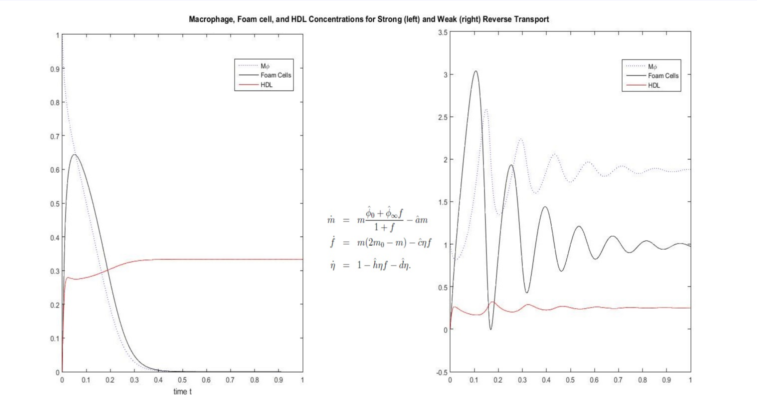
Ordinary Differential Equations

Lake Ritter, Kennesaw State University



2024

MATH 2306: Ordinary Differential Equations

Lake Ritter, Kennesaw State University

This manuscript is a *text-like* version of the lecture slides I have created for use in my ODE classes at KSU. It is not intended to serve as a complete text or reference book, but is provided as a supplement to my students. I strongly advise against using this text as a substitute for regular class attendance. In particular, most of the computational details have been omitted. However, my own students are encouraged to read (and read ahead) as part of preparing for class.

All others are welcomed to use this document for any noncommercial use. To report errata or to request TEXand figure files please email me at lritter@kennesaw.edu.

Table of Contents

Section 1: Concepts and Terminology

In the first couple of sections of this document, we introduce the main ideas and the language inherent to the study of *differential equations*. What is a differential equation? What sorts of characteristics do these things have, and how do we talk about this subject?

Differential Equations arise in a broad array of mathematical modeling scenarios (physics and engineering, biology, finance, etc.), especially when we have information about how some quantity changes. As the name suggests, dealing with differential equations requires the tools of calculus, differentiation and integration. So we assume that these are familiar concepts.

Rather than trying to start by defining what a differential equation is, let's look at an example of how a mathematical model might be built from information about how some quantity changes. This example is coming from elementary biology.

Why are most cells microscopic?

A *cell* is the smallest individual unit from which living organisms are composed. Why is it that most cells are microscopic[1]?

Let's consider how we can model a single cell. To keep things simple, let's suppose that our cell is a sphere so that we only have to consider one characteristic length, a radius. To determine what sort of limitations there could be on a cells size, we need to model cell growth—i.e., how the cell mass changes. A cell takes in nutrients through its cell wall surface, and metabolizes nutrients in various ways. A reasonable growth assumption is that cell mass can increase when nutrient uptake exceeds metabolic need.

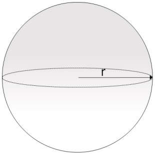


Figure: An idealized cell as a sphere of radius r.

[1] While macroscopic cells certainly exists, most cells are measured on the scale of micrometers.

A model of cell mass

For our idealized spherical cell, we model the mass, m, as $m = \rho V$ where ρ is the density (mass per unit volume) and V is the volume of the cell. If we assume that ρ is constant (say the average density over the whole cell), then in terms of the radius, we have

$$m = \frac{4\pi}{3}\rho r^3.$$

The rate at which nutrients enter the cell is proportional to the surface area S, and the rate of metabolism is proportional to the cell volume. So the rate of change of mass with respect to time can be modeled by the *balance equation*

$$\frac{dm}{dt} = \alpha S - \beta V.$$

where α and β are nonnegative rates of nutrient uptake and metabolism, respectively.

An equation for change in cell mass

Since we know about factors that change mass, as opposed to having direct information about what the mass is at some moment, our equation involves a derivative. This is the defining feature of a differential equation, it involves derivative(s).

The goal was to understand what sort of constraints might keep cells from growing arbitrarily large. We can rephrase our equation in terms of the cell radius which characterizes the size of the cell. The surface area $S = 4\pi r^2$, and stating mass, surface area, and volume in terms of r, our equation becomes

$$\frac{d}{dt}\left(\frac{4\pi}{3}\rho r^3\right) = 4\pi\alpha r^2 - \frac{4}{3}\pi\beta r^3. \tag{1}$$

An equation for change in cell radius

Since r is changing in time, we can use implicit differentiation to write

$$\frac{d}{dt}\left(\frac{4\pi}{3}\rho r^3\right) = \frac{4\pi}{3}\rho(3r^2)\frac{dr}{dt} = 4\pi\rho r^2\frac{dr}{dt}.$$

Our equation becomes

$$4\pi\rho r^2\frac{dr}{dt} = 4\pi\alpha r^2 - \frac{4}{3}\pi\beta r^3.$$

We're not interested in $r = 0$ (since that corresponds to no cell), so we can cancel the common $4\pi r^2$ and obtain an equation for the rate of change of the radius.

$$\rho\frac{dr}{dt} = \alpha - \frac{1}{3}\beta r.$$

Analysis

We can get some insight from this equation. While it isn't wholly necessary, let's divide both sides by ρ and factor out the $\frac{\beta}{3}$ to write our equation in the form

$$\frac{dr}{dt} = \frac{\beta}{3\rho}\left(\frac{3\alpha}{\beta} - r\right). \tag{2}$$

Equation (2) is an example of a differential equation. It's a fairly simple example of a differential equation and one that we'll be able to deal with very soon. Something important to note here is that the rate of change of the radius depends on the radius! So this isn't a simple "Calc II" problem in which you can integrate directly (undo the derivative on the left side). We can't integrate the right side with respect to t because we don't even know what sort of function r is! But even without knowing how to *solve*[2] the equation, we can say something about cell size.

[2] Soon, we'll be able to conclude that $r(t) = \frac{3\alpha}{\beta} - Ce^{-\frac{\beta t}{3\rho}}$, where C is some constant (think of it as a constant of integration).

Analysis

Recall from early calculus that the sign of a derivative indicates whether a quantity is increasing (positive derivative) or decreasing (negative derivative). If we look at equation (2), we see something interesting. The parameter $\frac{\beta}{3\rho}$ is positive, so the sign of the right hand side is determined by the difference $\left(\frac{3\alpha}{\beta} - r\right)$. If

$$r < \frac{3\alpha}{\beta}, \quad \text{then} \quad \frac{dr}{dt} > 0,$$

and the radius (and hence the mass) increases. If

$$r > \frac{3\alpha}{\beta}, \quad \text{then} \quad \frac{dr}{dt} < 0,$$

and the radius (hence the mass) decreases. The number $\frac{3\alpha}{\beta}$ seems to represent some sort of biological *sweet spot* governing cell size. This ratio [3] is a measure of the nutrient uptake (α) to metabolic needs (β) of the cell.

[3] The number 3 is a characteristic of the geometry. We would expect it to change if we consider a different cell shape.

Analysis

Without even finding an explicit function for the radius, $r(t)$, our model predicts that there is a limit on cell size. Cell growth facilitates greater surface area through which nutrients can be absorbed, but it also necessitates greater metabolic need. This places a limit on cell size that balances these two factors. A similar process can be followed if we first assume a different cell shape (e.g., capsular).

So there it is. Without actually defining what it is, we see an example of a differential equation. The example also highlights a couple of interesting things:

1. it's unlikely that integration alone will be useful to solve for some unknown function based only on information about its derivative,
2. it may be possible to glean useful information about some process even without actually solving (whatever that means) a differential equation.

Now let's proceed with our initial terms and concepts.

What is a Differential Equation?

We know that if $y = \phi(x)$ is some differentiable function, then its derivative, which we could write as y', $\frac{dy}{dx}$, or $\phi'(x)$ is some related function (often different from y).

To pick a familiar example, if $y = \cos(2x)$, then $\frac{dy}{dx} = -2\sin(2x)$. The derivative is itself a new differentiable function, and in fact $y'' = -4\cos(2x)$.

Notice that this particular function ($y = \cos(2x)$) happens to have a special relationship to its rates of change, specifically $y'' = -4y$ or what we're more likely to write as the equation

$$y'' + 4y = 0.$$

This equation that involves a second derivative is another example of a differential equation.

What is a Differential Equation?

In equations such as

$$y'' + 4y = 0 \quad \text{or} \quad \frac{dr}{dt} = \frac{\beta}{3\rho}\left(\frac{3\alpha}{\beta} - r\right),$$

the unknown is some function. Well, the equation $y'' + 4y = 0$ was constructed artificially by starting with a known function $y = \cos(2x)$, but we can certainly ask:

1. If we started with the equation $y'' + 4y = 0$, could we figure out that $\cos(2x)$ is a function that y could be?
2. Are there other functions besides $\cos(2x)$ that y could be—or more generally, can we characterize all the possible functions that y could be?

Much of the goal of this course is to explore some types of equations for which we have strategies to find the unknown function(s). Let's give a broad definition of what a differential equation is.

Definition

Definition

A **Differential Equation** is an equation containing the derivative(s) of one or more dependent variables, with respect to one or more indendent variables.

We can characterize the *variables* in calculus terms:

Independent and Dependent Variables

An **Independent Variable:** will appear as one that derivatives are taken with respect to. (The x in $y = f(x)$.)
A **Dependent**[a] **Variable:** will appear as one that derivatives are taken of. (The y in $y = f(x)$.)

[a] Think of dependent variables as the functions.

If we consider the expressions below, we can identify related independent and dependent variables. Leibniz notation is convenient for this.

$$\frac{dy}{dx} \qquad \frac{du}{dt} \qquad \frac{dx}{dr}$$

Recall that the expression $\frac{dy}{dx}$ is the derivative **OF** the function y **WITH RESPECT TO** the variable x.

Exercise: With that in mind, identify the independent and dependent variable in each expression. (Exercise left to the reader.)

We can start by catagorizing all differential equations as falling into one of two categories based on independent variables.

Classifications: Type ODE or PDE

ODEs

An **ordinary differential equation (ODE)** has exactly one independent variable[a]. For example

$$\frac{dy}{dx} - y^2 = 3x, \quad \text{or} \quad \frac{dy}{dt} + 2\frac{dx}{dt} = t, \quad \text{or} \quad y'' + 4y = 0$$

[a] These are the subject of this course.

PDEs

A **partial differential equation (PDE)** has two or more independent variables. For example

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}, \quad \text{or} \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

For those unfamiliar with the notation, ∂ is the symbol used when taking a derivative with respect to one variable keeping the remaining variable(s) constant. $\frac{\partial u}{\partial t}$ is read as the "partial derivative of u with respect to t."

Note on Notation

The subject of this course is **Ordinary Differential Equations.** So we won't be considering PDEs. The symbol ∂ is called a *partial* symbol. It is used to express derivatives when there are two or more independent variables. It's similar to d but indicates that one variable is being held fixed. If the function $u(x,t)$ depends on two variables x and t, then the expression

$$\frac{\partial u}{\partial t}$$

is read as "the "partial derivative of u with respect to t." It is defined by

$$\lim_{\Delta t \to 0} \frac{u(x, t+\Delta t) - u(x,t)}{\Delta t}.$$

Classifications: Order

Now, let's categorize differential equations according to the derivatives involved.

Definition: Order of a Differential Equation

The order of a differential equation is defined to be the same as the highest order derivative appearing anywhere in the equation.

Examples:

$$\frac{dy}{dx} - y^2 = 3x \tag{3}$$

$$y''' + (y')^4 = x^3 \tag{4}$$

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} \tag{5}$$

These are (3) **first order**, (4) **third order**, and (5) **second order**, respectively. Note that powers on expressions (such as the fourth power $(y')^4$ or the third power on x^3) are not relevant to the order of a differential equation.

Notations and Symbols

In writing derivatives, we'll mostly use standard notation familiar to you from Calculus:

$$\text{Leibniz:} \quad \frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \ldots \quad \frac{d^ny}{dx^n}, \quad \text{or}$$

$$\text{Prime \& superscripts:} \quad y', \quad y'', \quad \ldots \quad y^{(n)}.$$

There is another popular, yet less common, notation for ordinary derivatives reserved for use when the independent variable denotes **time.** Newton's **dot notation** is similar to prime notation, except that the dot is placed on top of the dependent variable. For example, if $s(t)$ represents the position of a body, then

$$\text{velocity is } \frac{ds}{dt} = \dot{s}, \quad \text{and acceleration is } \frac{d^2s}{dt^2} = \ddot{s}$$

Notations and Symbols

On occasion, we'll want to reference a generic ODE. We have a couple of formats for that.

An n^{th} order ODE, with independent variable x and dependent variable y can always be expressed as an equation

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

where F is some real valued function of $n+2$ variables.

Normal Form

Normal Form: If it is possible to isolate the highest derivative term, then we can write a **normal form** of the equation

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}).$$

Examples

Let's consider the equation we looked at before.

The equation $y'' + 4y = 0$ has the form $F(x, y, y', y'') = 0$ where

$$F(x, y, y', y'') = y'' + 4y.$$

This equation is second order. In normal form it is $y'' = f(x, y, y')$ where

$$f(x, y, y') = -4y.$$

Equations in normal form when $n = 1$ or $n = 2$ look like

$$\frac{dy}{dx} = f(x, y) \quad \text{or} \quad \frac{d^2y}{dx^2} = f(x, y, y').$$

Notations and Symbols

Differential Form

If $M(x,y)$ and $N(x,y)$ are functions of the variables x and y, then an expression of the form

$$M(x,y)\,dx + N(x,y)\,dy$$

is called a **Differential Form.** A first order equation may be written in terms of a differential form as follows:

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

Note that this can be rearranged into a couple[4] of different normal forms

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} \quad \text{or} \quad \frac{dx}{dy} = -\frac{N(x,y)}{M(x,y)}$$

[4] We have to assume that N or M is nonzero as needed.

Classifications: Linearity

One of the most important characteristics a differential equation has is that of being a **linear** differential equation or a nonlinear one.

Linear Differential Equation

Linearity: An n^{th} order differential equation is said to be **linear** if it can be written in the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

Example First Order:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Example Second Order:

$$a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Properties of a Linear ODE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

Properties of Linear ODEs

- Each of the coefficients $a_0, \ldots, a_n$ and the right hand side g may depend on the independent variable but not the dependent one.
- y, and its derivatives can only appear as themselves (not squared, square rooted, inside some other function).
- The characteristic structure of the left side is that

  $$y, \quad \frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \ldots, \frac{d^n y}{dx^n}$$

  are multiplied by functions of the independent variable and added together.

Examples (Linear -vs- Nonlinear)

$$y'' + 4y = 0 \qquad\qquad t^2\frac{d^2x}{dt^2} + 2t\frac{dx}{dt} - x = e^t$$

Convince yourself that the top two equations are linear. (Hint: Try to identify the parts such as a_2, a_1, a_0 and g.)

$$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 = x^3 \qquad\qquad u'' + u' = \cos u$$

The terms $\left(\frac{dy}{dx}\right)^4$ and $\cos u$ make these nonlinear. Can you see what property is violated?

Solutions

Let's define what we mean by a *solution* to a differential equation. These come in two broad flavors. Here, we consider the generic ODE

$$F(x, y, y', \ldots, y^{(n)}) = 0. \tag{6}$$

Definition: Solution (*Explicit* Solution)

A function ϕ defined on an interval I[a] and possessing at least n continuous derivatives on I is a **solution** of (6) on I if upon substitution (i.e. setting $y = \phi(x)$) the equation reduces to an identity.

[a] The interval is called the *domain of the solution* or the *interval of definition*.

Definition: Implicit Solutions

Definition: An **implicit solution** of (6) is a relation $G(x, y) = 0$ provided there exists at least one function $y = \phi$ that satisfies both the differential equation (6) and this relation.

Domain of Definition

The solution to a differential equation is a function, and as such will have some domain which we can all its *domain* or *interval of definition*. Here, we use the term in a more restrictive sense than what you may be used to from Calculus. In particular, the domain is required to be an *interval*. It need not be finite, but it can't contain holes or jumps.

What do we mean by that?

To illustrate, consider the differential equation $\frac{dy}{dx} = -y^2$. You can readily convince yourself that $y = \frac{1}{x}$ is a function that satisfies this ODE. What is the domain of this function?

Function vs Solution

Outside of the context of differential equations, it's reasonable to say that the domain of $y = \dfrac{1}{x}$ is $(-\infty, 0) \cup (0, \infty)$. That is, it is defined on all reals except for zero.

But $(-\infty, 0) \cup (0, \infty)$ is NOT an interval. In fact, the graph of $y = \dfrac{1}{x}$ on this domain has two distinct pieces and is not defined (much less continuously differentiable) on any interval that includes zero. Since we require our solution to be sufficiently continuously differentiable on the interval of definition, the domain of $y = \dfrac{1}{x}$ **as the solution to the differential equation** $y' = -y^2$ could be $(0, \infty)$ or $(-\infty, 0)$, or any interval (a, b) that does not contain zero.

In the absence of additional information, we'll usually take the interval of definition to be the largest possible one (or one of the largest possible ones).

Function vs Solution

The graph of the solution to a differential equation will be an unbroken, single *smooth* curve.

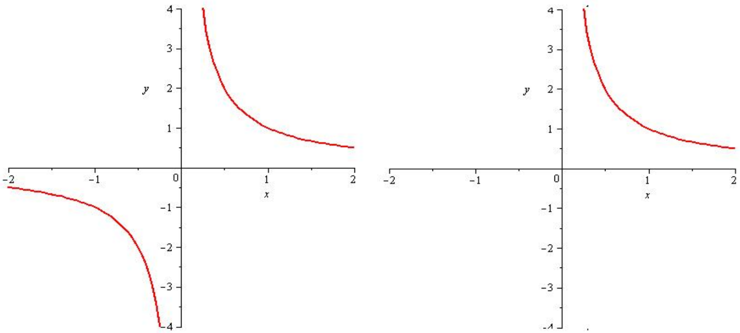


Figure: Left: Plot of $f(x) = \frac{1}{x}$ as a **function** without reference to a differential equation. Right: A possible plot of $f(x) = \frac{1}{x}$ as a **solution** of an ODE.

