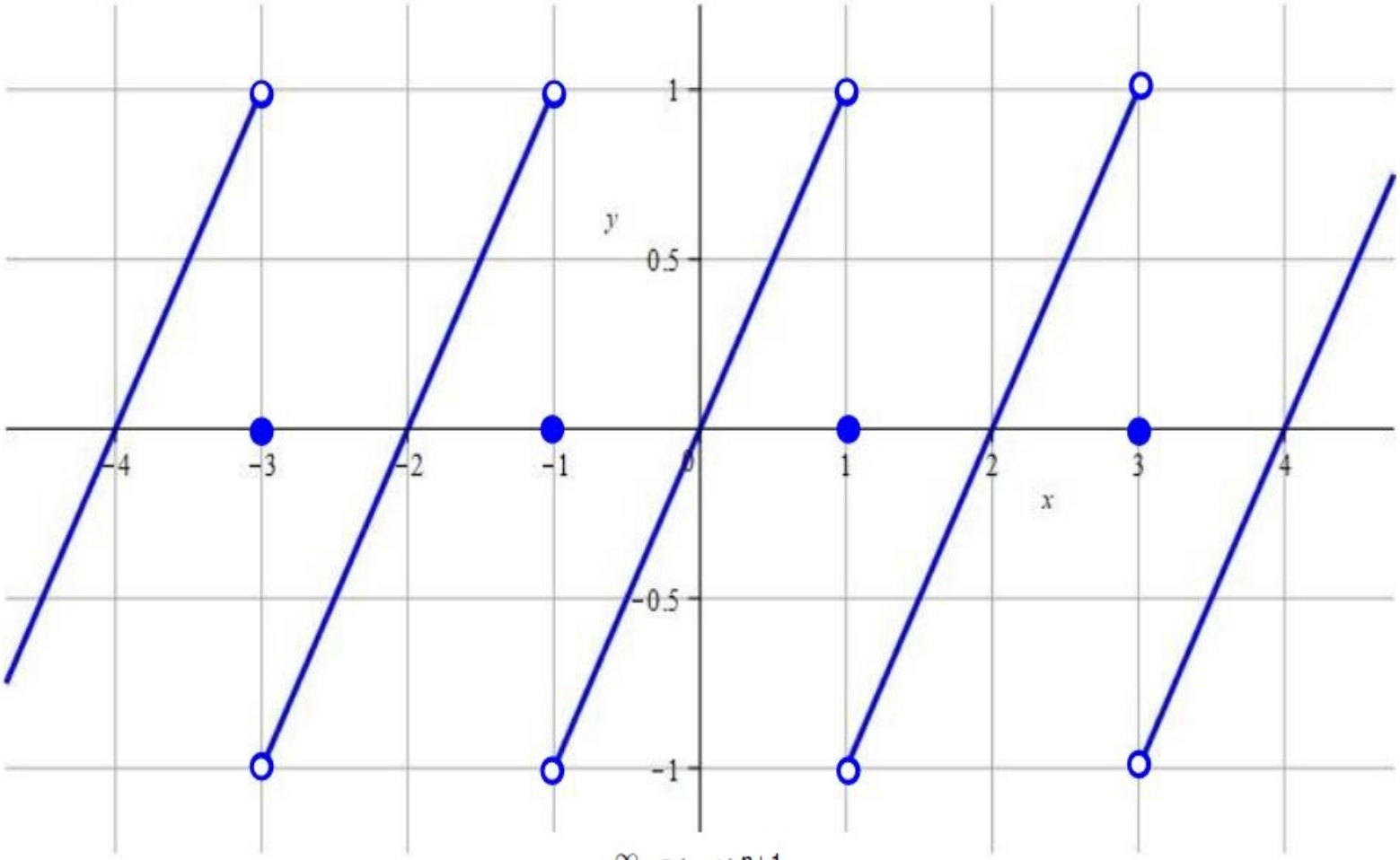


Figure: Plot of $f(x) = x$ for $-1 < x < 1$ with the Fourier series plotted on $(-3, 3)$. Note that the series repeats the profile every 2 units. At the jumps, the series converges to $(-1 + 1)/2 = 0$.



$$S_f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi x)$$

Figure: Here is a plot of the series (what it converges to). We see the periodicity and convergence in the mean. **Note:** A plot like this is determined by our knowledge of the generating function and Fourier series, not by analyzing the series itself.

Section 18: Sine and Cosine Series

Functions with Symmetry

Recall some definitions:

Suppose f is defined on an interval containing x and $-x$.

If $f(-x) = f(x)$ for all x, then f is said to be **even**.

If $f(-x) = -f(x)$ for all x, then f is said to be **odd**.

For example, $f(x) = x^n$ is even if n is even and is odd if n is odd. The trigonometric function $g(x) = \cos x$ is even, and $h(x) = \sin x$ is odd.