Solution Example

Solutions of differential equations will often include unknown constants. To illustrate, let's show that for any choice of constants c_1 and c_2, the function $y = c_1 x + \frac{c_2}{x}$ is a solution of the differential equation

$$x^2 y'' + xy' - y = 0.$$

We can find the first and second derivatives and substitute them into the ODE. Note that

$$y' = c_1 - \frac{c_2}{x^2}, \quad \text{and} \quad y'' = \frac{2c_2}{x^3}$$

So

$$\begin{aligned} x^2 y'' + xy' - y &= x^2\left(\frac{2c_2}{x^3}\right) + x\left(c_1 - \frac{c_2}{x^2}\right) - \left(c_1 x + \frac{c_2}{x}\right) \\ &= \frac{2c_2}{x} + c_1 x - \frac{c_2}{x} - c_1 x - \frac{c_2}{x} \\ &= (2c_2 - c_2 - c_2)\frac{1}{x} + (c_1 - c_1)x \\ &= 0 \end{aligned}$$

as required.

Additional Common Terms

Let's summarize some of the terms and phrases that we will frequently use when talking about differential equations and their solutions.

- A **parameter** is an unspecified constant. A parameter is assumed constant, unlike a variable. The c_1 and c_2 in $y = c_1 x + \frac{c_2}{x}$ are examples of parameters.

- A **family of solutions** is a collection of solution functions that only differ by one or more parameters. For example, $y = c_1 x + \frac{c_2}{x}$ is a family of solutions. Different members of this family correspond to different choices for c_1 and c_2.

- An ***n*-parameter family of solutions** is one containing n parameters, The family $y = c_1 x + \frac{c_2}{x}$ has two parameters, c_1 and c_2, hence this is a 2-parameter family of solutions (to the ODE in the previous example).

Additional Common Terms

- A **particular solution** is a solution that does not contain any arbitrary constants (i.e., has no nonfixed parameters). For example, $y = 2x + \frac{3}{x}$ is a particular solution to the ODE in the last example.
- An **integral curve** is the graph of one solution—typically a particular solution from a family of solutions.
- When the simple, constant function $y(x) = 0$ is the solution to a differential equation, it is called the **trivial solution**.

Remark: Note that the function $y(x) = 0$ solves the equation $x^2y'' + xy' - y = 0$ from our recent example. In fact, $y(x) = 0$ is the member of our family $y = c_1x + \frac{c_2}{x}$ in which $c_1 = c_2 = 0$. Not all differential equations will admit the trivial solution. As the name suggest, the trivial solution is of little interest when it does solve an equation.

Systems of ODEs

Sometimes we want to consider two or more dependent variables that are functions of the same independent variable. The ODEs for the dependent variables can depend on one another. Some examples of relevant situations are

- predator and prey
- competing species
- two or more masses attached to a system of springs
- two or more composite fluids in attached tank systems

Such systems can be **linear** or **nonlinear**. A system is linear if each equation in the system is a linear equation.

Example of Nonlinear System

Lotka-Volterra Model

$$\frac{dx}{dt} = -\alpha x + \beta xy$$
$$\frac{dy}{dt} = \gamma y - \delta xy$$

This is known as the **Lotka-Volterra** predator-prey model. $x(t)$ is the population (density) of predators, and $y(t)$ is the population of prey. The numbers α, β, γ and δ are nonnegative constants.

This model is built on the assumptions that

- in the absence of predation, prey increase exponentially
- in the absence of predation, predators decrease exponentially,
- predator-prey interactions increase the predator population and decrease the prey population.

Example of a Linear System

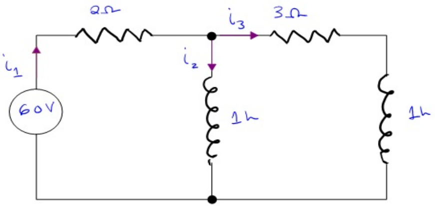


Figure: An example of a simple circuit with two loops that includes a voltage source and resisters and inductors in series.

The currents, i_2 and i_3, in this circuit will satisfy a system of two differential equations. Later, we'll talk about how the differential equations are actually constructed. The equation details come from application of Kirchhoff's (loop) Law along with empirical formulas for potential drops across various elements.

Example of a Linear System

LR-Circuit Network Example

The system governing the currents i_2 and i_3 is

$$\begin{aligned}\frac{di_2}{dt} &= -2i_2 - 2i_3 + 60\\ \frac{di_3}{dt} &= -2i_2 - 5i_3 + 60\end{aligned}$$

Note that the rates of change of i_2 and i_3 depend on themselves and each other. Also notice that both equations have the properties necessary to be called **linear** (we can always rearrange them to pull all dependent variables to one side). The equations are constructed by multiplying the dependent variables and their derivatives by coefficients that depend only on t and are added. Compare that to the nonlinear Lotka Volterra equations in which the product xy of dependent variables appears.

Solutions to Systems of ODE

A **solution** to a system of equations requires a function for each dependent variable. These functions must reduce every equation[a] in the system to an identity upon substitution.

[a]The point being that all equations are considered together so that all must be satisfied.

Example: Show that the pair of functions $i_2(t) = 30 - 24e^{-t} - 6e^{-6t}$ and $i_3(t) = 12e^{-t} - 12e^{-6t}$ are a solution to the system

$$\begin{aligned} \frac{di_2}{dt} &= -2i_2 - 2i_3 + 60 \\ \frac{di_3}{dt} &= -2i_2 - 5i_3 + 60 \end{aligned}$$

Exercise left to the reader.

Systems of ODEs

Solution Methods

There are various approaches to solving a system of differential equations. These can include

- elimination (try to eliminate a dependent variable),
- matrix techniques,
- Laplace transforms
- numerical approximation techniques

We will use Laplace transforms to solve select systems of linear equations later in the course.

Section 2: Initial Value Problems

Definition: Initial Value Problem

An **Initial Value Problem (IVP)** consists of a differential equation coupled with a certain type of additional conditions. For Example: Solve the equation [a]

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}) \tag{7}$$

subject to the *initial conditions*

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, y^{(n-1)}(x_0) = y_{n-1}. \tag{8}$$

The problem (7)–(8) is called an *initial value problem*.

[a] on some interval I containing x_0.

Note that y and its derivatives are evaluated at the same initial x value of x_0.

Examples for $n = 1$ or $n = 2$

First order case:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

If there is a solution $y(x)$, we can think of the ODE part as providing information about the shape of its curve while the initial condition indicates that the curve must contain the point (x_0, y_0).

Second order case:

$$\frac{d^2y}{dx^2} = f(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = y_1$$

Note that the number of IC matches the order of the ODE. For example, if $y(x)$ indicates the position of a particle moving along a line. The ODE describes the acceleration profile of the particle, and the values y_0 and y_1 indicate the starting position and velocity of the particle, respectively.

Example

Given that $y = c_1 x + \frac{c_2}{x}$ is a 2-parameter family of solutions of the ODE $x^2 y'' + xy' - y = 0$, solve the IVP

$$x^2 y'' + xy' - y = 0, \quad y(1) = 1, \quad y'(1) = 3$$

We already know that $y = c_1 x + \frac{c_2}{x}$ solves the ODE for any choice of the parameters c_1 and c_2. We need to determine the values of those parameters so that y satisfies the initial conditions. Note that

$$y(1) = c_1(1) + \frac{c_2}{1} = 1 \quad \Longrightarrow \quad c_1 + c_2 = 1$$

$$y'(1) = c_1 - \frac{c_2}{1^2} = 3 \quad \Longrightarrow \quad c_1 - c_2 = 3$$

Solving this algebraic system, one finds that $c_1 = 2$ and $c_2 = -1$.

The solution to the IVP is $\quad y = 2x - \dfrac{1}{x}.$

Graphical Interpretation: Direction Fields

A **direction field** gives a visual interpretation of the differential equation $\frac{dy}{dx} = f(x, y)$. On a portion of the xy-plane, a lattice of points is considered, and at each point a small directed line segment (a vector) is placed with its direction determined by the function f. So at each point (a, b) on the lattice, a little arrow having slope $f(a, b)$ is placed to fill the plane. Any solution curve (x, y) to the ODE will be tangent to these little arrows.

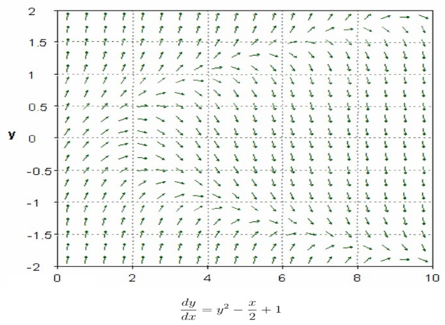


Figure: A direction field for the ODE $y' = y^2 - \frac{x}{2} + 1$ in the rectangle $0 \le x \le 10$ and $-2 \le y \le 2$.

Graphical Interpretation: Direction Fields

To illustrate, we can consider a few of these arrows in the direction field for the differential equation $y' = y^2 - \frac{x}{2} + 1$. Note that here, $f(x, y) = y^2 - \frac{x}{2} + 1$ is the function that determines the slope of each vector.

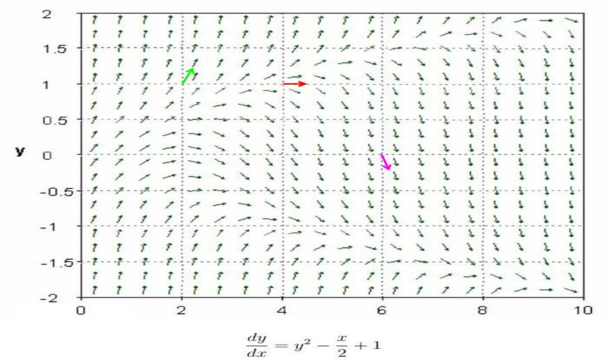


Figure: The little green arrow is at $(2, 1)$ having slope $f(2, 1) = 1^2 - \frac{2}{2} + 1 = 1$. The little red arrow at $(4, 1)$ has slope $f(4, 1) = 1^2 - \frac{4}{2} + 1 = 0$, and the little pink arrow at $(6, 0)$ has slope $f(6, 0) = 0^2 - \frac{6}{2} + 1 = -2$

Graphical Interpretation: Direction Fields

The inclusion of an initial condition determines a specific curve within the direction field. Here, we see the solutions to the IVP

$$\frac{dy}{dx} = y^2 - \frac{x}{2} + 1, \quad y(0) = y_0$$

for three choices of the value y_0 (the value $x_0 = 0$ was chosen for all three examples).

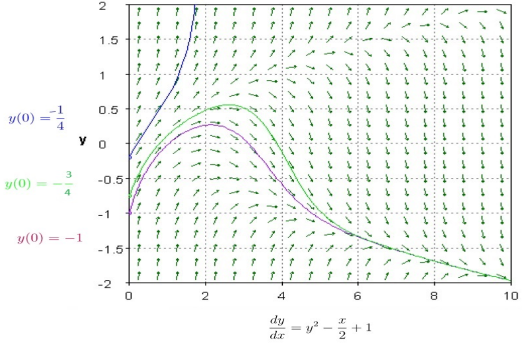


Figure: The purple curve solves the IVP with the condition $y(0) = -1$. The green curve is the solution to the IVP with $y(0) = -\frac{3}{4}$, and the blue curve is the solution to the IVP with $y(0) = -\frac{1}{4}$.

Graphical Interpretation: Direction Fields

As the images of direction fields suggest, we may be able to create a curve representing the solution to a first order initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

by somehow generating a graph from knowledge about its tangent line at a collection of points. Next, we will consider a method for generating such a curve numerically.

A Numerical Solution

Consider a first order initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

In the coming sections, we'll see methods for solving some of these problems analytically (e.g. by hand). The method will depend on the type of equation. But not all ODEs are readily solved by hand. We can ask whether we can at least obtain an approximation to the solution, for example as a table of values or in the form of a curve. In general, the answer is that we can get such an approximation. Various algorithms have been developed to do this. We're going to look at a method known as **Euler's Method**.

Euler's Method

The strategy behind Euler's method is to construct the solution starting with the known initial point (x_0, y_0) and using the tangent line to *find* (approximate) the next point on the solution curve. We will consider an example, and then derive the general formula used for Euler's method.

For the next few slides, we will consider the example

$$\frac{dy}{dx} = xy, \quad \text{with initial condition} \quad y(0) = 1$$

Note that

$$f(x, y) = xy, \quad x_0 = 0, \quad \text{and} \quad y_0 = 1$$

We will build the solution in increments of 0.25. (This number is chosen for this example and can be changed.)

The true solution for this simple example is well know, so the true curve can be plotted along with the approximations. But keep in mind that, in general, the exact solution isn't known. (It it was, you wouldn't need to approximate it.)

Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

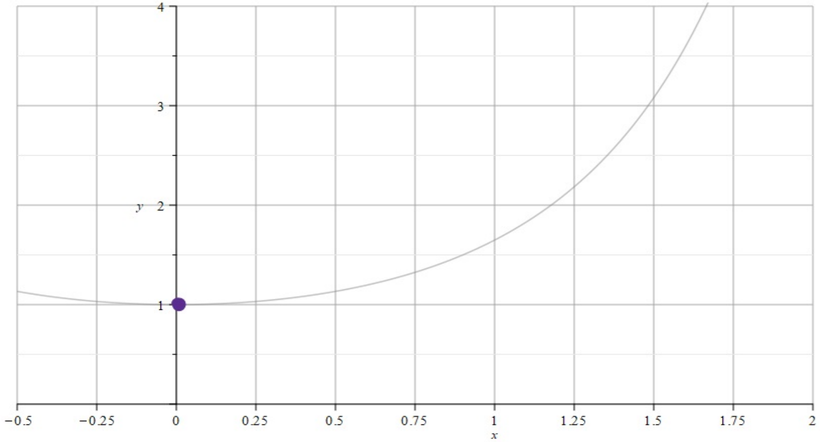


Figure: We know that the point $(x_0, y_0) = (0, 1)$ is on the curve. And the slope of the curve at $(0, 1)$ is $m_0 = f(0, 1) = 0 \cdot 1 = 0$.
Note: The gray curve is the true solution to this IVP. It's shown for reference.

Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

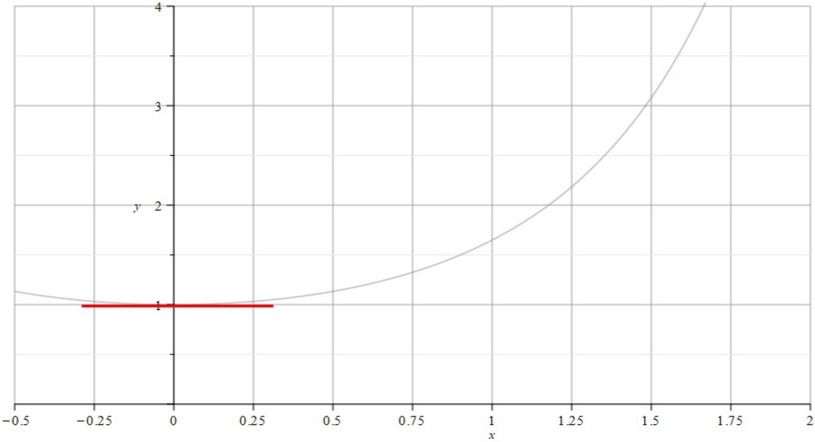


Figure: So we draw a little tangent line (we know the point and slope). Then we increase x, say $x_1 = x_0 + h$, and approximate the solution value $y(x_1)$ with the value on the tangent line y_1. So $y_1 \approx y(x_1)$. (I'm taking $h = 0.25$.)

Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

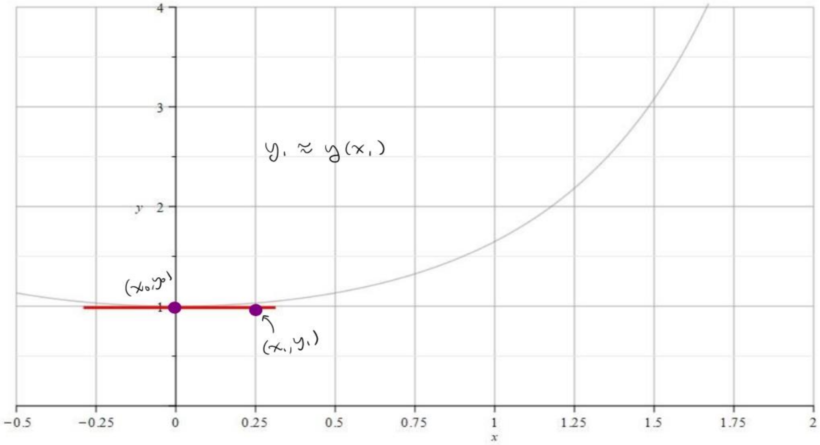


Figure: We take the approximation to the true function y at the point $x_1 = x_0 + h$ to be the point on the tangent line.

Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

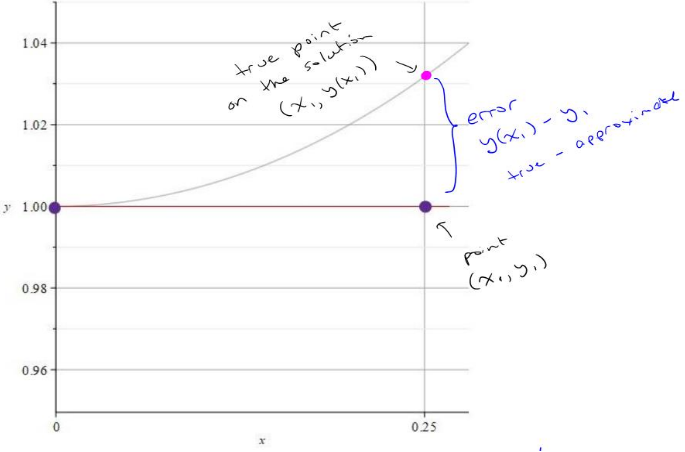


Figure: When h is very small, the true solution and the tangent line point will be close. Here, we've zoomed in to see that there is some error between the exact y value and the approximation from the tangent line.

Example $\frac{dy}{dx} = xy, \quad y(0) = 1$

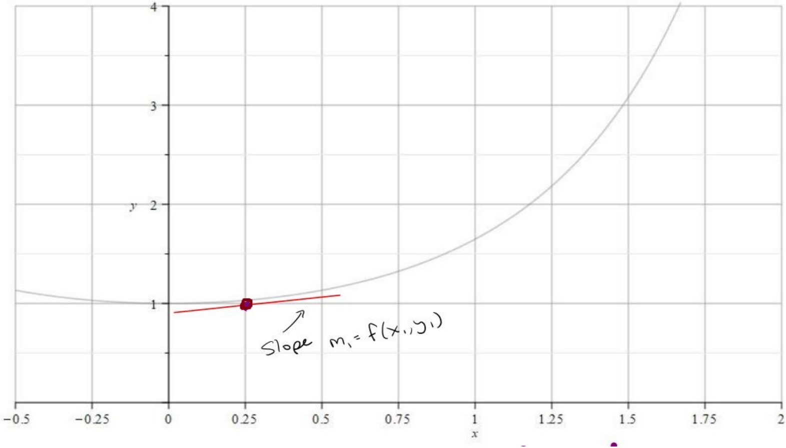


Figure: Now we start with the point (x_1, y_1) and repeat the process. We get the slope $m_1 = f(x_1, y_1)$ and draw a tangent line through (x_1, y_1) with slope m_1.

Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

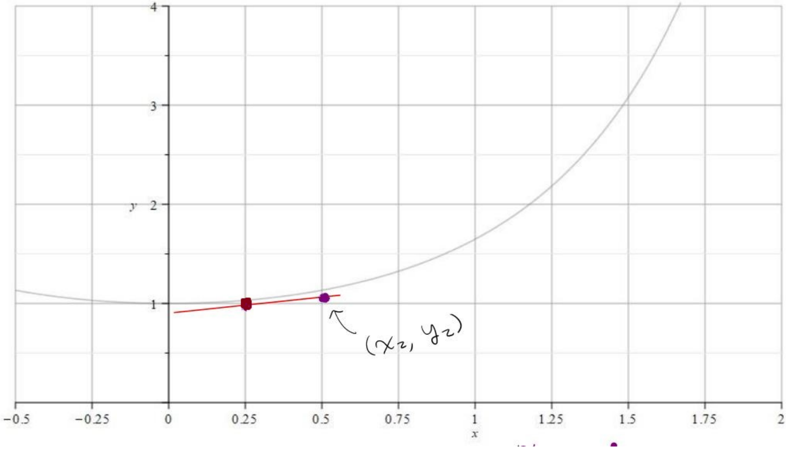


Figure: We go out h more units to $x_2 = x_1 + h$. Pick the point on the tangent line (x_2, y_2), and use this to approximate $y(x_2)$. So $y_2 \approx y(x_2)$

Example $\quad \dfrac{dy}{dx} = xy, \quad y(0) = 1$

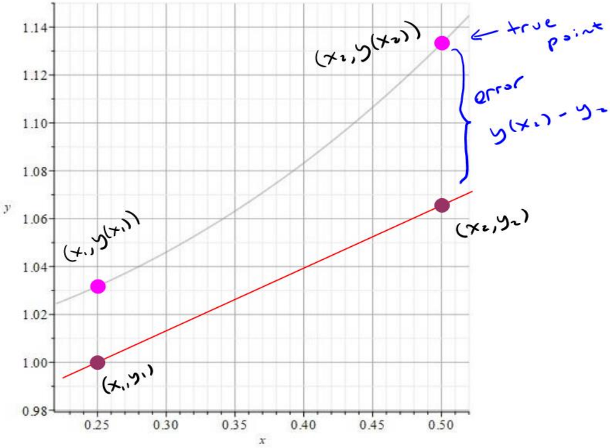


Figure: If we zoom in, we can see that there is some error. But as long as *h* is small, the point on the tangent line approximates the point on the actual solution curve.

Example $\dfrac{dy}{dx} = xy, \quad y(0) = 1$

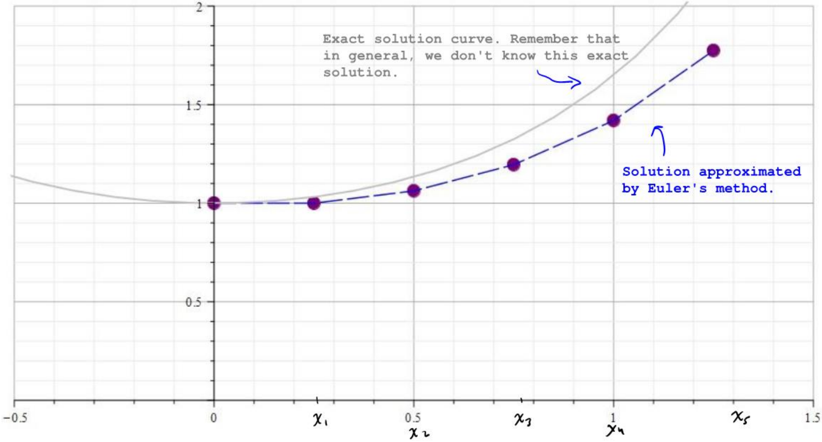


Figure: We can repeat this process at the new point to obtain the next point. We build an approximate solution by advancing the independent variable and connect the points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$.

Euler's Method: An Algorithm & Error

We start with the IVP

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0.$$

We build a sequence of points that approximates the true solution y

$$(x_0, y_0),\ (x_1, y_1),\ (x_2, y_2),\ \dots, (x_N, y_N).$$

We'll take the x values to be equally spaced with a common difference of h. That is

$$\begin{aligned}
x_1 &= x_0 + h \\
x_2 &= x_1 + h = x_0 + 2h \\
x_3 &= x_2 + h = x_0 + 3h \\
&\vdots \\
x_n &= x_0 + nh
\end{aligned}$$

Euler's Method: An Algorithm

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0.$$

Notation:

- The approximate value of the solution will be denoted by y_n,
- and the exact values (that we don't expect to actually know) will be denoted $y(x_n)$.

To build a formula for the approximation y_1, let's approximate the derivative at (x_0, y_0).

$$f(x_0, y_0) = \left.\frac{dy}{dx}\right|_{(x_0,y_0)} \approx \frac{y_1 - y_0}{x_1 - x_0}$$

(Notice that's the standard formula for slope.)

Euler's Method: An Algorithm

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

We know x_0 and y_0, and we also know that $x_1 = x_0 + h$ so that $x_1 - x_0 = h$. Thus, we can solve for y_1.

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{y_1 - y_0}{h} = f(x_0, y_0)$$

$$\implies \quad y_1 - y_0 = hf(x_0, y_0)$$

$$\implies \quad y_1 = y_0 + hf(x_0, y_0)$$

The formula to approximate y_1 is therefore

$$\boxed{y_1 = y_0 + hf(x_0, y_0).}$$

Euler's Method: An Algorithm

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

We can continue this process. So we use

$$\frac{y_2 - y_1}{h} = f(x_1, y_1) \quad \Longrightarrow \quad y_2 = y_1 + hf(x_1, y_1)$$

and so forth. We have

Euler's Method Formula: The n^{th} approximation y_n to the exact solution $y(x_n)$ is given by

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$$

with (x_0, y_0) given in the original IVP and h the choice of step size.

Euler's Method Example: $\frac{dy}{dx} = xy, \quad y(0) = 1$

Problem: Using a step size of $h = 0.25$, find the first three iterates y_1, y_2, and y_3 using Euler's method.

Solution We have $x_0 = 0$ and $y_0 = 1$. So

$$y_1 = y_0 + hf(x_0, y_0) = 1 + 0.25(0 \cdot 1) = 1$$

Now we repeat to find y_2. We have $x_1 = 0.25$ and $y_1 = 1$.

$$y_2 = y_1 + hf(x_1, y_1) = 1 + 0.25(0.25 \cdot 1) = 1.0625$$

Now we repeat to find y_3. We have $x_2 = 0.5$ and $y_2 = 1.0625$.

$$y_3 = y_2 + hf(x_2, y_2) = 1.0625 + 0.25(0.5 \cdot 1.0625) = 1.19531$$

This completes the first three terms.

Euler's Method Example: $\dfrac{dx}{dt} = \dfrac{x^2 - t^2}{xt}, \quad x(1) = 2$

Problem: Using a step size of $h = 0.2$, use Euler's method to approximate $x(1.4)$.

Solution: With $h = 0.2$, we'll move from $t = 1$ to $t = 1.2$ and then to $t = 1.4$. So we will need two steps. First, let's determine the formula using Euler's method. We have $f(t, x) = \frac{x^2 - t^2}{xt}$, so the general formula will be

$$x_n = x_{n-1} + hf(t_{n-1}, x_{n-1}) = x_{n-1} + 0.2\left(\frac{x_{n-1}^2 - t_{n-1}^2}{x_{n-1}t_{n-1}}\right)$$

We also have

$$t_0 = 1 \quad \text{and} \quad x_0 = 2.$$

Euler's Method Example: $\dfrac{dx}{dt} = \dfrac{x^2 - t^2}{xt}, \quad x(1) = 2$

First, we approximate $x(1.2)$

$$x_1 = x_0 + 0.2\left(\frac{x_0^2 - t_0^2}{x_0 t_0}\right) = 2 + 0.2\left(\frac{2^2 - 1^2}{2 \cdot 1}\right) = 2.3$$

So we have the point $(1.2, 2.3)$.

Euler's Method Example: $\dfrac{dx}{dt} = \dfrac{x^2 - t^2}{xt}, \quad x(1) = 2$

Now we move on to the next point to approximate $x(1.4)$. From the last step, we have $t_1 = 1.2$ and $x_1 = 2.3$ so

$$x_2 = x_1 + 0.2\left(\frac{x_1^2 - t_1^2}{x_1 t_1}\right) = 2.3 + 0.2\left(\frac{2.3^2 - 1.2^2}{2.3 \cdot 1.2}\right) = 2.579$$

So we have the point $(1.4, 2.579)$.

The approximation

$$x(1.4) \approx 2.579.$$

It is possible to solve this IVP exactly to obtain the solution $x = \sqrt{4t^2 - 2t^2\ln(t)}$. The true value $x(1.4) = 2.554$ to four decimal digits.

Euler's Method: Error

As the previous examples suggest, the approximate solution obtained using Euler's method has error. Moreover, the error can be expected to become more pronounced, the farther away from the initial condition we get.

First, let's define what we mean by the term *error*. There are a couple of types of error that we can talk about. These are[5]

$$\text{Absolute Error} = |\text{True Value} - \text{Approximate Value}|$$

and

$$\text{Relative Error} = \frac{\text{Absolute Error}}{|\text{True value}|}$$

[5]Some authors will define absolute error without use of absolute value bars so that absolute error need not be nonnegative.

Euler's Method: Error

We expect to get better results taking smaller steps. We can ask, how does the error depend on the step size? Let's look at some error for one of the examples.

$$\frac{dy}{dx} = xy, \quad y(0) = 1$$

I programed Euler's method into Matlab and used different h values to approximate $y(1)$. The number of iterations depends on the step size. For example, if $h = 0.2$, it takes five steps to get from $x_0 = 0$ to $x_5 = 1$. In general, the number of steps is $n = \frac{1}{h}$.

h	$y(1) - y_n$	$\frac{y(1)-y_n}{y(1)}$
0.2	0.1895	0.1149
0.1	0.1016	0.0616
0.05	0.0528	0.0320
0.025	0.0269	0.0163
0.0125	0.0136	0.0082

Euler's Method: Error

We notice from this example that cutting the step size in half, seems to cut the error and relative error in half. This suggests the following:

The absolute error in Euler's method is proportional to the step size.

There are two sources of error for Euler's method (not counting numerical errors due to machine rounding).

- The error in approximating the curve with a tangent line, and
- using the approximate value y_{n-1} to get the slope at the next step.

Euler's Method: Error

For numerical schemes of this sort, we often refer to the *order* of the scheme. If the error satisfies

$$\text{Absolute Error} = Ch^p$$

where C is some constant, then the order of the scheme is p.

Since the error in Euler's method is proporational to h—i.e., h^1,

Euler's method is an order 1 scheme.

Euler's Method

Euler's method is simple and intuitive. However, it is rarely used in practice because of its error. There are more widely used schemes. Other methods tend to use multiple tangent lines for each iteration and are sometimes referred as mutli-step methods. The two most common are

- Improved Euler[6] which is order 2, and
- Runge-Kutta[7] which is order 4.

---

[6] a.k.a. RK2
[7] a.k.a. RK4

Improved Euler's Method: $\dfrac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$

Euler's method approximated y_1 using the slope $m_0 = f(x_0, y_0)$ for the tangent line. An initial improvement on the method can be made by using this as an intermediate point to give a second approximation to the slope. That is, let

$$m_0 = f(x_0, y_0)$$

as before, and now let

$$\hat{m}_0 = f(x_1, y_0 + m_0 h).$$

Then we take y_1 to be the point on the line that has the average of these two slopes

$$y_1 = y_0 + \frac{1}{2}(m_0 + \hat{m}_0)h.$$

Other methods will use the weighted averages of 3, 4 or more tangent lines.

Existence and Uniqueness

Existence & Uniqueness Questions

Two important questions we can always pose (and sometimes answer) are

1. Does an IVP have a solution? (existence) and
2. If it does, is there just one? (uniqueness)

As a silly example, consider whether the following can be solved[8]

$$\left(\frac{dy}{dx}\right)^2 + 1 = -y^2.$$

[8] If we only wish to consider real valued functions.

Uniqueness

Consider the IVP

$$\frac{dy}{dx} = x\sqrt{y} \quad y(0) = 0$$

Exercise 1: Verify that $y = \frac{x^4}{16}$ is a solution of the IVP.

Exercise 2: Can you find a second solution of the IVP by inspection—i.e. by clever guessing? (Hint: What's the simplest type of function you can think of. Is there one of that type that satisfies both the ODE and the initial condition?)

This IVP has two distinct solutions. We'll see how to solve the ODE in the next section. The solution technique will give us a 1-parameter family of solutions. We'll find that one of the solutions is a member of the family, and one is not.

Section 3: First Order Equations: Separation of Variables

Given all the integration studied in Calculus, it's reasonable to wonder whether there's a need to study differential equations. Some ODEs can be solved using nothing new beyond basic Calculus. For example, the simplest type of equation we could encounter would be of the form

$$\frac{dy}{dx} = g(x).$$

If $G(x)$ is any antiderivative of $g(x)$, the solutions to this ODE would be

$$y = G(x) + c$$

obtained by simply integrating. But what if the ODE is $\dfrac{dy}{dx} = f(x, y)$, where the right side depends on the unknown function y? Basic integration isn't expected to apply here. We'll consider a special form of equation that allows us to extend using basic integration.

Separable Equations

Definition:

The first order equation $y' = f(x, y)$ is said to be **separable** if the right side has the form

$$f(x, y) = g(x)h(y).$$

That is, a separable equation is one that has the form

$$\frac{dy}{dx} = g(x)h(y).$$

Separable -vs- Nonseparable

For example,

$$\frac{dy}{dx} = x^3 y$$

is separable as the right side is the product of $g(x) = x^3$ and $h(y) = y$.

However, the equation

$$\frac{dy}{dx} = 2x + y$$

is not separable. It is just not possible to write $2x + y$ as the **product** of two functions in which one depends only on x and the other only on y.

Let's see how we can extend integration to solve separable equations.

Solving Separable Equations

Let's assume that it's safe[9] to divide by $h(y)$ and let's set $p(y) = 1/h(y)$. The following is called **separating the variables**.

We will start with $\frac{dy}{dx} = g(x)h(y)$, divide by h and multiply by the differential dx. Note that

$$\frac{1}{h(y)}\frac{dy}{dx} = g(x) \quad \Longrightarrow \quad p(y)\frac{dy}{dx}\,dx = g(x)\,dx.$$

Since $\frac{dy}{dx}\,dx = dy$, we integrate both sides

$$\int p(y)\,dy = \int g(x)\,dx \quad \Longrightarrow \quad P(y) = G(x) + c$$

where P and G are any antiderivatives of p and g, respectively. The expression

$$P(y) = G(x) + c$$

defines a one parameter family of solutions implicitly.

---

[9] We'll come back to address this assumption later.