Integrals on symmetric intervals

If f is an even function on $(-p, p)$, then

$$\int_{-p}^{p} f(x)\,dx = 2\int_{0}^{p} f(x)\,dx.$$

If f is an odd function on $(-p, p)$, then

$$\int_{-p}^{p} f(x)\,dx = 0.$$

Products of Even and Odd functions

$$\text{Even} \quad \times \quad \text{Even} \quad = \quad \text{Even},$$

and

$$\text{Odd} \quad \times \quad \text{Odd} \quad = \quad \text{Even}.$$

While

$$\text{Even} \quad \times \quad \text{Odd} \quad = \quad \text{Odd}.$$

So, suppose f **is even** on $(-p, p)$. This tells us that $f(x)\cos(nx)$ is even for all n and $f(x)\sin(nx)$ is odd for all n.

And, if f **is odd** on $(-p, p)$. This tells us that $f(x)\sin(nx)$ is even for all n and $f(x)\cos(nx)$ is odd for all n

Fourier Series of an Even Function

If f is even on $(-p, p)$, then the Fourier series of f has only constant and cosine terms. Moreover

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right)$$

where

$$a_0 = \frac{2}{p}\int_0^p f(x)\,dx$$

and

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)\,dx.$$

Fourier Series of an Odd Function

If f is odd on $(-p, p)$, then the Fourier series of f has only sine terms. Moreover

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$$

where

$$b_n = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)\,dx.$$

Find the Fourier series of f

$$f(x) = \begin{cases} x + \pi, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases}$$

An assessment of f (e.g. by plotting) tells us that f is even. So we know that the Fourier series will not have any sine terms. We can simplify the work of finding the coefficients by making use of the symmetry. We have

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\,dx = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\cos(nx)\,dx =$$

$$= \frac{2(1-(-1)^n)}{n^2\pi}$$

Example Continued...

The series is therefore

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n^2\pi} \cos(nx)$$

By recognizing and using the symmetry, we avoided the work of computing four integrals—those from $-\pi$ to 0 and then 0 to π—instead of two to obtain the a's as well as computing the integrals for the b's which would just end up being zero.

Half Range Sine and Half Range Cosine Series

Suppose f is only defined for $0 < x < p$. We can **extend** f to the left, to the interval $(-p, 0)$, as either an even function or as an odd function. Then we can express f with **two distinct** series:

Half range cosine series $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right)$$

where $$a_0 = \frac{2}{p}\int_0^p f(x)\,dx \quad \text{and} \quad a_n = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)\,dx.$$