Example

Let's solve the initial value problem $\frac{dy}{dx} = x\sqrt{y}, \quad y(0) = 0.$

Before starting any solution technique, always confirm that you're working with the right *kind* of equation. To that end, note that the ODE is separable with $g(x) = x$ and $h(y) = \sqrt{y}$. So we will divide both side by $\sqrt{y}$ and multiply by the differential dx.

$$\frac{1}{\sqrt{y}}\frac{dy}{dx}\,dx = x\,dx \quad \Longrightarrow \quad y^{-1/2}\,dy = x\,dx.$$

Note that one side has only y and one only x. We have **separated the variables.** Now we can integrate each side, the left with respect to y and the right with respect to x.

$$\int y^{-1/2}\,dy = \int x\,dx \quad \Longrightarrow \quad 2\sqrt{y} = \frac{x^2}{2} + C$$

$$\frac{dy}{dx} = x\sqrt{y}, \quad y(0) = 0$$

You'll note that I only put a constant of integration on one side. You can put constants on both sides, but they need to be different constants, e.g., C and K or C_1 and C_2. Since we can combine them, it is only necessary to include one[10] added constant. It's not always possible to isolate y, but here, we can.

$$2\sqrt{y} = \frac{x^2}{2} + C \quad \Longrightarrow \quad y = \left(\frac{x^2}{4} + \frac{C}{2}\right)^2.$$

This is a one-parameter family of solution to the ODE $y' = x\sqrt{y}$.

[10] I usually put it on the side with the independent variable, but that's just a preference.

$$\frac{dy}{dx} = x\sqrt{y}, \quad y(0) = 0$$

The last step will be to apply[11] the initial condition.

$$y = \left(\frac{x^2}{4} + \frac{C}{2}\right)^2 \implies y(0) = \left(\frac{0^2}{4} + \frac{C}{2}\right)^2 = 0 \implies C = 0$$

We are ready to complete the problem. Having found C,

$$y = \left(\frac{x^2}{4} + \frac{0}{2}\right)^2 = \left(\frac{x^2}{4}\right)^2 = \frac{x^4}{16}.$$

The solution we found to the IVP is $y = \dfrac{x^4}{16}$

[11] We could have applied the condition without isolating y first. We would simply substitute $x = 0$ and $y = 0$ and solve for C.

Caveat regarding division by $h(y)$.

We made the assumption that we can divide by the expression $h(y)$ in the equation $y' = g(x)h(y)$. This rests on the assumption that $h(y) \neq 0$ over the domain of the solution. It's an assumption that we have to make in order to proceed, but we should be aware that we are making it!

For example, you may recall that the IVP $\frac{dy}{dx} = x\sqrt{y}, \quad y(0) = 0$ actually has two distinct solution

$$(1) \quad y = \frac{x^2}{16} \quad \text{and} \quad (2) \quad y = 0.$$

The family of solutions we found separating the variables was $y = \left(\frac{x^2}{4} + \frac{C}{2}\right)^2$. There is no value for C such that $y = 0$ is a member of this family. We lost this solution in the process.

Can you identify which step in the process caused us to lose track of the possible solution $y = 0$?

Missed Solutions $\frac{dy}{dx} = g(x)h(y)$.

We can state the following theorem about possible missed, constant solutions to separable ODEs.

Theorem:

If the number c is a zero of the function h, i.e. $h(c) = 0$, then the constant function $y(x) = c$ is a solution to the differential equation $\frac{dy}{dx} = g(x)h(y)$.

Remark: Such a constant solution may or may not be recovered by separation of variables. We can always look for such solutions in addition to separation of variables by looking for solutions to the equation $h(y) = 0$.

Solutions Defined by Integrals

The Fundamental Theorem of Calculus tells us that: If g and $\frac{dy}{dx}$ are continuous on an interval $[x_0, b)$ and x is in this interval, then

$$\frac{d}{dx}\int_{x_0}^{x} g(t)\,dt = g(x) \quad \text{and} \quad \int_{x_0}^{x} \frac{dy}{dt}\,dt = y(x) - y(x_0).$$

Theorem: If g is continuous on some interval containing x_0, then the function

$$y = y_0 + \int_{x_0}^{x} g(t)\,dt$$

is a solution of the initial value problem

$$\frac{dy}{dx} = g(x), \quad y(x_0) = y_0$$

Example

Express the solution of the IVP in terms of an integral.

$$\frac{dy}{dx} = \sin(x^2), \quad y(\sqrt{\pi}) = 1$$

First we'll note that the equation has the form $\frac{dy}{dx} = g(x)$ (which is separable with $h(y) = 1$). The function $\sin(x^2)$ does not have an antiderivative in terms of elementary functions. So we can't integrate it with only Calculus II tools. Nevertheless, it fits the previous theorem with

$$g(t) = \sin(t^2), \quad x_0 = \sqrt{\pi}, \quad \text{and} \quad y_0 = 1.$$

Hence, the solution to the IVP can be expressed in terms of an integral

$$\boxed{y = 1 + \int_{\sqrt{\pi}}^{x} \sin(t^2)\, dt}$$

Exercise for the reader: use the FTC to show that $y' = \sin(x^2)$ and confirm that $y(\sqrt{\pi}) = 1$.

Section 4: First Order Equations: Linear & Special

Recall that a first order linear equation has the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

If $g(x) = 0$ the equation is called **homogeneous**. Otherwise it is called **nonhomogeneous**.

Standard Form

Provided $a_1(x) \neq 0$ on the interval I of definition of a solution, we can write the **standard form** of the equation

$$\frac{dy}{dx} + P(x)y = f(x).$$

We'll be interested in equations (and intervals I) for which P and f are continuous on I.

The General Solution $\frac{dy}{dx} + P(x)y = f(x)$

We will consider the first order linear equation in standard form with P and f continuous on some interval. It turns out the solution will always have a basic structure, $y = y_c + y_p$ where

- y_c is called the **complementary** solution. It is a solution to the **associated homogeneous** equation
$$\frac{dy}{dx} + P(x)y = 0, \quad \text{and}$$
- y_p is called the **particular** solution. The particular solution depends heavily on f and is zero if $f(x) = 0$.

With higher order equations, we'll have to find y_c and y_p separately, but for first order equations we have a process for finding both at one time.

Motivating Example

Before deriving the solution process, let's consider a simple example. Let's solve

$$x^2\frac{dy}{dx}+2xy=e^x$$

First, we confirm that the equation is first order linear. It is not in standard form, but I'm going to accept that fact for now. This equation has the amazing featuer that the left hand side is the derivative of a product. By the product rule, note that

$$\frac{d}{dx}\left[x^2y\right]=x^2\frac{dy}{dx}+2xy.$$

Hence the equation can be collapsed and integrated.

$$\frac{d}{dx}\left[x^2y\right]=e^x \quad\Longrightarrow\quad \int\frac{d}{dx}\left[x^2y\right]\,dx=\int e^x\,dx$$

Integrate and divide by x^2 to obtain the general solution

$$\boxed{y=\frac{e^x}{x^2}+\frac{c}{x^2}} \qquad \left(\text{Here, } y_p=\frac{e^x}{x^2} \quad \text{and} \quad y_c=\frac{c}{x^2}.\right)$$

Derivation of Solution via Integrating Factor

This example has the fortuitous property that the left side of the equation is the derivative of a product. So, while it is a sum of two terms, we can collapse it into one derivative term, allowing us to integrate.

We don't expect an arbitrary, first order linear ODE to have this feature. So the question becomes, can we somehow induce this feature?

The answer is "yes." We will use something called an *integrating factor*. We seek a function that I'll call $\mu(x)$ such that if we multiply our ODE $\frac{dy}{dx} + P(x)y = f(x)$ (in standard form) by this function, the left side becomes the derivative of a product.

Derivation of Solution via Integrating Factor

We seek to solve the equation in standard form

$$\frac{dy}{dx} + P(x)y = f(x)$$

We seek a function $\mu(x)$ such that when we multiply the above equation by this new function, the left side collapses as a product rule. That is, μ should result in

$$\mu\frac{dy}{dx} + \mu P y = \mu f \quad \Longrightarrow \quad \frac{d}{dx}[\mu y] = \mu f.$$

Use the product rule to evaluate $\frac{d}{dx}[\mu y]$ and match to the left side of the ODE

$$\mu\frac{dy}{dx} + \mu P y = \frac{d}{dx}[\mu y] = \mu\frac{dy}{dx} + \frac{d\mu}{dx}y.$$

The first terms match automatically. The second terms give us an equation for μ, namely

$$\frac{d\mu}{dx} = P\mu.$$

$$\frac{d\mu}{dx} = P\mu.$$

Notice here that we've obtained a separable equation for the sought after function μ! Fortunately, we know how to solve this separable equation. We find that

Integrating Factor

For the first order, linear ODE in standard form $\frac{dy}{dx} + P(x)y = f(x)$, the integrating factor

$$\mu(x) = \exp\left(\int P(x)\,dx\right).$$

Solution Process 1^{st} Order Linear ODE

- Put the equation in standard form $y' + P(x)y = f(x)$, and correctly identify the function $P(x)$.
- Obtain the integrating factor $\mu(x) = \exp\left(\int P(x)\,dx\right)$.
- Multiply both sides of the equation (in standard form) by the integrating factor μ. The left hand side **will always** collapse into the derivative of a product

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

- Integrate both sides, and solve for y.

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)f(x)\,dx$$

$$y(x) = e^{-\int P(x)\,dx}\left(\int e^{\int P(x)\,dx} f(x)\,dx + C\right)$$

Example

Find the general solution to the ODE

$$x\frac{dy}{dx} - y = 2x^2$$

First, observe that the ODE is 1^{st} order linear. Putting it in standard form gives

$$\frac{dy}{dx} - \frac{1}{x}y = 2x, \quad \text{so that} \quad P(x) = -\frac{1}{x}.$$

Then[12]

$$\mu(x) = \exp\left(\int -\frac{1}{x}\,dx\right) = \exp(-\ln|x|) = x^{-1}$$

The equation becomes

$$\frac{d}{dx}[x^{-1}y] = x^{-1}(2x) = 2$$

[12]We can take any constant of integration we want when finding μ, so I'm taking it to be zero.

$$\frac{d}{dx}[x^{-1}y] = x^{-1}(2x) = 2$$

Next we integrate both sides—μ makes this possible, hence the name *integrating factor*—and solve for our solution y.

$$\int \frac{d}{dx}[x^{-1}y]\,dx = \int 2\,dx \quad \Longrightarrow \quad x^{-1}y = 2x + C$$

and finally

$$\boxed{\text{the general solution to the ODE is} \quad y = 2x^2 + Cx.}$$

Remark: Note that this solution has the form $y = y_p + y_c$ where $y_c = Cx$ and $y_p = 2x^2$. The complementary part comes from the constant of integration and is independent of the right side of the ODE $2x$. The particular part comes from the right hand side integration of $x^{-1}(2x)$.

The reader should take a moment to verify that our solution does in fact solve the ODE.

Steady and Transient States

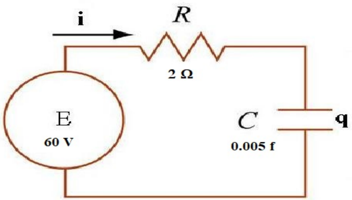


Figure: The charge $q(t)$ on the capacitor in the given curcuit satisfies a first order linear equation.

$$2\frac{dq}{dt} + 200q = 60, \quad q(0) = 0.$$

We'll see in section 5 how this equation is derived. For now, let's notice that the ODE is a first order linear equation. It is not in standard form (WHY?). Let's solve this IVP. We will find the general solution to the ODE first, then we will apply the initial condition.

$$2\frac{dq}{dt} + 200q = 60, \quad q(0) = 0$$

Let's put the ODE in standard form and identify $P(t)$. Divide by 2 to get

$$\frac{dq}{dt} + 100q = 30 \quad \Longrightarrow \quad P(t) = 100.$$

Compute the integrating factor (we'll take the intermediate constant of integration to be zero)

$$\mu = e^{\int P(t)\,dt} = e^{\int 100\,dt} = e^{100t}.$$

Multiply the ODE by μ and collapse.

$$e^{100t}\left(\frac{dq}{dt} + 100q\right) = 30e^{100t} \quad \Longrightarrow \quad \frac{d}{dt}\left[e^{100t}q\right] = 30e^{100t}.$$

$$\frac{d}{dt}\left[e^{100t}q\right] = 30e^{100t}$$

Now, we integrate with respect to t and isolate q.

$$\int \frac{d}{dt}\left[e^{100t}q\right], dt = \int 30e^{100t}\,dt \quad \Longrightarrow \quad e^{100t}q = \frac{30}{100}e^{100t} + K.$$

Note that at this last integration step, we have to include a constant of integration. We have the general solution to the ODE

$$q(t) = \frac{\frac{3}{10}e^{100t} + K}{e^{100t}} = \frac{3}{10} + Ke^{-100t}.$$

Applying the initial condition $q(0) = 0$, we find $K = -\frac{3}{10}$ so that

the charge on the capacitor is $\quad q(t) = \dfrac{3}{10} - \dfrac{3}{10}e^{-100t}.$

Steady and Transient States

Note that the solution, the charge, consists of a complementary and a particular solution, $q = q_p + q_c$.

$$q(t) = \frac{3}{10} - \frac{3}{10}e^{-100t}$$

$$q_c(t) = -\frac{3}{10}e^{-100t} \quad \text{and} \quad q_p(t) = \frac{3}{10}$$

Notice that due to the decaying exponential,

$$\lim_{t\to\infty} q_c(t) \quad = \lim_{t\to\infty}\left(-\frac{3}{10}e^{-100t}\right) = 0.$$

So after a long time, the charge on the capacitor

$$q(t) \approx q_p(t).$$

Steady and Transient States

The complementary solution contains the information given by the initial condition, and for some physical systems like this the complementary solution decays.

Definition: Such a decaying complementary solution is called a **transient state**.

Note that due to this decay, after a while $q(t) \approx q_p(t)$.

Definition: Such a corresponding particular solution is called a **steady state**.

Bernoulli Equations

Bernoulli 1^{st} Order Equation

Suppose $P(x)$ and $f(x)$ are continuous on some interval (a, b) and n is a real number different from 0 or 1 (not necessarily an integer). An equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

is called a **Bernoulli** equation.

Observation: This equation has the flavor of a linear ODE, but since $n \neq 0, 1$ it is necessarily nonlinear. So our previous approach involving an integrating factor does not apply directly. Fortunately, we can use a change of variables to obtain a related linear equation.

Solving the Bernoulli Equation

$$\frac{dy}{dx} + P(x)y = f(x)y^n \tag{9}$$

We will derive a general solution technique for a Bernoulli equation. This is done by a change of variables. We will consider a new *dependent* variable that I'll call u. This new variable will solve a first order linear ODE. Then we can substitute back to solve the original Bernoulli equation.

Let $u = y^{1-n}$. Then

$$\frac{du}{dx} = (1-n)y^{-n}\frac{dy}{dx} \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{y^n}{1-n}\frac{du}{dx}.$$

Substituting into (9) and dividing through by $y^n/(1-n)$

$$\frac{y^n}{1-n}\frac{du}{dx} + P(x)y = f(x)y^n \quad \Longrightarrow \frac{du}{dx} + (1-n)P(x)y^{1-n} = (1-n)f(x)$$

Solving the Bernoulli Equation

From our choice of u, we have arrived at the first order linear equation

$$\frac{du}{dx} + P_1(x)u = f_1(x), \quad \text{where} \quad P_1 = (1-n)P, \quad f_1 = (1-n)f.$$

Now, we solve this equation using an integrating factor as needed. We can subsitute back to obtain the original dependent variable y. Note that

$$u = y^{1-n} \quad \Longrightarrow \quad y = u^{\frac{1}{1-n}}.$$

Let's consider an example.

Bernoulli Equation Example

Solve the initial value problem $y' - y = -e^{2x}y^3$, subject to $y(0) = 1$.

First, confirm that this is a Bernoulli equation. We have $n = 3$, $P(x) = -1$ and $f(x) = -e^{2x}$. So our new variable

$$u = y^{1-n} = y^{1-3} = y^{-2}.$$

We have $1 - n = -2$, so the equation for u is

$$\frac{du}{dx} + (-2)(-1)u = (-2)(-e^{2x}) \quad \Longrightarrow \quad \frac{du}{dx} + 2u = 2e^{2x}$$

Example Continued

Now we solve the first order linear equation for u using an integrating factor. Omitting the details, we obtain

$$u(x) = \frac{1}{2}e^{2x} + Ce^{-2x}$$

Of course, we need to remember that our goal is to solve the original equation for y. But the relationship between y and u is known. From $u = y^{-2}$, we know that $y = u^{-1/2}$. Hence

$$y = \frac{1}{\sqrt{\frac{1}{2}e^{2x} + Ce^{-2x}}}$$

Applying $y(0) = 1$ we find that $C = 1/2$.

The solution to the IVP is $\quad y = \dfrac{\sqrt{2}}{\sqrt{e^{2x} + e^{-2x}}}$.

Note that we substituted back to y prior to applying the IC. We could translate the IC into a condition for u, find C first, then sub back to y. Both approaches are valid.

Exact Equations

We considered first order equations of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0. \tag{10}$$

The left side is called a *differential form*. We will assume here that M and N are continuous on some (shared) region in the plane.

Definition: The equation (10) is called an **exact equation** on some rectangle R if there exists a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial F}{\partial y} = N(x,y)$$

for every (x,y) in R.

Exact Equation Solution

If $M(x,y)\,dx + N(x,y)\,dy = 0$ happens to be exact, then it is equivalent to

$$\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = 0$$

This implies that the function F is constant on R and solutions to the

DE are given by the relation

$$F(x,y) = C$$

Recognizing Exactness

There is a theorem from calculus that ensures that if a function F has first partials on a domain, and if those partials are continuous, then the second mixed partials are equal. That is,

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}.$$

If it is true that

$$\frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N$$

this provides a condition for exactness, namely

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Exact Equations

Theorem: Let M and N be continuous on some rectangle R in the plane. Then the equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Example

Show that the equation is exact and obtain a family of solutions.

$$(2xy - \sec^2 x)\,dx + (x^2 + 2y)\,dy = 0$$

First note that for $M = 2xy - \sec^2 x$ and $N = x^2 + 2y$ we have

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}.$$

Hence the equation is exact. We obtain the solutions (implicitly) by finding a function F such that $\partial F/\partial x = M$ and $\partial F/\partial y = N$. Using the first relation we get[13]

$$F(x,y) = \int M(x,y)\,dx = \int (2xy - \sec^2 x)\,dx = x^2 y - \tan x + g(y)$$

[13] Holding y constant while integrating with respect to x means that the *constant* of integration may well depend on y

Example Continued

We must find g to complete our solution. We know that

$$F(x,y) = x^2y - \tan x + g(y) \quad \text{and} \quad \frac{\partial F}{\partial y} = N(x,y) = x^2 + 2y$$

Differentiating the expression on the left with respect to y and equating

$$\frac{\partial F}{\partial y} = x^2 + g'(y) = x^2 + 2y$$

from which if follows that $g'(y) = 2y$. An antiderivative is given by $g(y) = y^2$. Since our solutions are $F = C$, we arrive at the family of solutions

$$x^2y - \tan x + y^2 = C$$

Special Integrating Factors

Suppose that the equation $M\,dx + N\,dy = 0$ is not exact. Clearly our approach to exact equations would be fruitless as there is no such function F to find. It may still be possible to solve the equation if we can find a way to morph it into an exact equation. As an example, consider the DE

$$(2y - 6x)\,dx + (3x - 4x^2y^{-1})\,dy = 0$$

Note that this equation is NOT exact. In particular

$$\frac{\partial M}{\partial y} = 2 \neq 3 - 8xy^{-1} = \frac{\partial N}{\partial x}.$$

Special Integrating Factors

But note what happens when we multiply our equation by the function $\mu(x,y) = xy^2$.

$$xy^2(2y - 6x)\,dx + xy^2(3x - 4x^2y^{-1})\,dy = 0, \quad \Longrightarrow$$

$$(2xy^3 - 6x^2y^2)\,dx + (3x^2y^2 - 4x^3y)\,dy = 0$$

Now we see that

$$\frac{\partial(\mu M)}{\partial y} = 6xy^2 - 12x^2y = \frac{\partial(\mu N)}{\partial x}$$

The new equation[14] IS exact!

Of course this raises the question: **How would we know to use $\mu = xy^2$?**

---

[14]The solutions sets for these equations are almost the same. However, it is possible to introduce or lose solutions employing this approach.We won't worry about this here.

Special Integrating Factors

The function μ is called a *special integrating factor*. Finding one (assuming one even exists) may require ingenuity and likely a bit of luck. However, there are certain cases we can look for and perhaps use them to solve the occasional equation. A useful method is to look for μ of a certain *form* (usually $\mu = x^n y^m$ for some powers n and m). We will restrict ourselves to two possible cases:

There is an integrating faction $\mu = \mu(x)$ depending only on x, or there is an integrating factor $\mu = \mu(y)$ depending only on y.

Special Integrating Factor $\mu = \mu(x)$

Suppose that

$$M\,dx + N\,dy = 0$$

is NOT exact, but that

$$\mu M\,dx + \mu N\,dy = 0$$

IS exact where $\mu = \mu(x)$ does not depend on y. Then

$$\frac{\partial(\mu(x)M)}{\partial y} = \frac{\partial(\mu(x)N)}{\partial x}.$$

Let's use the product rule in the right side.

Special Integrating Factor $\mu = \mu(x)$

Since μ is *constant* in y,

$$\mu \frac{\partial M}{\partial y} = \frac{d\mu}{dx} N + \mu \frac{\partial N}{\partial x} \tag{11}$$

Rearranging (11), we get both a condition for the existence of such a μ as well as an equation for it. The function μ must satisfy the separable equation

$$\frac{d\mu}{dx} = \mu \left(\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \right) \tag{12}$$

Note that this equation is solvable, insofar as μ depends only on x, only if

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

depends only on x!

Special Integrating Factor

When solvable, equation (12) has solution

$$\mu = \exp\left(\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\, dx\right)$$

A similar manipulation assuming a function $\mu = \mu(y)$ depending only on y leads to the existence condition requiring

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$$

depend only on y.

Special Integrating Factor

$$M\,dx + N\,dy = 0 \tag{13}$$

Theorem: If $(\partial M/\partial y - \partial N/\partial x)/N$ is continuous and depends only on x, then

$$\mu = \exp\left(\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\,dx\right)$$

is a special integrating factor for (13). If $(\partial N/\partial x - \partial M/\partial y)/M$ is

continuous and depends only on y, then

$$\mu = \exp\left(\int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\,dy\right)$$

is a special integrating factor for (13).

Example

Solve the equation $2xy\,dx + (y^2 - 3x^2)\,dy = 0$.

Note that $\partial M/\partial y = 2x$ and $\partial N/\partial x = -6x$. The equation is not exact. Looking to see if there may be a special integrating factor, note that

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{8x}{y^2 - 3x^2}$$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-8x}{2xy} = \frac{-4}{y}$$

The first does not depend on x alone. But the second does depend on y alone. So there is a special integrating factor

$$\mu = \exp\left(\int -\frac{4}{y}\,dy\right) = y^{-4}$$

Example Continued

The new equation obtained by multiplying through by μ is

$$2xy^{-3}\,dx + (y^{-2} - 3x^2y^{-4})\,dy = 0.$$

Note that

$$\frac{\partial}{\partial y}2xy^{-3} = -6xy^{-4} = \frac{\partial}{\partial x}(y^{-2} - 3x^2y^{-4})$$

so this new equation is exact. Solving for F

$$F(x,y) = \int 2xy^{-3}\,dx = x^2y^{-3} + g(y)$$

and $g'(y) = y^{-2}$ so that $g(y) = -y^{-1}$. The solutions are given by

$$\frac{x^2}{y^3} - \frac{1}{y} = C.$$

Section 5: First Order Equations: Models and Applications

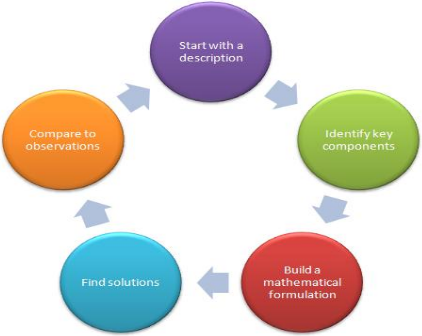


Figure: Mathematical Models give Rise to Differential Equations

In this section, we will consider select models involving first order ODEs. Let's see the process in action.

Population Dynamics

A population of dwarf rabbits grows at a rate proportional to the current population. In 2021, there were 58 rabbits. In 2022, the population was up to 89 rabbits. Estimate the number of rabbits expected in the population in 2031.

We can translate this into a mathematical statement then hope to solve the problem and answer the question. Letting the population of rabbits (say population density) at time t be given by $P(t)$, to say that the rate of change is proportional to the population is to say

$$\frac{dP}{dt} = kP(t) \quad \text{for some constant } k.$$

This is a differential equation! To answer the question, we will require the value of k as well as some initial information—i.e. we will need an IVP.

Example Continued...[15]

We can choose units for time t. Based on the statement, taking t in years is well advised. Letting $t = 0$ in year 2021, the second and third sentences translate as

$$P(0) = 58, \quad \text{and} \quad P(1) = 89$$

Without knowing k, we can solve the IVP

$$\frac{dP}{dt} = kP, \quad P(0) = 58$$

by separation of variables to obtain

$$P(t) = 58e^{kt}.$$

[15] Taking P as the population density, i.e. number of rabbits per unit habitat, allows us to consider non-integer P values. Thus fractional and even irrational P values make sense.

Example Continued...

To evaluate the population function (for a real number), we still need to know k. We have the additional information $P(1) = 89$. Note that this gives

$$P(1) = 89 = 58e^{1k} \quad \Longrightarrow \quad k = \ln\left(\frac{89}{58}\right).$$

Hence the function

$$P(t) = 58e^{t\ln(89/58)}.$$

Finally, the population in 2031 is approximately

$$P(10) = 58e^{10\ln(89/58)} \approx 4200$$

Note that we took the statement given involving a rate of change, and interpreted it as a *differential equation*. Additional data gave us *initial conditions*. We solved the resulting problem, and used it to make a prediction. This is a simple example, but it illustrates the power of mathematical modeling.

Exponential Growth or Decay

Exponential Growth/Decay

If a quantity P changes continuously at a rate proportional to its current value, then it will be governed by a differential equation of the form

$$\frac{dP}{dt} = kP \quad \text{i.e.} \quad \frac{dP}{dt} - kP = 0.$$

Note that this equation is both separable and first order linear.

If $k > 0$, P experiences **exponential growth**. If $k < 0$, then P experiences **exponential decay**.

In practice, we typically take $k > 0$ and in the case of decay, we write $\frac{dP}{dt} = -kP$.

Series Circuits: RC-circuit

With the restriction that we are considering only models involving first order equations, we can consider two types of simple circuits, an *RC*-series circuit or an *LR*-series circuit.

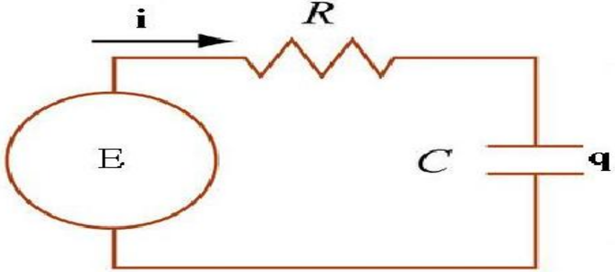


Figure: Series Circuit with Applied Electromotive force E, Resistance R, and Capcitance C. The charge on the capacitor is q and the current $i = \frac{dq}{dt}$. Both q and i are functions of time.

Series Circuits: LR-circuit

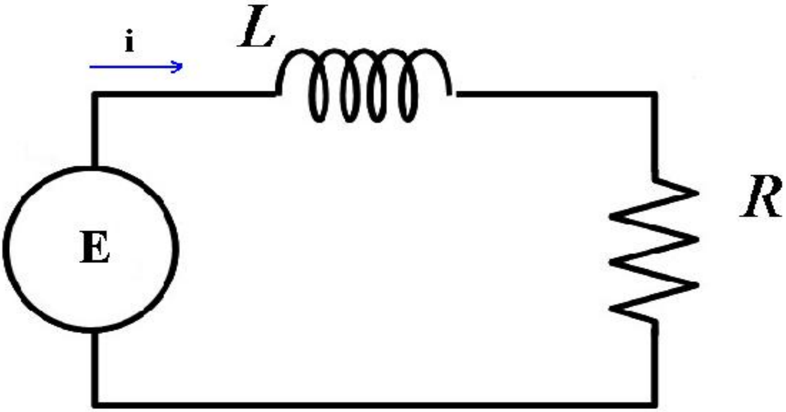


Figure: Series Circuit with Applied Electromotive force E, Inductance L, and Resistance R. We track the current i as a function of time.

Measurable Quantities:

In these problems, there are several measurable quantities. These are listed here along with the relevant units of measure.

Resistance R in ohms (Ω),	Implied voltage E in volts (V),
Inductance L in henries (h),	Charge q in coulombs (C),
Capacitance C in farads (f),	Current i in amperes (A)

Current is the rate of change of charge with respect to time: $i = \dfrac{dq}{dt}$.

Component	Potential Drop
Inductor	$L\frac{di}{dt}$
Resistor	Ri i.e. $R\frac{dq}{dt}$
Capacitor	$\frac{1}{C}q$

Table: The potential drop across various elements is know empirically.

Kirchhoff's Law

Kirchhoff's Law

Kirchhoff's Law states that:
The sum of the voltages around a closed circuit is zero.

In other words, the sum of potential drops across the passive components is equal to the applied electromotive force. We can use this to arrive at a differential equation for the charge $q(t)$ in an RC circuit or the current $i(t)$ in and LR circuit.

Both of these result in a first order linear differential equation.

RC Series Circuit

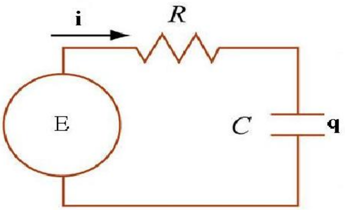


Figure: Series Circuit with Applied Electromotive force E, Resistance R, and Capcitance C. The charge of the capacitor is q and the current $i = \frac{dq}{dt}$.

drop across resistor	+	drop across capacitor	=	applied force
$R\frac{dq}{dt}$	$+$	$\frac{1}{C}q$	$=$	$E(t)$

$$\boxed{R\frac{dq}{dt} + \frac{1}{C}q = E(t)}$$

If $q(0) = q_0$, the IVP can be solved to find $q(t)$ for all $t > 0$.

LR Series Circuit

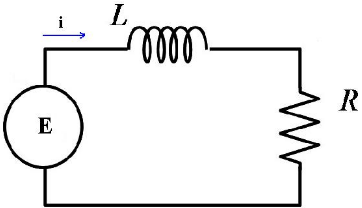


Figure: Series Circuit with Applied Electromotive force E, Inductance L, and Resistance R. The current is i.

drop across inductor	$+$	drop across resistor	$=$	applied force
$L\frac{di}{dt}$	$+$	Ri	$=$	$E(t)$

$$\boxed{L\frac{di}{dt} + Ri = E(t)}$$

If $i(0) = i_0$, the IVP can be solved to find $i(t)$ for all $t > 0$.

Summary of First Order Circuit Models

Before considering an example, let's summarize our two circuit models.

The charge $q(t)$ at time t on the capacitor in an RC-series circuit with resistance R ohm, capacitance C farads, and applied voltage $E(t)$ volts satisfies

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t), \quad q(0) = q_0$$

where q_0 is the initial charge on the capacitor.

The current $i(t)$ at time t in an LR-series circuit with resistance R ohm, inductance L henries, and applied voltage $E(t)$ volts satisfies

$$L\frac{di}{dt} + Ri = E(t), \quad i(0) = i_0$$

where i_0 is the initial current in the circuit.

Example

A 200 volt battery is applied to an RC series circuit with resistance 1000Ω and capacitance 5×10^{-6} f. Find the charge $q(t)$ on the capacitor if $i(0)=0.4A$. Determine the charge as $t\to\infty$.

The problem clearly refers to an RC circuit, so we use the appropriate model. Note that

$$R=1000,\quad C=5\times 10^{-6},\quad \text{and}\quad E=200.$$

The initial condition in this problem is irregular. Since $i=q'$, the condition can be read as $q'(0)=0.4$. Using the RC model, we have

$$1000\frac{dq}{dt}+\frac{1}{5\cdot 10^{-6}}q=200,\quad q'(0)=0.4$$

In standard form the equation is $q'+200q=1/5$ with integrating factor $\mu=\exp(200t)$.

Example Continued...

Continuing,

$$\frac{dq}{dt} + 200q = \frac{1}{5}, \quad q'(0) = 0.4.$$

Using the integrating factor, we find the general solution to the ODE

$$q(t) = \frac{1}{1000} + Ke^{-200t}.$$

Finally, we apply the initial condition and

$$\boxed{\text{the charge on the capacitor} \quad q(t) = \frac{1}{1000} - \frac{e^{-200t}}{500}.}$$

The long time charge on the capacitor (i.e., the **steady state charge**) is therefore

$$\lim_{t\to\infty} q(t) = \frac{1}{1000}.$$

A Classic Mixing Problem

The next type of model we will derive governs the amount of some identifiable substance in a composite mixture. Examples of such mixing problems could include tracking the mass of ethanol in an ethanol-gasoline mixture, or tracking the mass of pollutant in a volume of water. We will derive a general ODE model for classical mixing by considering an example involving the mass of salt in a volume of salt water.

The general scenario typically involves a volume of composite mixture in which fluid is flowing in and flowing out. We assume that the composition is spatially homogeneous so that the independent variable is time. Such information as fluid flow rates and the concentration of the tracked substance in the inflow are generally given in the problem description. Let's get a visual idea of this process, and then consider a specific example.

A Classic Mixing Problem

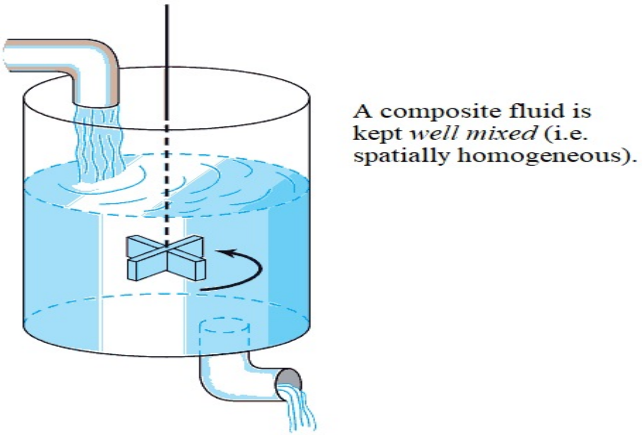


Figure: We wish to track the amount of some substance in a composite mixture such as salt and water, gas and ethanol, polutant and water, etc. Fluid may flow in and out of the composition, and we assume instant mixing so that the mass of some substance is dependent on time, but not on space.

A Classic Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

In order to answer such a question, we need to convert the problem statement into a mathematical one.