Half range sine series $$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$$

where $$b_n = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)\,dx.$$

Extending a Function to be Odd

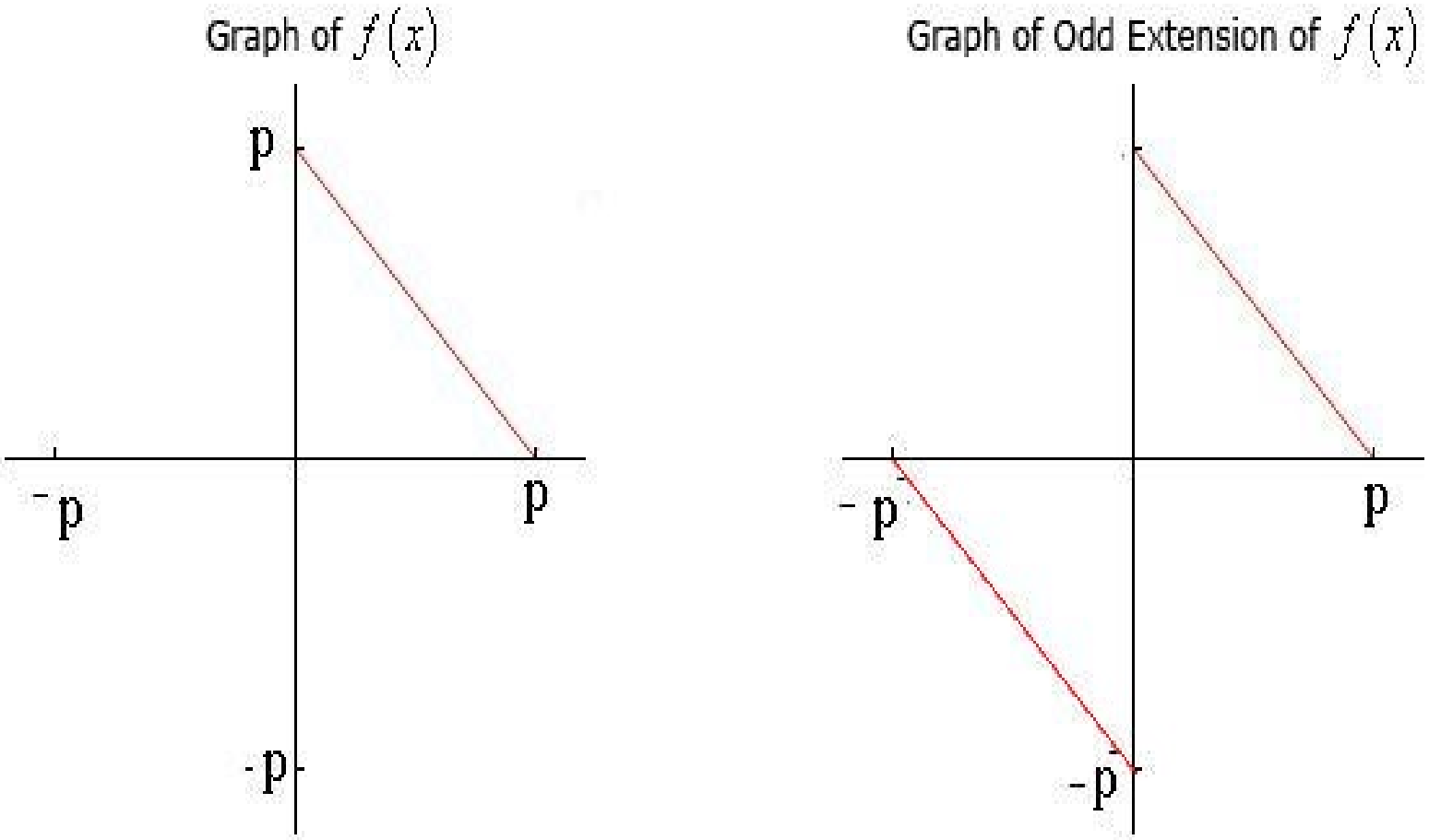


Figure: $f(x) = p - x$, $0 < x < p$ together with its **odd** extension.

Extending a Function to be Even

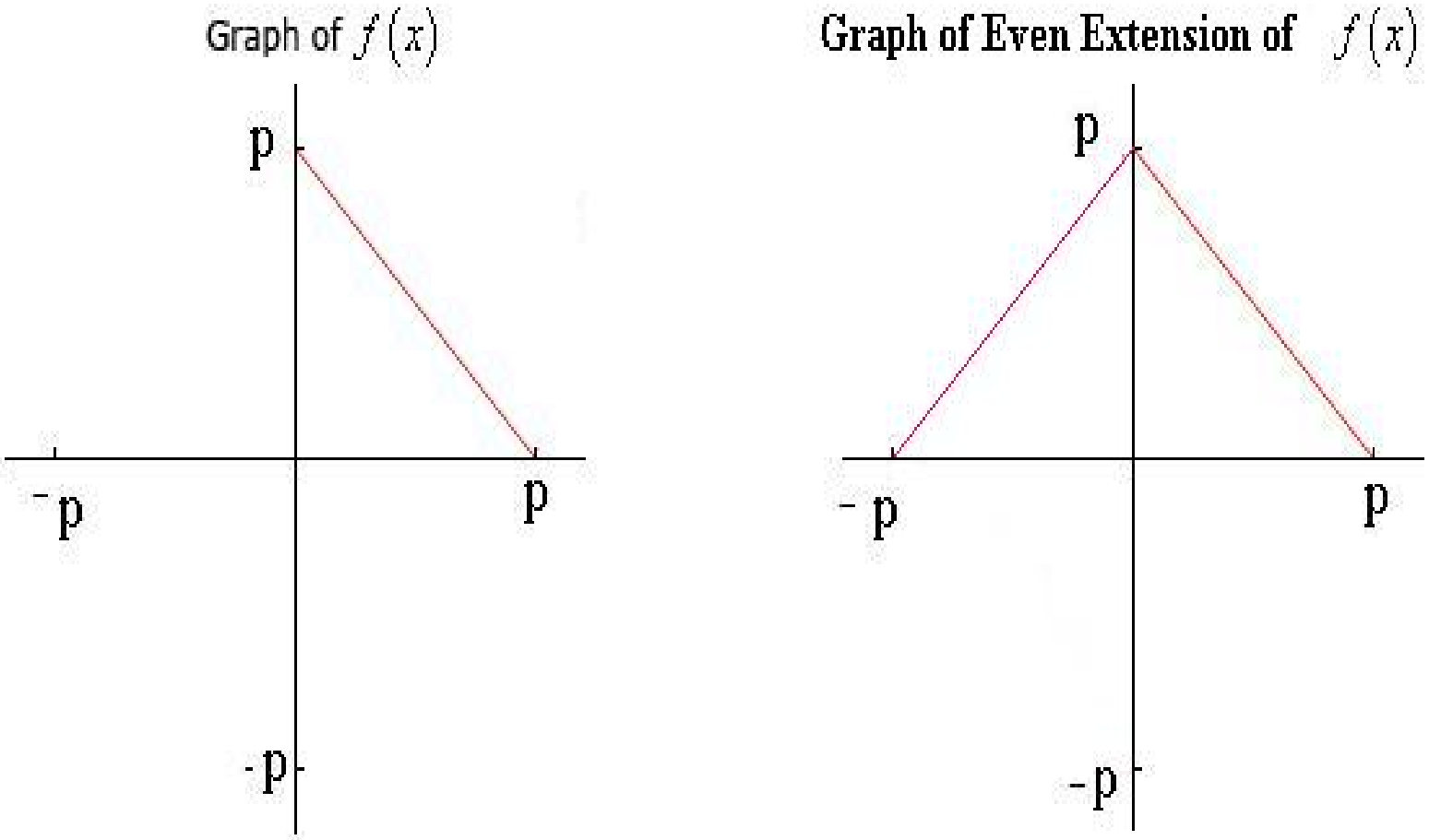


Figure: $f(x) = p - x$, $0 < x < p$ together with its **even** extension.