Some Notation

In addition to the amount of salt, $A(t)$, at time t we have several variables or parameters. Let

- r_i be the rate at which fluid enters the tank (rate in),
- r_o be the rate at which fluid leaves the tank (rate out),
- c_i be the concentration of substance (salt) in the in-flowing fluid (concentration in),
- c_o be the concentration of substance (salt) in the out-flowing fluid (concentration out),
- $V(t)$ be the total volume of fluid in the tank at time t,
- V_0 be the volume of fluid in the tank at time $t = 0$, i.e., $V_0 = V(0)$

A Classic Mixing Problem Illustrated

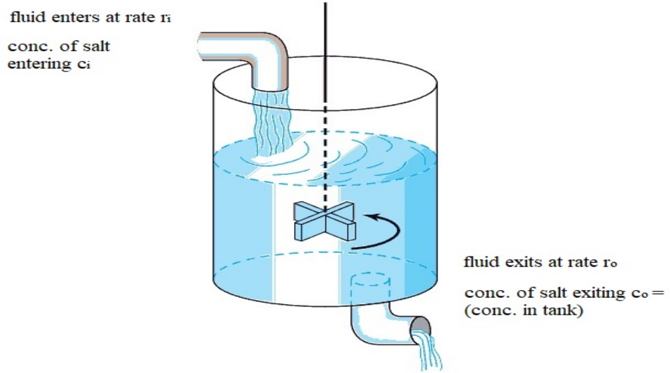


Figure: The concentration of salt in the tank can change with time (but not space). The well mixed solution assumption ensures that the concentration of salt in the outflow, c_o, matches the concentration of salt in the tank at each time t. Note, this makes c_o time dependent through its dependence on both the volume and the amount of salt at time t.

Building an Equation

We consider the rate at which the amount of salt changes with time, and create a *balance* equation. The rate of change of the amount of salt

$$\frac{dA}{dt} = \left(\begin{array}{c} \text{input rate} \\ \text{of salt} \end{array}\right) - \left(\begin{array}{c} \text{output rate} \\ \text{of salt} \end{array}\right)$$

We characterize these rates as products of flow rates and concentrations.

The input rate of salt is

$$\text{fluid rate in} \cdot \text{concentration of inflow} \quad = \quad r_i \cdot c_i.$$

The output rate of salt is

$$\text{fluid rate out} \cdot \text{concentration of outflow} \quad = \quad r_o \cdot c_o.$$

The parameters r_i, c_i, and r_o are part of the problem statement. We must determine c_o.

Building an Equation

The well mixed solution assumption means that the concentration of salt in the out-flowing fluid is equal to the concentration in the tank. This is a variable that depends on t and A.

$$c_o = \frac{\text{total salt}}{\text{total volume}} = \frac{A(t)}{V(t)} = \frac{A(t)}{V(0)+(r_i-r_o)t}.$$

Note that the volume

$$V(t) = \textit{initial volume} + \textit{rate in} \times \textit{time} - \textit{rate out} \times \textit{time}.$$

If $r_i = r_o$, then $V(t) = V(0)$ a constant.

Pulling this together, the amount A satisfies the first order linear ODE

$$\boxed{\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V}.}$$

Solve the Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

We can take A in pounds, V in gallons, and t in minutes. Here, $V(0) = 500$ gal, $r_i = 5$ gal/min, $c_i = 2$ lb/gal, and $r_o = 5$ gal/min. Since the incoming and outgoing rates are the same, the volume $V(t) = 500$ gallons for all t. This gives an outgoing concentration of

$$c_o = \frac{A(t)}{V(t)} = \frac{A(t)}{500 + 5t - 5t} = \frac{A(t)}{500}.$$

Since the tank originally contains pure water (no salt), we have $A(0) = 0$.

Mixing Example

Our IVP is

$$\frac{dA}{dt} = 5\text{gal/min} \cdot 2\text{lb/gal} - 5\text{gal/min} \cdot \frac{A}{500}\text{lb/gal}, \quad A(0) = 0$$

$$\frac{dA}{dt} + \frac{1}{100}A = 10, \quad A(0) = 0.$$

The ODE is a first order linear equation[16] that can be solved using an integrating factor. The solution to the IVP is

$$A(t) = 1000\left(1 - e^{-t/100}\right).$$

The concentration c of salt in the tank after five minutes is therefore

$$c = \frac{A(5)\,\text{lb}}{V(5)\,\text{gal}} = \frac{1000(1 - e^{-5/100})}{500}\,\text{lb/gal} \approx 0.01\,\text{lb/gal}.$$

[16] Since the volume is constant, this ODE is also separable. If the volume is not constant, the ODE will still be linear but not separable.

A Nonlinear Modeling Problem

The last model we will consider is a nonlinear population model. Let's consider it through an example.

A population $P(t)$ of tilapia changes at a rate jointly proportional to the current population and the difference between the constant carrying capacity[17] M of the environment and the current population. Determine the differential equation satsified by P.

To say that P has a rate of change jointly proportional to P and the difference between P and M is

$$\frac{dP}{dt} \propto P(M-P) \quad \text{i.e.} \quad \frac{dP}{dt} = kP(M-P)$$

for some constant of proportionality k.

[17] The carrying capacity is the maximum number of individuals that the environment can support due to limitation of space and resources.

Logistic Differential Equation

Logistic Growth Model

The equation $\frac{dP}{dt} = kP(M - P)$, where $k, M > 0$ is called a **logistic growth equation**.

Suppose the intial population $P(0) = P_0$. Solve the resulting initial value problem. Show that if $P_0 > 0$, the population tends to the carrying capacity M.

The equation is separable and Bernoulli[18]. It can be solved by separation of variables or by using the change of variables for a Bernoulli equation[19]. The general solution to the DE is

$$P(t) = \frac{MCe^{Mkt}}{1 + Ce^{Mkt}}.$$

[18] As written, it's clearly separable. If we rearrange it as $\frac{dP}{dt} - kMP = -kP^2$, we see that it is also a Bernoulli equation.

[19] I'd recommend trying both methods to determine which approach you think is most desirable.

Logistic Growth: $P'(t) = kP(M-P)\ P(0) = P_0$

Applying the condition $P(0) = P_0$ to find the constant C, we obtain the solution to the IVP

$$P(t) = \frac{P_0 M e^{Mkt}}{M - P_0 + P_0 e^{Mkt}} = \frac{P_0 M}{(M - P_0)e^{-Mkt} + P_0}.$$

If $P_0 = 0$, then $P(t) = 0$ for all t (i.e., if there are no fish to start with, there will always be no fish). Otherwise, we can take the limit as $t \to \infty$ to obtain

$$\lim_{t\to\infty} P(t) = \frac{P_0 M}{0 + P_0} = M$$

as expected.

Long Time Solution of Logistic Equation

$$\frac{dP}{dt} = kP(M - P)$$

If we assume that the constant $k > 0$, we can see from the ODE why the long time population tends to M. Recall that $\frac{dP}{dt}$ is the rate of population change, and its sign indicates whether the population is increasing (+) or decreasing (-). We can consider two cases assuming that $P > 0$.

- If $0 < P < M$, then $\dfrac{dP}{dt} > 0$ and P increases.
- IF $P > M$, then $\dfrac{dP}{dt} < 0$ and P decreases.

We see that the differential equation itself, aside from any solution, tells us that any positive population should tend towards the carrying capacity. If it's below the capacity, it grows. If it's above the capacity, it shrinks.

Note that this is a qualitative analysis of the ODE and its solution that doesn't require us to actually solve the differential equation.

Qualitative Analysis

While the models considered here give rise to ODEs that we can solve in terms of *nice* expressions, it is often the case in practice that a solution is difficult (perhaps impossible) to state explicitly. An option is to rely on a numerical scheme (e.g., Euler's Method). As our observations about the logistic growth model imply, we may be able to make some predictions about the solution to a model in the absence of an actual solution. In particular, we may be able to determine nature of a time varying process after a *long time* has past. When the rate of change does not depend explicitly on time, there is a process we can follow.

Qualitative Analysis

Autonomous Equation

The differential equation $\dfrac{dy}{dt} = f(t, y(t))$ is called **autonomous** if the right hand side does not depend explicitly on t—i.e., an autonomous equation has the form

$$\frac{dy}{dt} = F(y).$$

For example, $\dfrac{dy}{dt} = y(2-y)$ is autonomous, whereas the equation $\dfrac{dy}{dt} = 2ty$ is NOT autonomous. In the first example, there is no explicit mention of the variable t (even though it is understood that y depends on t). In the second example, the factor t appears on the right side making the equation not autonomous (or we could say nonautonomous).

Equilibrium Solutions

Equilibrium Solutions

If y_0 is a value such that $F(y_0) = 0$, then the constant function $y(t) = y_0$ is called an **equilibrium** solution (or equilibrium point) of the autonomous differential equation $\dfrac{dy}{dt} = F(y)$.

Note that if $y(t) = y_0$ is a constant function, then $\dfrac{dy}{dt} = 0$. If $F(y_0) = 0$, then it is obvious upon substitution that $y = y_0$ solves the ODE $y' = F(y)$. If we have an initial condition $y(0) = y_0$, we expect that $y(t) = y_0$ for all $t > 0$ since there's no propensity for y to change! An interesting question is *what if y is not actually equal to y_0 but is somehow "close" to y_0?*

Equilibrium Solutions

Consider the example of the logistic equation $\frac{dy}{dt} = ky(M-y)$ which is autonomous (assuming that k and M are constants). Here, $F(y) = ky(M-y)$ has two roots, $y_{0,1} = 0$ and $y_{0,2} = M$. So $y(t) = 0$ and $y(t) = M$ are two equilibrium solutions.

From our previous analysis, we saw that if $y(0) = 0$, the $y(t) = 0$ for all $t > 0$. But, for any nonzero initial condition, $y(t) \to M$. There is a fundamental difference between these two equilibria. If y is not **exactly** $y_{0,1} = 0$, then it tends away from this equilibrium solution. If y is not equal to $y_{0,2} = M$, whether it's larger or smaller, it tends to this equilibrium solution. We have a name for this observation. We say that $y_{0,1} = 0$ is an **unstable** equilibrium solution, and $y_{0,2} = M$ is a **stable** (or asymptotically stable) equilibrium solution.

Stability of Equilibrium Solutions

In general, we may classify an equilibrium solution of a given autononous ODE as being

- **unstable**: solutions close, but not exact, will tend away from the equilibrium value,
- **stable**: solutions close, but not exact, will tend towards the equilibrium value[20], or
- **semi-stable**: solutions close, but not exact, may tend towards or away from the equilibrium value depending on whether the solution is greater than or less than the equilibrium value.

The nature of equilibria and stability is more complex than I'm letting on here, especially if we consider the case of nonautonomous equations or systems involving two or more dependent variables. For example, we might consider a type of stability in which solutions that start near an equilibrium value stay near the value without actually tending towards it.

[20] This is more accurately referred to as asymptotically stable

Stability of Equilibrium Solutions

To determine the nature of an equilibrium solution y_0 for an ODE $y' = F(y)$, we can analyze the sign of F in the neighborhood of y_0. Suppose F is continuous on an open interval about y_0.

- If F changes sign from positive (+) to negative (-) as y passes through y_0 (from left to right), then y_0 is a stable equilibrium.
- If F changes signs from negative (-) to positive (+) as y passes through y_0, then then y_0 is an unstable equilibrium.
- If F doesn't change signs, then y_0 is semi-stable.

If this reminds you of a *derivative test*, there's a good reason for that. Fortunately, it's easy to visualize the cases if you can obtain even a crude drawing of the graph of F.

Stability of Equilibrium Solutions

If we can plot $F(y)$, then we can create a flow diagram from which the nature of the solution to $y' = F(y)$ can be inferred. Just remember that if $F(y) > 0$, then $y(t)$ will tend to the right whereas it will tend to the left when $F(y) < 0$. Consider the following example.

Example: The equation $\frac{dy}{dt} = ky(M - y)(y - N)$ models the population of some species that is subject to both logistic growth considerations coupled with a threshold criterion. The threshold criterion states that there is a minimum population N necessary for successful reproduction. The threshold criterion is captured by the added factor $y - N$. Note that by itself, the factor $y - N$ will result in a decrease if $y < N$ and an increase if $y > N$.

We see that F has three roots, $y_{0,1} = 0$, $y_{0,2} = N$ and $y_{0,2} = M$. We can plot

$$F(y) = ky(M - y)(y - N) = -kMNy + kMy^2 + kNy^2 - ky^3$$

and look at the sign of y' near each of the three equilibria.

Stability of Equilibrium Solutions

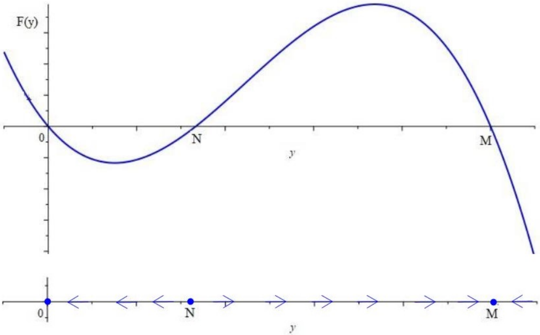


Figure: In the top of the figure, we plot the cubic $F(y) = ky(M - y)(y - N)$. The graph passes through the horizontal (y) axis at the three equilibrium values 0, N, and M. Negative values correspond to leftward motion by the solution of $y' = F(y)$ and positive values correspond to rightward motion. In the bottom figure, this is illustrated by a flow diagram showing leftward or rightward arrows.

Stability of Equilibrium Solutions

From the diagram, we see that on the interval $(0, N)$, the solution $y(t)$ tends to the equilibrium value $y_{0,1} = 0$. On the intervals (N, M) and (M, ∞), the solution tends toward the equilibrium value $y_{0,3} = M$. The two equilibria $y_{0,1} = 0$ and $y_{0,3} = M$ are **asymptotically stable**, whereas the equilibrium $y_{0,2} = N$ is unstable.

The analysis suggests that if the initial condition $y(0)$ is such that $0 < y(0) < N$, the population will experience extinction (tend to zero). Any initial condition satisfying $N < y(0)$ will result in the solution tending to the carrying capacity M.

Remark: The ODE is separable, so it's possible to obtain (implicit) solutions[21] with the tools we have. But it is much easier to deduce the long term behavior from the ODE than from the general solution.

[21] Solutions are defined implicitly by $\ln\left|y^{N-M}(y-N)^M(M-y)^{-N}\right| - kMN(M-N)x = C.$

Example

Consider the IVP $y' = 2(y+1)(2-y)^2(y-3), \quad y(0) = k$. Determine the long time behavior, $\lim_{t\to\infty} y(t)$, if

(a) $k = -2$, (b) $k = 0$, (c) $k = 1$,
(d) $k = 2$, (e) $k = 2.5$, (f) $k = 4$.

The ODE is autonomous and $F(y) = 2(y+1)(2-y)^2(y-3)$ is a fourth degree polynomial with three roots at $y_{0,1} = -1$, $y_{0,2} = 2$ and $y_{0,3} = 3$. So these are the equilibrium values. It's not really necessary to plot F because we can simply determine the sign in each of the intervals $(-\infty, -1)$, $(-1, 2)$, $(2, 3)$, and $(3, \infty)$, and infer the nature of each equilibrium.

Example

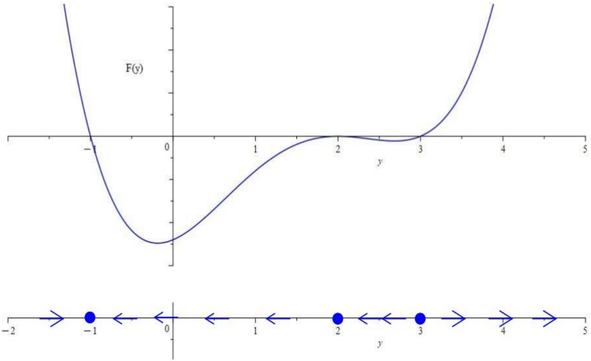


Figure: The figure shows a plot of $F(y) = 2(y+1)(2-y)^2(y-3)$ (top) along with the associated flow diagram based on the sign of $F(y)$ in each interval determined by its zeros. We see that $y_{0,1} = -1$ is a stable equilibrium, $y_{0,2} = 2$ is semi-stable, and $y_{0,3} = 3$ is unstable. Solutions that start close to 2 on its left side will tend away from 2, but solutions that start close to 2 on its right side will tend towards 2.

Example

Consider the IVP $y' = 2(y+1)(2-y)^2(y-3), \quad y(0) = k$. Determine the long time behavior, $\lim_{t\to\infty} y(t)$, if

(a)	$k = -2$,	(b)	$k = 0$,	(c)	$k = 1$,
(d)	$k = 2$,	(e)	$k = 2.5$,	(f)	$k = 4$.

From the flow diagram, we see that

(a)	$y(t) \to -1$,	(b)	$y(t) \to -1$,	(c)	$y(t) \to -1$,
(d)*	$y(t) \to 2$,	(e)	$y(t) \to 2$,	(f)	$y(t) \to \infty$.

* Note that if $k = 2$, then $y(t) = 2$ for all $t > 0$. That is, the solution is constant.

Models Derived in this Section

To wrap up, let's list the various models discussed here that involve first order ODEs.

Exponential Growth/Decay

$$\frac{dP}{dt} = kP$$

RC-Series Circuit

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

LR-Series Circuit

$$L\frac{di}{dt} + Ri = E(t)$$

Classical Mixing

$$\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A(t)}{V(0) + (r_i - r_o)t}$$

Logistic Growth

$$\frac{dP}{dt} = kP(M - P)$$

Section 6: Linear Equations: Theory and Terminology

Recall that an n^{th} order linear IVP consists of an equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

to solve subject to conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}.$$

The problem is called **homogeneous** if $g(x) \equiv 0$. Otherwise it is called **nonhomogeneous.**

Theorem: Existence & Uniqueness

We have the following important theorem regarding the existence and uniqueness of solutions to the IVP

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}.$$

Theorem:

If $a_0, \ldots, a_n$ and g are continuous on an interval I, $a_n(x) \neq 0$ for each x in I, and x_0 is any point in I, then for any choice of constants $y_0, \ldots, y_{n-1}$, the IVP has a unique solution $y(x)$ on I.

Put differently, we're guaranteed to have a solution exist, and it is the only one there is!