Find the Half Range Sine Series of f

$$f(x) = 2 - x, \quad 0 < x < 2$$

Here, the value $p = 2$. Using the formula for the coefficients of the sine series

$$\begin{aligned} b_n &= \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right)\,dx \\ &= \int_0^2 (2-x)\sin\left(\frac{n\pi x}{2}\right)\,dx \\ &= \frac{4}{n\pi} \end{aligned}$$

The series is $\quad f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi}\sin\left(\frac{n\pi x}{2}\right).$

Find the Half Range Cosine Series of f

$$f(x) = 2 - x, \quad 0 < x < 2$$

Using the formulas for the cosine series

$$\begin{aligned}
a_0 &= \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^2 (2-x)\,dx = 2 \\
a_n &= \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)\,dx \\
&= \int_0^2 (2-x)\cos\left(\frac{n\pi x}{2}\right)\,dx \\
&= \frac{4(1-(-1)^n)}{n^2\pi^2}
\end{aligned}$$

Example Continued...

We can write out the half range cosine series

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right).$$

We have two different series representations for this function each of which converge to $f(x)$ on the interval $(0, 2)$. The following plots show graphs of f along with partial sums of each of the series. When we plot over the interval $(-2, 2)$ we see the two different symmetries. Plotting over a larger interval such as $(-6, 6)$ we can see the periodic extensions of the two symmetries.

Plots of f with Half range series

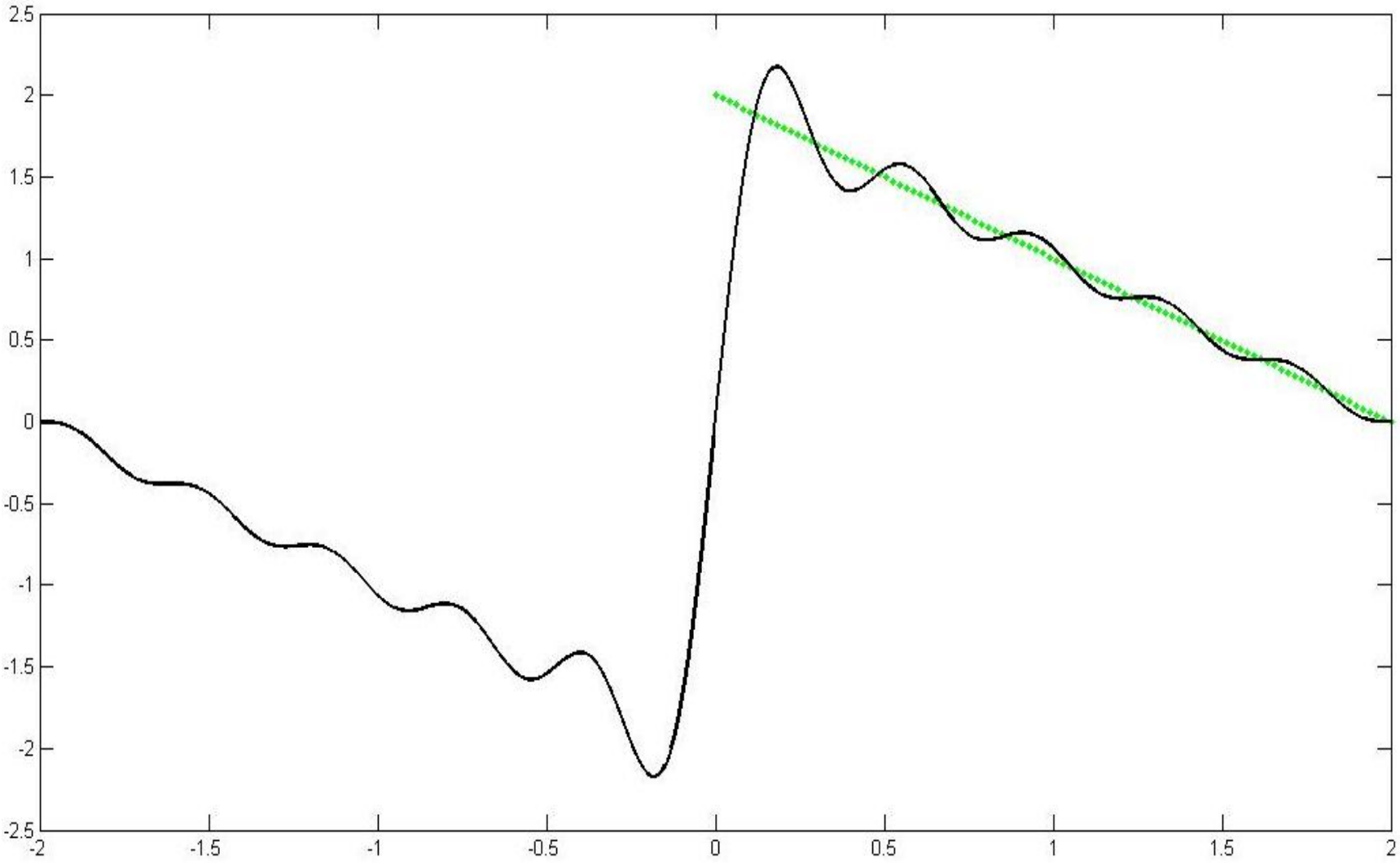


Figure: $f(x) = 2 - x$, $0 < x < 2$ with 10 terms of the sine series.

Plots of f with Half range series

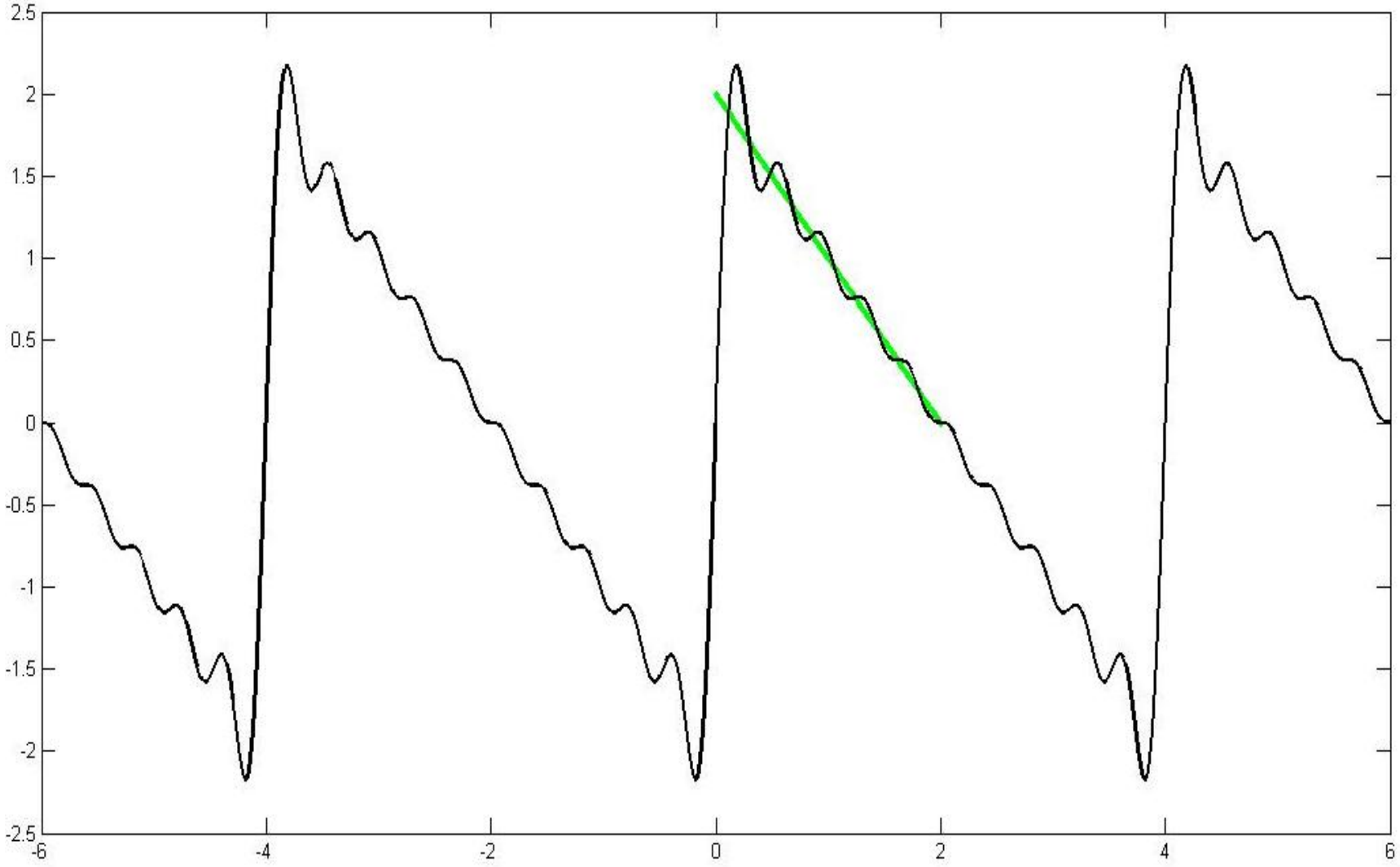


Figure: $f(x) = 2 - x$, $0 < x < 2$ with 10 terms of the sine series, and the series plotted over $(-6, 6)$