Example

Use only a little clever intuition to solve the IVP

$$y'' + 3y' - 2y = 0, \quad y(0) = 0, \quad y'(0) = 0$$

Exercise left to the reader. (Hint: Think *simple*. While we usually consider the initial conditions at the end, it may help to think about them first.)

A Second Order Linear Boundary Value Problem

While we will primarily consider IVPs, there is a similar problem called a BVP (boundary value problem).

Boundary Value Problem

A second order BVP consists of a differential equation

$$a_2(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x), \quad a<x<b$$

to solve subject to a pair of conditions[a]

$$y(a)=y_0, \quad y(b)=y_1.$$

[a]Other conditions on y and/or y' can be imposed. The key characteristic is that conditions are imposed at both end points $x=a$ and $x=b$.

Remark: However similar this is in appearance, the existence and uniqueness result **does not hold** for this BVP!

BVP Example

Let's consider three similar BVPs.

$$(1) \quad y'' + 4y = 0, \quad 0 < x < \frac{\pi}{4} \quad y(0) = 0, \quad y\left(\frac{\pi}{4}\right) = 0.$$

$$(2) \quad y'' + 4y = 0, \quad 0 < x < \frac{\pi}{2} \quad y(0) = 0, \quad y\left(\frac{\pi}{2}\right) = 0.$$

$$(3) \quad y'' + 4y = 0, \quad 0 < x < \frac{\pi}{2} \quad y(0) = 0, \quad y\left(\frac{\pi}{2}\right) = 1.$$

BVP Examples

It can be shown that the solution to the ODE $y'' + 4y = 0$ is given by the two-parameter family of functions

$$y = c_1 \cos(2x) + c_2 \sin(2x).$$

It can also be shown (verify this for yourself) that

- Problem (1) has exactly one solution $y = 0$.
- Problem (2) has infinitely many solutions $y = c_2 \sin(2x)$ where c_2 is any real number.
- And problem (3) has no solutions.

Remark: The only difference between these problems is the choice of interval and the values prescribed at the end points. However, the existence and uniqueness properties of the solutions are completely different!

Homogeneous Equations

We will consider n^{th} order, linear ODEs and come up with general statements about the solution(s) and the form a solution can take. We will define some terms such as **general solution**, **complementary solution**, and **particular solution**. Context is critical as these terms can have slightly different meanings depending on the equation considered.

First, we will focus on **homogeneous** equations. We will consider the ODE

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0.$$

In what follows, we will assume that $a_i(x)$ is continuous on some interval I and that $a_n(x)\neq 0$ for all x in I.

Superposition

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

Theorem: The Principle of Superposition

If y_1, $y_2, \ldots, y_k$ are all solutions of this homogeneous equation on an interval I, then the *linear combination*

$$y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x)$$

is also a solution on I for any choice of constants $c_1, \ldots, c_k$.

Remark 1: This result, known as the principle of superposition says that new solutions to the homogeneous equation can be constructed by multiplying solutions by constants and adding them together.

Remark 2: This is the principle of superposition for **homogeneous**, linear ODEs. We will state another principle for nonhomogeneous equations.

Corollaries

These two results follow directly from the principle of superposition for linear, homogeneous ODEs.

Corollaries

(i) If y_1 solves the homogeneous equation, the any constant multiple $y = cy_1$ is also a solution.

(ii) The solution $y = 0$ (called the trivial solution) is always a solution to a homogeneous equation.

This raies a couple of **Big Questions:**

- Does an equation have any **nontrivial** solution(s), and
- since y_1 and cy_1 aren't truly *different* solutions, what criteria will be used to call solutions distinct? (The next definition will address this.)

Linear Dependence

Definition:

A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ are said to be **linearly dependent** on an interval I if there exists a set of constants $c_1, c_2, \ldots, c_n$ with at least one of them being nonzero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all} \quad x \text{ in } I.$$

A set of functions that is not linearly dependent on I is said to be **linearly independent** on I.

Note: It is always possible to form the above linear combination to obtain zero by simply taking all of the coefficients $c_i = 0$. The question here is whether it is possible to have at least one of the c's be nonzero. If so, the functions are **Linearly Dependent**. If all c's **must** be zero, then the functions are **Linearly Independent**.

Example: A linearly Dependent Set

The functions $f_1(x) = \sin^2 x$, $f_2(x) = \cos^2 x$, and $f_3(x) = 1$ are linearly dependent on $I = (-\infty, \infty)$.

Making use of our knowledge of Pythagorean IDs, we can take $c_1 = c_2 = 1$ and $c_3 = -1$ (this isn't the only choice, but it will do the trick). Note that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = \sin^2 x + \cos^2 x - 1 = 0 \quad \text{for all real } x.$$

Since we've found a set of coefficients with at least one nonzero such that $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$ for all real x, we have demonstrated that the functions are **linearly dependent**.

Example: A linearly Independent Set

The functions $f_1(x) = \sin x$ and $f_2(x) = \cos x$ are linearly independent on $I = (-\infty, \infty)$.

We will show that the only linear combination of f_1 and f_2 that is zero for all x must have zero coefficients. To do this, suppose $c_1 f_1(x) + c_2 f_2(x) = 0$ for all real x. Then the equation must hold when $x = 0$, and it must hold when $x = \pi/2$. Consequently

$$c_1 \cos(0) + c_2 \sin(0) = 0 \quad \Longrightarrow \quad c_1 = 0 \quad \text{and}$$

$$0 \cdot \cos(\pi/2) + c_2 \sin(\pi/2) = 0 \quad \Longrightarrow \quad c_2 = 0.$$

We see that the only way for our linear combination to be zero is for both coefficients to be zero. Hence the functions are **linearly independent**.

Determine if the set is Linearly Dependent or Independent

$$f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2 \quad \text{for} \quad -\infty < x < \infty$$

Looking at the functions, we should suspect that they are linearly dependent. Why? Because f_3 is a linear combination of f_1 and f_2. In fact,

$$f_3(x) = \frac{1}{4}f_2(x) - f_1(x) \quad \text{i.e.} \quad f_1(x) - \frac{1}{4}f_2(x) + f_3(x) = 0$$

for all real x. The latter is our linear combination with $c_1 = c_3 = 1$ and $c_2 = -\frac{1}{4}$ (not all zero).

With only two or three functions, we may be able to intuit linear dependence/independence. We have an object that will allow us to test for linear dependence under certain circumstances. This is the next topic.

Definition of Wronskian

Definition: Wronskian

Let f_1, $f_2, \dots, f_n$ posses at least $n-1$ continuous derivatives on an interval I. The **Wronskian** of this set of functions is the determinant

$$W(f_1, f_2, \dots, f_n)(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

Note that, in general, this Wronskian is a real valued function of the independent variable x. The notation allows us to indicate what functions the Wronskian depends on as well as the independent variable. We'll often shorten it to $W(x)$ or just W as long as it's clear from the context.

Recall

Determinant Formulas (2×2 and 3×3)

If A is a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then its determinant

$$\det(A) = ad - bc.$$

If A is a 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then its determinant

$$\det(A) = a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Wronskian Examples

Let's do some small examples. Determine the Wronskian of the pair $\{f_1, f_2\}$ where

$$f_1(x) = \sin x, \quad f_2(x) = \cos x$$

$$W(f_1, f_2)(x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$$= \sin x(-\sin x) - \cos x(\cos x)$$

$$= -\sin^2 x - \cos^2 x = -1$$

. So we have

$$W(\sin x, \cos x)(x) = -1.$$

Determine the Wronskian of the Functions

$$f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2$$

$$W(f_1, f_2, f_3)(x) = \begin{vmatrix} x^2 & 4x & x - x^2 \\ 2x & 4 & 1 - 2x \\ 2 & 0 & -2 \end{vmatrix}$$

$$2 \begin{vmatrix} 4x & x - x^2 \\ 4 & 1 - 2x \end{vmatrix} + (-2) \begin{vmatrix} x^2 & 4x \\ 2x & 4 \end{vmatrix}$$

$$= 2(-4x^2) - 2(-4x^2) = 0.$$

That is,

$$W(x^2, 4x, x - x^2)(x) = 0.$$

Recall that this set of functions was linearly dependent. This property can be connected to the value of the Wronskian. This is the subject of the following theorems.

Theorem (a test for linear independence)

Theorem

Let $f_1,\ f_2, \ldots, f_n$ be $n-1$ times continuously differentiable on an interval I. If there exists x_0 in I such that $W(f_1, f_2, \ldots, f_n)(x_0) \neq 0$, then the functions are **linearly independent** on I.

Remark 1: We can compute the Wronskian W as a test[22]:

$$W = 0 \Longrightarrow \text{dependent} \quad \text{or} \quad W \neq 0 \Longrightarrow \text{independent}$$

Remark 2: If the functions $y_1, y_2, \ldots, y_n$ all solve the same linear, homogeneous ODE on some interval I, then their Wronskian is either everywhere zero or nowhere zero on I.

---

[22] I'm over simplifying here. There are outlier examples of linearly independent functions with a zero Wronskian, but we will not be considering such functions here.

Example

Determine whether the set of functions $\{y_1, y_2\}$ is linearly dependent or linearly independent on the indicated interval.

$$y_1 = e^x, \quad y_2 = e^{-2x} \quad I = (-\infty, \infty)$$

Let's compute the Wronskian. We have

$$W(y_1, y_2)(x) = \begin{vmatrix} e^x & e^{-2x} \\ e^x & -2e^{-2x} \end{vmatrix}$$

$$= -2e^{-2x}e^x - e^x e^{-2x} = -3e^{-x}.$$

Since $W(y_1, y_2)(x) \neq 0$ (in fact it is never zero) the functions are **linearly independent**.

Fundamental Solution Set

We continue to consider the n^{th} order, linear, homogeneous ODE

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0$$

We're ready to get at what the solution to a homogeneous linear ODE will be. First, a definition.

Definition: Fundamental Solution Set

A set of functions $y_1, y_2, \dots, y_n$ is a **fundamental solution set** of the n^{th} order homogeneous equation provided they

(i) are solutions of the equation,

(ii) there are n of them, and

(iii) they are linearly independent.

Fundamental Solution Set

Theorem

If $a_1, a_2, \ldots, a_n$ are continuous on an interval I and $a_n(x) \neq 0$ for every x in I, then the homogeneous equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

possess a fundamental solution set.

So under the conditions on the coefficients that we've stated, a fundamental solution exists. The next definition tells us what the general solution to the ODE is.

General Solution of n^{th} order Linear Homogeneous Equation

We continue to consider the n^{th} order, linear, homogeneous ODE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

Definition: General Solution of Homogeneous, Linear ODE

Let y_1, $y_2, \ldots, y_n$ be a fundamental solution set of the n^{th} order linear homogeneous equation. Then the **general solution** of the equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where c_1, $c_2, \ldots, c_n$ are arbitrary constants.

Remark: This indicates that the task of solving an n^{th} order linear **homogeneous** ODE is to find a fundamental solution set, i.e., n, linearly independent solutions. We build the general solution by creating a linear combination.

Example

Verify that $y_1 = e^x$ and $y_2 = e^{-x}$ form a fundamental solution set of the ODE

$$y'' - y = 0 \quad \text{on} \quad (-\infty, \infty),$$

and determine the general solution.

We have to show that (i) they are solutions, (ii) that the number matches the order of the ODE, and (iii) they are linearly independent. First,

$$(i) \quad y_1'' - y_1 = e^x - e^x = 0 \quad \text{and} \quad y_2'' - y_2 = e^{-x} - e^{-x} = 0.$$

So they are solutions. We can count that there are two of them. And finally (iii)

$$W(y_1, y_2)(x) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2 \neq 0.$$

Hence the functions are linearly independent. We have a fundamental solution set. Hence

$$\boxed{\text{the general solution is} \quad y = c_1 e^x + c_2 e^{-x}.}$$

Nonhomogeneous Equations

Now we turn our attention to nonhomogeneous equations. We will consider the equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad (14)$$

where g is not the zero function. We'll continue to assume that a_n doesn't vanish and that a_i and g are continuous.

The **associated homogeneous equation** of (14) is

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

This equation has the same left hand side as (14). It's simply the homogeneous version of (14).

Theorem: General Solution of Nonhomogeneous Equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Definition: General Solution of Nonhomogeneous, Linear ODE

Let y_p be any solution of the nonhomogeneous equation (14), and let $y_1, y_2, \dots, y_n$ be any fundamental solution set of the associated homogeneous equation.

Then the general solution of the (14) is

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x) + y_p(x)$$

where $c_1, c_2, \dots, c_n$ are arbitrary constants.

Notice that this has the form $y = y_c + y_p$ where $y_c = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$.

Superposition Principle (for nonhomogeneous eqns.)

Consider the nonhomogeneous equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g_1(x) + g_2(x) \quad (15)$$

Theorem: Superposition Principle Nonhomogeneous ODE

Theorem: If y_{p_1} is a particular solution for

$$a_n(x)\frac{d^ny}{dx^n} + \cdots + a_0(x)y = g_1(x),$$

and y_{p_2} is a particular solution for

$$a_n(x)\frac{d^ny}{dx^n} + \cdots + a_0(x)y = g_2(x),$$

then

$$y_p = y_{p_1} + y_{p_2}$$

is a particular solution for the nonhomogeneous equation (15).

Superposition Principle (for nonhomogeneous eqns.)

Consider the nonhomogeneous equation

$$a_n(x)\frac{d^ny}{dx^n} + \cdots + a_0(x)y = g_1(x) + \dots g_k(x) \tag{16}$$

Second Principle of Superposition

The principle applies to more than two functions in the sum on the right side of (16). More generally, if y_{p_i} is any particular solution of the equation

$$a_n(x)\frac{d^ny}{dx^n} + \cdots + a_0(x)y = g_i(x)$$

for each $i = 1, \dots, k$, then

$$y_p = y_{p_1} + y_{p_2} + \cdots y_{p_k}$$

is a particular solution to (16).

This property will allow us to break problems up into parts as needed and construct the solution in pieces.

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(a) Verify that

$$y_{p_1} = 6 \quad \text{solves} \quad x^2y'' - 4xy' + 6y = 36.$$

Exercise left to the reader.

(b) Verify that

$$y_{p_2} = -7x \quad \text{solves} \quad x^2y'' - 4xy' + 6y = -14x.$$

Exercise left to the reader.

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(c) Verify that $y_1 = x^2$ and $y_2 = x^3$ constitute a fundamental solution set of

$$x^2y'' - 4xy' + 6y = 0.$$

Exercise left to the reader.

(d) Use (c) along with results (a) and (b) to write the general solution of $x^2y'' - 4xy' + 6y = 36 - 14x$.

The second principle of superposition says that $y_p = y_{p_1} + y_{p_2}$. By the first principle of superposition, we have $y_c = c_1y_1 + c_2y_2$. Using the definition of the general solution to a nonhomogeneous equation

the general solution is $\quad y = c_1x^2 + c_2x^3 + 6 - 7x.$

Section 7: Reduction of Order

In this section, we will restrict attention to second order, linear, homogeneous equations. Recall that such an equation has the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

Let us assume that $a_2(x) \neq 0$ on the interval of interest. We will write our equation in **standard form**

$$\boxed{\textbf{standard form}\ \frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0}$$

where $P = a_1/a_2$ and $Q = a_0/a_2$. We will assume that P and Q are continuous on the domain of definition.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Recall that every fundmantal solution set will consist of two linearly independent solutions y_1 and y_2, and the general solution will have the form

$$y = c_1 y_1(x) + c_2 y_2(x).$$

Suppose we happen to know one solution $y_1(x)$. **Reduction of order** is a method for finding a second linearly independent solution $y_2(x)$ that starts with the assumption that

$$y_2(x) = u(x)y_1(x)$$

for some function $u(x)$. The method involves finding the function u.

Remark: Because we insist that $\{y_1, y_2\}$ is linearly independent, we know that $u(x)$ cannot be a constant function.

Reduction of Order

Consider the equation **in standard form** with one known solution. Determine a second linearly independent solution.

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0, \quad y_1(x)--\text{is known.}$$

We begin by assuming that $y_2 = u(x)y_1(x)$ for some yet to be determined function $u(x)$. Note then that

$$y_2' = u'y_1 + uy_1', \quad \text{and} \quad y_2'' = u''y_1 + 2u'y_1' + uy_1''.$$

Since y_2 must solve the homogeneous equation, we can substitute the above into the equation to obtain a condition on u.

$$y_2''+Py_2'+Qy_2 = u''y_1 + (2y_1'+Py_1)u' + (y_1''+Py_1'+Qy_1)u = 0.$$

Reduction of Order

Since y_1 is a solution of the homogeneous equation, the last expression in parentheses is zero. So we obtain an equation for u

$$u''y_1 + (2y_1' + Py_1)u' = 0.$$

While this appears as a second order equation, the absences of u makes this equation first order in u' (hence the name *reduction of order*). If we let $w = u'$, we can express the equation in standard form (assuming y_1 doesn't vanish)

$$w' + (2y_1'/y_1 + P)w = 0.$$

This first order linear equation has a solution

$$w = \frac{\exp\left(-\int P(x)\,dx\right)}{y_1^2}.$$

Reduction of Order

With w determined, we integrate once to obtain u. We find that

$$u = \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx.$$

Finally, the second solution $y_2 = y_1(x)u(x)$, that is

$$y_2(x) = y_1(x)\int \frac{\exp\left(-\int P(x)\,dx\right)}{(y_1(x))^2}\,dx.$$

Summary of Reduction of Order

Reduction of Order

For the second order, homogeneous equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

in standard form with one known solution y_1, a second linearly independent solution y_2 is given by

$$y_2(x) = y_1(x)u(x) \quad \text{where} \quad u(x) = \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx$$

Remark: It's easy to recognize when reduction of order should be used. It requires (1) a second order, linear, homogeneous ODE, and (2) **that one solution is somehow already known**. (This second condition is a give-away.)