Plots of f with Half range series

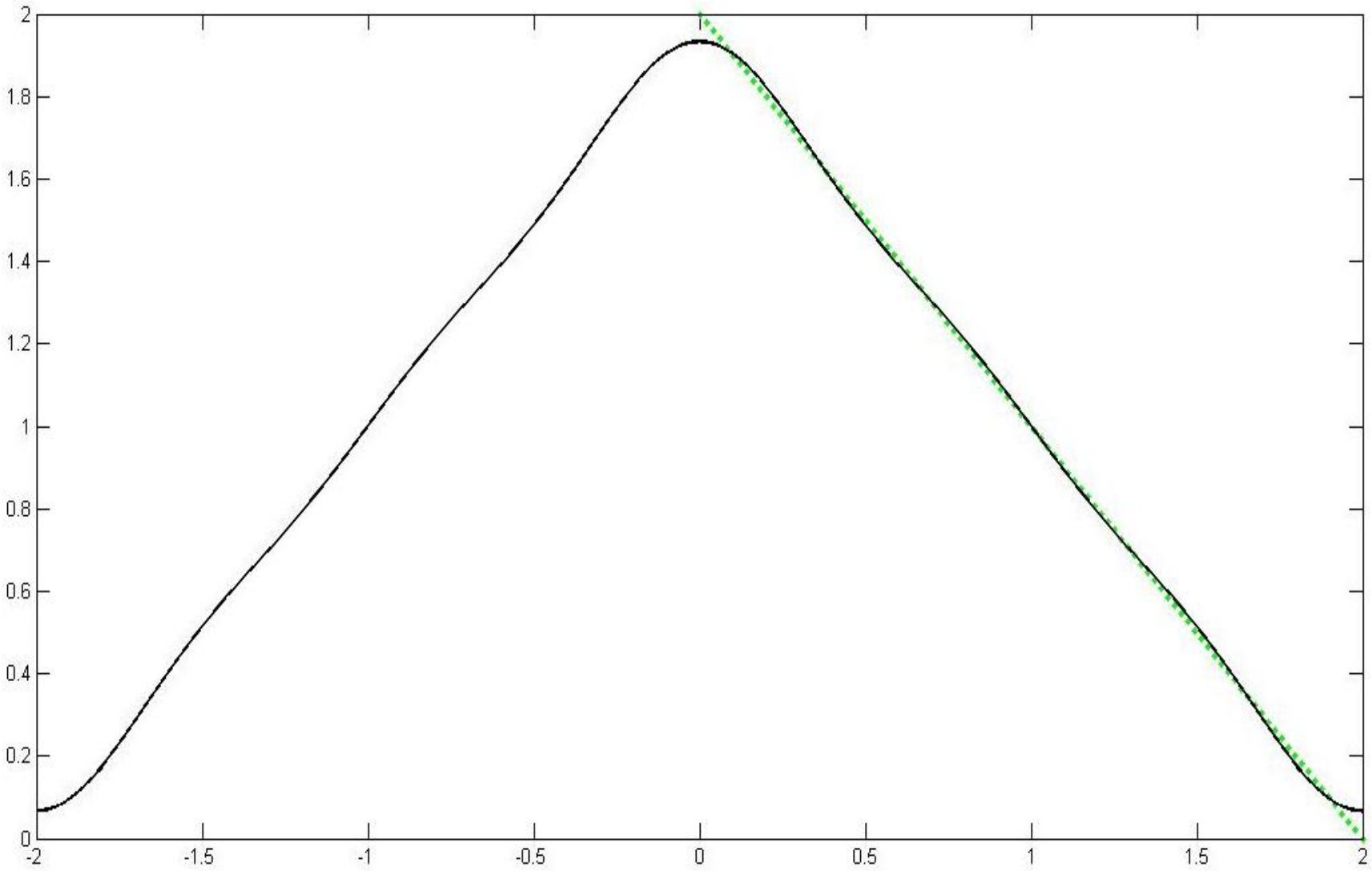


Figure: $f(x) = 2 - x$, $0 < x < 2$ with 5 terms of the cosine series.

Plots of f with Half range series

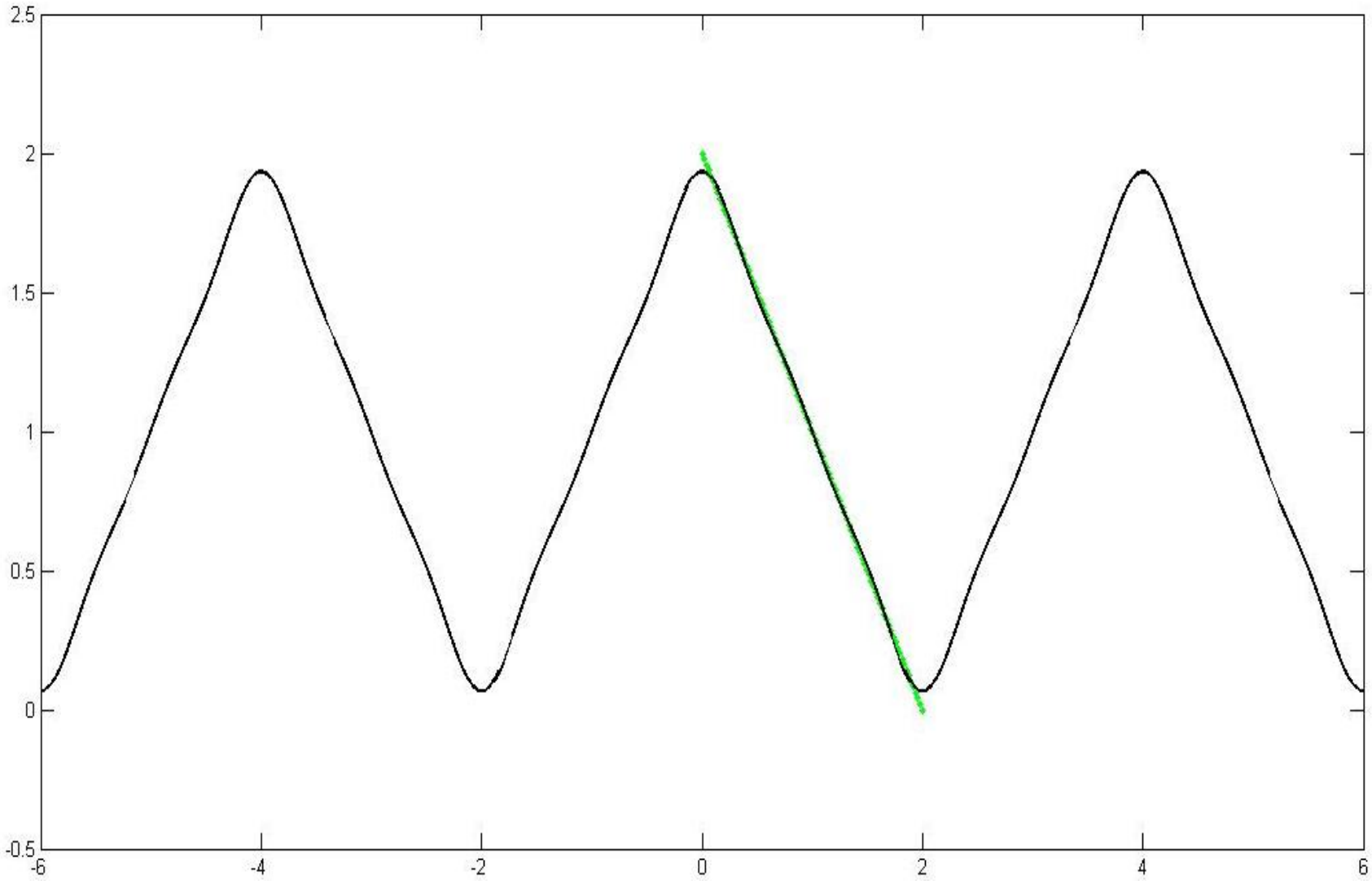


Figure: $f(x) = 2 - x$, $0 < x < 2$ with 5 terms of the cosine series, and the series plotted over $(-6, 6)$

Solution of a Differential Equation

An undamped spring mass system has a mass of 2 kg attached to a spring with spring constant 128 N/m. The mass is driven by an external force $f(t) = 2t$ for $-1 < t < 1$ that is 2-periodic so that $f(t+2) = f(t)$ for all $t > 0$. Determine a particular solution x_p for the displacement for $t > 0$.

We are only interested in $t > 0$, but since f is an odd function that is 2-periodic, we can express it conveniently as a sine series

$$f(t) = 2\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t).$$

Our differential equation is therefore

$$2x'' + 128x = f(t) \quad \Longrightarrow \quad x'' + 64x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t).$$

Solution of a Differential Equation

Let us assume that x_p can similarly be determined as a sine series (note that this is the Method of Undetermined Coefficients!) via

$$x_p = \sum_{n=1}^{\infty} B_n \sin(n\pi t).$$

To determine the coefficients B_n, we substitute this into the left side of our DE. Observe that (assuming we can differentiate term by term)

$$x_p'' = \sum_{n=1}^{\infty} -n^2\pi^2 B_n \sin(n\pi t).$$

Solution of a Differential Equation

Upon substitution we get

$$x_p'' + 64x_p = \sum_{n=1}^{\infty} -n^2\pi^2 B_n \sin(n\pi t) + 64 \sum_{n=1}^{\infty} B_n \sin(n\pi t) =$$

$$= \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t).$$

Collecting the series on the left side produces

$$\sum_{n=1}^{\infty} (64 - n^2\pi^2) B_n \sin(n\pi t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t).$$

Finally, comparing coefficients of $\sin(n\pi t)$ for each value of n yields the formulas for the B's

$$B_n = \frac{2(-1)^{n+1}}{n\pi(64 - n^2\pi^2)}.$$

Solution of a Differential Equation

We should be careful to determine whether our formula is well defined for every value of n. Since $64 - n^2\pi^2$ is never zero, our expression is always valid. The particular solution can now be expressed

$$x_p = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi(64 - n^2\pi^2)} \sin(n\pi t).$$

If the specifics of the problem had resulted in a value of n, say n_k, for which B_{n_k} could not be solved (i.e. if $\omega^2 - n_k^2\pi^2 = 0$), this would indicated a pure resonance term. The above approach would still yield the remaining B values. The resonance term would have to be considered separately. We could assume, using the principle of superposition, that

$$x_p = A_{n_k} t \cos(n_k\pi t) + B_{n_k} t \sin(n_k\pi t) + \sum_{\substack{n=1 \\ n \neq n_k}}^{\infty} B_n \sin(n\pi t).$$

Table of Laplace Transforms

$f(t) = \mathscr{L}^{-1}\{F(s)\}$	$F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$
1	$\frac{1}{s} \quad s > 0$
$t^n \quad n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}} \quad s > 0$
$t^r \quad r > -1$	$\frac{\Gamma(r+1)}{s^{r+1}} \quad s > 0$
e^{at}	$\frac{1}{s-a} \quad s > a$
$\sin(kt) \quad k \neq 0$	$\frac{k}{s^2+k^2} \quad s > 0$
$\cos(kt)$	$\frac{s}{s^2+k^2} \quad s > 0$
$e^{at} f(t)$	$F(s-a)$
$\mathscr{U}(t-a) \quad a > 0$	$\frac{e^{-as}}{s} \quad s > 0$
$\mathscr{U}(t-a) f(t-a) \quad a > 0$	$e^{-as} F(s)$
$\mathscr{U}(t-a) g(t) \quad a > 0$	$e^{-as} \mathscr{L}\{g(t+a)\}$
$f'(t)$	$sF(s) - f(0)$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$
$tf(t)$	$-\frac{d}{ds} F(s)$
$t^n f(t) \quad n = 1, 2, \ldots$	$(-1)^n \frac{d^n}{ds^n} F(s)$

$$\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathscr{L}\{f(t)\} + \beta \mathscr{L}\{g(t)\}$$

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha \mathscr{L}^{-1}\{F(s)\} + \beta \mathscr{L}^{-1}\{G(s)\}$$

Appendix: Crammer's Rule

Crammer's rule is an approach to solving a linear system of equations under the conditions that

a. the number of equations matches the number of unknowns (i.e. the system is square), and
b. the system is uniquely solvable (i.e. there is exactly one solution).

While Crammer's rule can be used with any size system, we'll restrict ourselves to the 2×2 case. We obtain the solution in terms of ratios of determinants. First, let's see how the method plays out in general, and then we illustrate with an example.

Note: Crammer's rule will produce the same solution as any other approach. Its advantage is in its computational simplicity (which gets lost the larger the system is).

Appendix: Crammer's Rule

We begin with a 2×2 (two equations in two variables) system

$$\begin{array}{ccccc} ax & + & by & = & e \\ cx & + & dy & = & f \end{array}$$

The unknowns are x and y, and the parameters a, b, c, d, e, and f are constants[25].

We're going to form 3 matrices. The first is the coefficient matrix for the system. I'll call that A. So

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

[25] We can allow any of a through f to be unknown parameters, but they don't depend on x or y so they will still be considered *constant*.

Appendix: Crammer's Rule

$$\begin{array}{ccccc} ax & + & by & = & e \\ cx & + & dy & = & f \end{array}$$

Next, we form two more matrices that I'll call A_x and A_y. These matrices are obtained by replacing one column of A with the values from the right side of the system. For A_x we replace the first column (the one with x's coefficients), and for A_y we replace the second column (the one with y's coefficients). We have

$$A_x = \begin{bmatrix} e & b \\ f & d \end{bmatrix}, \quad \text{and} \quad A_y = \begin{bmatrix} a & e \\ c & f \end{bmatrix}.$$

Appendix: Crammer's Rule

$$\begin{array}{ccccc} ax & + & by & = & e \\ cx & + & dy & = & f \end{array}$$

Now we have the three matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A_x = \begin{bmatrix} e & b \\ f & d \end{bmatrix}, \quad \text{and} \quad A_y = \begin{bmatrix} a & e \\ c & f \end{bmatrix}.$$

The condition that the system is uniquely solvable[26] guarantees that $\det(A) \neq 0$. Here's the punch line: The solution to the system is

$$x = \frac{\det(A_x)}{\det(A)} \quad \text{and} \quad y = \frac{\det(A_y)}{\det(A)}.$$

[26] This is a well known result that can be found in any elementary discussion of Linear Algebra.

Appendix: Crammer's Rule

Let's look at a simple example. Solve the system of equations

$$\begin{array}{rcrcr} 2x & - & 3y & = & -4 \\ 3x & + & 7y & = & 2 \end{array}$$

Let's form the three matrices and verify that the determinant of the coefficient matrix isn't zero.

$$A = \begin{bmatrix} 2 & -3 \\ 3 & 7 \end{bmatrix}, \quad A_x = \begin{bmatrix} -4 & -3 \\ 2 & 7 \end{bmatrix}, \quad \text{and} \quad A_y = \begin{bmatrix} 2 & -4 \\ 3 & 2 \end{bmatrix}.$$

$$\det(A) = 2(7) - (-3)(3) = 23.$$

Appendix: Crammer's Rule

Okay, we can proceed by finding the determinants of the two matrices A_x and A_y. We get

$$\det(A_x) = -4(7) - 2(-3) = -22 \quad \text{and} \quad \det(A_y) = 2(2) - 3(-4) = 16.$$

Together with $\det(A) = 23$, the solution to the system is

$$x = -\frac{22}{23} \quad \text{and} \quad y = \frac{16}{23}.$$

It's worth taking a moment to substitute those values back into the system to verify that it does indeed solve it. It's not hard to imagine, looking at the solution, that solving it with substitution or elimination is probably more tedious.

Appendix: Crammer's Rule

This process can be extended in the obvious way to larger systems of equations provided they are square and uniquely solvable. You form the coefficient matrix. Then for each variable, form another matrix by replacing that variable's coefficient column with the values on the right side of the system. Each variable's solution value will be the ratio of the corresponding determinants.

For larger systems (perhaps bigger than 3×3) one must weigh the computational intensity of computing determinants with that of other options such as elimination or substitution. The approach also breaks down if the coefficient matrix has zero determinant. The system may have solutions (or not), but another approach is needed to characterize them.