Example

Find the general solution of the ODE given one known solution

$$x^2y'' - 3xy' + 4y = 0, \quad x > 0, \quad y_1 = x^2$$

First note that we have a 2^{nd} order, linear, homogeneous ODE, and we know one solution. In standard form, the equation is

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0 \quad \text{so that} \quad P(x) = -\frac{3}{x}.$$

Hence

$$-\int P(x)\,dx = -\int \left(-\frac{3}{x}\right)\,dx = 3\ln(x) = \ln x^3.$$

A second solution is therefore

$$y_2 = x^2 \int \frac{\exp(\ln x^3)}{(x^2)^2}\,dx = x^2 \int \frac{x^3}{x^4}\,dx = x^2 \int \frac{dx}{x} = x^2 \ln x.$$

Note that we can take the constant of integration here to be zero (why?).

The general solution of the ODE is $\quad y = c_1x^2 + c_2x^2 \ln x.$

Section 8: Homogeneous Equations with Constant Coefficients

In this section, we will consider homogeneous, linear ODEs with the restriction that the coefficients are all constants. That is, the equations we'll consider now are of the form

$$a_n\frac{d^ny}{dx^n}+a_{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1\frac{dy}{dx}+a_0y=0.$$

The coefficients, $a_0, \dots, a_n$ are constant function (i.e., numbers), and the right hand side is zero.

First, we'll restrict our attention to the second order case. Once we investigate all the possible results for second order equation, we'll see how these ideas extend to general n^{th} order equations.

Second Order Homogeneous Equations with Constant Coefficients

Let's consider a second order, linear, homogeneous ODE with constant coefficients. An equation of the form

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

where we assume $a\neq 0$ (why?). To proceed, let's ask a question:

Question

What sorts of functions to we expect (using our knowledge of calculus) to have the property

$$y''=\text{constant}\,y'+\text{constant}\,y?$$

There are a few reasonable answers. The most prominent is probably *exponentials*. But you might also think of sines and cosines or perhaps polynomials.

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

We will start by assuming[23] that the solution(s) are exponentials and determine for what value of the constant r does the function $y=e^{rx}$ satisfy this ODE.

So we'll start with $y=e^{rx}$. Since this is supposed to solve our ODE, we'll substitute it in. We have[24]

$$y'=re^{rx} \quad \text{and} \quad y''=r^2e^{rx}$$

Plugging in

$$\begin{aligned} 0 &= ay''+by'+cy \\ &= ar^2e^{rx}+bre^{rx}+ce^{rx} \\ &= (ar^2+br+c)e^{rx} \end{aligned}$$

[23]An assumption about the form a solution takes in mathematics is called an *ansatz*. This is a game we often play in the study of differential equations.

[24]Knowing that r is a constant is enough information to allow us to take derivatives.

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

We arrive at the equation

$$(ar^2 + br + c)e^{rx} = 0$$

which requires that one of the two factors is zero. Noting that the exponential is never zero, the truth of the above equation requires r to satisfy

$$ar^2 + br + c = 0.$$

This quadratic equation is called the ***characteristic*** or ***auxiliary*** equation. The polynomial on the left side is the characteristic (or auxiliary) polynomial.

We need to consider all the possible solution types of a quadratic equation, and interpret these in terms of our exponential functions $y = e^{rx}$.

Characteristic (a.k.a. Auxiliary) Equation

Characteristic Equation

The characteristic equation for the second order, linear, homogeneous ODE $ay'' + by' + cy = 0$ is the quadratic equation

$$ar^2 + br + c = 0$$

Based on the properties of quadratic equations with real coefficients, solutions to the characteristic equation come in three types. The type is determined by the discriminant $b^2 - 4ac$. If

I $b^2 - 4ac > 0$ then there are two distinct real roots $r_1 \neq r_2$

II $b^2 - 4ac = 0$ then there is one repeated real root $r_1 = r_2 = r$

III $b^2 - 4ac < 0$ then there are two roots that are complex conjugates $r_{1,2} = \alpha \pm i\beta$ where α and β are real numbers and $\beta > 0$.

Case I: Two distinct real roots

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0 \qquad (17)$$

Recall: Every fundamental solution set of a second order, linear, homogeneous ODE must contain two, linearly independent solutions.

Two Distinct Real Roots

Suppose $b^2-4ac>0$. Then the general solution to (17) is given by

$$y=c_1e^{r_1x}+c_2e^{r_2x} \quad \text{where} \quad r_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Exercise for the Reader: Show that $y_1=e^{r_1x}$ and $y_2=e^{r_2x}$ are linearly independent. (Hint: use the Wronskian.)

Case II: One repeated real root

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad (18)$$

Recall: Every fundamental solution set of a second order, linear, homogeneous ODE must contain two, linearly independent solutions.

Two Distinct Real Roots

Suppose $b^2 - 4ac = 0$. Then the general solution to (18) is given by

$$y = c_1 e^{rx} + c_2 x e^{rx} \quad \text{where} \quad r = \frac{-b}{2a}$$

Exercise for the Reader: Use reduction of order to show that if $y_1 = e^{\frac{-bx}{2a}}$, then $y_2 = xe^{\frac{-bx}{2a}}$.

Case III: Complex conjugate roots

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad (19)$$

Recall: Every fundamental solution set of a second order, linear, homogeneous ODE must contain two, linearly independent solutions.

Complex Conjugate Roots

Suppose $b^2 - 4ac < 0$. Then the general solution of (19) is given by

$$y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x), \quad \text{where the roots}^a$$

$$r = \alpha \pm i\beta, \quad \alpha = \frac{-b}{2a} \quad \text{and} \quad \beta = \frac{\sqrt{4ac - b^2}}{2a}$$

[a] Here, $i = \sqrt{-1}$.

It's not necessarily obvious how this solution comes about, but we can derive it.

Case III: Fundamental Solution Set Derivation

Because the coefficients of our characteristic polynomial are real numbers, complex roots must appear in conjugate pairs. So suppose our roots are

$$r_1 = \alpha + i\beta \quad \text{and} \quad r_2 = \alpha - i\beta, \quad \alpha, \beta \in \mathbb{R}, \quad \beta > 0.$$

We can start by writing the two solutions (I'll reserve lower case y_1 and y_2 for the final result) as

$$Y_1 = e^{(\alpha+i\beta)x} = e^{\alpha x} e^{i\beta x}, \quad \text{and} \quad Y_2 = e^{(\alpha-i\beta)x} = e^{\alpha x} e^{-i\beta x}.$$

Our variable is real, our coefficients are real, and we expect our solution to be real valued. We would like to express the functions in the fundamental solution set without explicit dependence on the complex unit i. We can use the **principle of superposition** to accomplish this! Recall that the principle of superposition tells us that if Y_1 and Y_2 are solutions, then every linear combination of them is also a solution.

Deriving the solutions Case III

We will use Euler's formula that states $e^{i\theta} = \cos\theta + i\sin\theta$.

We can express Y_1 and Y_2 as

$$Y_1 = e^{\alpha x} e^{i\beta x} = e^{\alpha x}(\cos(\beta x) + i\sin(\beta x))$$

$$Y_2 = e^{\alpha x} e^{-i\beta x} = e^{\alpha x}(\cos(\beta x) - i\sin(\beta x))$$

Now apply the principle of superposition and set

$$y_1 = \frac{1}{2}(Y_1 + Y_2) = e^{\alpha x}\cos(\beta x), \quad \text{and}$$

$$y_2 = \frac{1}{2i}(Y_1 - Y_2) = e^{\alpha x}\sin(\beta x).$$

Hence, a fundamental solution set is (confirmation of linear independence left as an exercise for the reader)

$$\boxed{y_1 = e^{\alpha x}\cos(\beta x) \quad \text{and} \quad y_2 = e^{\alpha x}\sin(\beta x)}$$

Examples

Solve the ODE

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 6y = 0.$$

The characteristic equation is

$$r^2 + 4r + 6 = 0 \quad \text{with roots} \quad -2 \pm \sqrt{2}i.$$

This is the complex conjugate case with $\alpha = -2$ and $\beta = \sqrt{2}$. The general solution is therefore

$$y = c_1 e^{-2x}\cos(\sqrt{2}x) + c_2 e^{-2x}\sin(\sqrt{2}x).$$

Examples

Solve the ODE

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} - 5x = 0.$$

The characteristic equation is

$$r^2 + 4r - 5 = 0 \quad \text{with roots} \quad -5, 1.$$

This is the two distinct real roots case. Hence $x_1 = e^{-5t}$, $x_2 = e^t$, and the general solution is therefore

$$x = c_1 e^{-5t} + c_2 e^t.$$

Examples

Solve the IVP

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0, \quad y(0) = 1, \quad y'(0) = -2$$

First, we solve the ODE. The characteristic equation is

$$r^2 + 4r + 4 = 0 \quad \text{with root} \quad -2 \text{ (repeated)}$$

This is the one repeated real root case. Hence $y_1 = e^{-2x}$, $y_2 = xe^{-2x}$, and the general solution is therefore

$$y = c_1 e^{-2x} + c_2 x e^{-2x}.$$

Examples

Solve the IVP

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0, \quad y(0) = 1, \quad y'(0) = 3$$

Now that we have the general solution, we can determine the parameters c_1 and c_2 that will satisfy the initial conditions.

$$y = c_1 e^{-2x} + c_2 x e^{-2x} \quad \Longrightarrow \quad y' = -2c_1 e^{-2x} + c_2 e^{-2x} - 2c_2 x e^{-2x}.$$

So

$$y(0) = c_1 \quad \text{and} \quad y'(0) = -2c_1 + c_2.$$

Applying the given IC, we see that $c_1 = 1$ and $c_2 = 3 + 2c_1 = 5$. Hence the solution to the IVP is

$$y = e^{-2x} + 5xe^{-2x}.$$

Example

Let's go back to day one and consider the example used to motivate the definition of a differential equation. Find the general solution of the ODE

$$y'' + 4y = 0.$$

The characteristic equation is

$$r^2 + 4 = 0 \quad \text{with roots} \quad \pm 2i.$$

This is the complex conjugate case with $\alpha = 0$ and $\beta = 2$. A fundamental solution set is give by the pair

$$y_1 = e^{0x}\cos(2x) = \cos(2x) \quad \text{and} \quad y_2 = e^{0x}\sin(2x) = \sin(2x).$$

We see that the general solution is

$$y = c_1\cos(2x) + c_2\sin(2x).$$

The real part of complex roots can be zero giving rise to sines and cosines. The imaginary part of complex roots won't be zero (why?).

Higer Order Linear Constant Coefficient ODEs

The three cases given here are all the possible solution types[25] to second order, linear, constant coefficient equations. These findings can be extended to higher order equations with constant coefficients.

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = 0.$$

- The same approach applies. For an n^{th} order equation, we obtain an n^{th} degree polynomial.
- Complex roots must appear in conjugate pairs (due to real coefficients) giving a pair of solutions $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$ for each pair of complex roots.
- It may require a computer algebra system to find the roots for a high degree polynomial.

[25]Note that this includes the case $r_1 = r_2 = 0$ in which case, $y_1 = 1$ and $y_2 = x$.

Higer Order Linear Constant Coefficient ODEs: Repeated roots.

- For an n^{th} degree polynomial, r may be a root of multiplicity k where $1 \leq k \leq n$.
- If a real root r is repeated k times, we get k linearly independent solutions
$$e^{rx}, \quad xe^{rx}, \quad x^2e^{rx}, \quad \ldots, \quad x^{k-1}e^{rx}$$
or in conjugate pairs cases $2k$ solutions
$$e^{\alpha x}\cos(\beta x),\ e^{\alpha x}\sin(\beta x), \quad xe^{\alpha x}\cos(\beta x),\ xe^{\alpha x}\sin(\beta x), \ldots,$$
$$x^{k-1}e^{\alpha x}\cos(\beta x),\ x^{k-1}e^{\alpha x}\sin(\beta x)$$

Example

Find the general solution of $y''' - 4y' = 0$.

The equation is third order, so every fundamental solution set must contain three linearly independent solutions. The characteristic equation is

$$r^3 - 4r = 0 \quad \text{with roots} \quad -2, 0, 2.$$

A fundamental solution set is $y_1 = e^{-2x}$, $y_2 = e^{0x} = 1$, and $y_3 = e^{2x}$. The general solution is therefore

$$y = c_1 e^{-2x} + c_2 + c_3 e^{2x}.$$

Note that as expected, this third order equation has a fundamental solution set consisting of three linearly independent functions.

Example

Find the general solution of $y''' - 3y'' + 3y' - y = 0$.

The characteristic equation is

$$r^3 - 3r^2 + 3r - 1 = 0 \quad \text{with one triple root} \quad 1 \text{ (repeated)}.$$

Every fundamental solution set must contain three linearly independent functions. The method for repeated roots in the second order case extends nicely to higher order equations. A fundamental solution set is

$$y_1 = e^x, \quad y_2 = xe^x, \quad y_3 = x^2 e^x.$$

The general solution is

$$y = c_1 e^x + c_2 xe^x + c_3 x^2 e^x.$$

Example

Consider the 9^{th} order homogeneous ODE

$$y^{(9)}+5y^{(8)}+2y^{(7)}+2y^{(6)}-75y^{(5)}-239y^{(4)}-440y^{(3)}-728y''-688y'-240y=0$$

which has characteristic equation (conveniently factored)

$$(r^2+4)^2(r-3)(r+5)(r+1)^3=0$$

Identify a fundamental solution set.

So this would be very challenging indeed if we did not have the characteristic polynomial in such a nice factored form. We do know that our fundamental solution set has to consist of nine functions. Fortunately, we can see the roots of the characteristic equation along with their multiplicities and apply the principles outlined here.

Example continued...

$$(r^2+4)^2(r-3)(r+5)(r+1)^3 = 0$$

First, we consider $(r^2+4)^2 = 0$. The roots are $r = \pm 2i$. But since the factor is squared, each of $2i$ and $-2i$ is a double root. So we get four solutions

$$y_1 = \cos(2x), \quad y_2 = \sin(2x), \quad y_3 = x\cos(2x), \quad \text{and} \quad y_4 = x\sin(2x).$$

The next two factors are not repeated. We have $r-3=0$ giving root $r=3$ and $r+5=0$ giving root $r=-5$. These two factors give us two more solutions, one each,

$$y_5 = e^{3x}, \quad \text{and} \quad y_6 = e^{-5x}.$$

Example continued...

$$(r^2+4)^2(r-3)(r+5)(r+1)^3=0$$

Then the last factor will give us one root that will be a triple root. $(r+1)^3=0$ gives $r=-1$, repeated three times. We get three solutions from this root.

$$y_7=e^{-x},\quad y_8=xe^{-x},\quad \text{and}\quad y_9=x^2e^{-x}.$$

This completes the set. Our fundamental solution set is populated by

$$\boxed{\begin{array}{lllllllll} y_1 &=& \cos(2x), & y_2 &=& \sin(2x), & y_3 &=& x\cos(2x)\\ y_4 &=& x\sin(2x), & y_5 &=& e^{3x}, & y_6 &=& e^{-5x}\\ y_7 &=& e^{-x}, & y_8 &=& xe^{-x}, & y_9 &=& x^2e^{-x} \end{array}}$$

The general solution will be a linear combination of these,

$$y=c_1y_1+c_2y_2+\cdots+c_9y_9.$$

Section 9: Method of Undetermined Coefficients

We are now ready to consider some types of **nonhomogeneous** equations. We should recall from our basic theory of linear ODEs that a nonhomogenous equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = g(x), \quad g \neq 0$$

will have a solution of the form

$$y = \underbrace{c_1 y_1 + \cdots + c_n y_n}_{y_c} + y_p$$

where $y_1, \ldots, y_n$ form a fundamental solution set to the associated homogeneous equation, and y_p is a particular solution (upon substitution into the ODE, we get g).

For the current method, we will only consider constant coefficient equations (i.e., a_0, a_1, etc. will be constants) and we'll restrict our attention to certain types of right hand sides, $g(x)$.

The Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions[26]

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Eventually, we will have to consider both the complementary and the particular solution. For the moment, I'm going to focus on y_p. Once the basic notion of the solution method has been covered, we'll bring y_c back into the discussion.

[26] It's worth noting that these sorts of functions are the types that could be solutions to linear, homogeneous ODEs with constant coefficients. Though now we're considering these types on the right hand side (as forcing functions).

Motivating Example

Rather than trying to define what the method of undetermined coefficients is, let's look at an example.

Example: Find a particular solution of the ODE
$y'' - 4y' + 4y = 8x + 1$.

First note that our left side is constant coefficient and the right side, $g(x) = 8x + 1$, is a first degree polynomial. So this problem has the properties mentioned in the previous slide. We're only focusing on a particular solution, so we want a function y_p that when subbed into the left would result in the polynomial expression "$8x + 1$". To proceed, let's make an educated guess[27] that since g is a first degree polynomial, perhaps y_p is as well. If y_p is a first degree polynomial, then we can write

$$y_p = Ax + B$$

for some pair of constants A, B.

---

[27] Nothing is lost by trying this. If it works out, great. If not, we try something else.

$$y'' - 4y' + 4y = 8x + 1$$

Since y_p is supposed to be a solution, we can substitute it into the DE. We don't know what, if any, values A and B will work, but since they are assumed to be constant, we can take the necessary derivatives. We have

$$y_p = Ax + B, \quad y_p' = A, \quad \text{and} \quad y_p'' = 0.$$

Plugging y_p into the ODE we get

$$\begin{aligned} 8x + 1 &= y_p'' - 4y_p' + 4y_p \\ &= 0 - 4(A) + 4(Ax + B) \\ &= 4Ax + (-4A + 4B) \end{aligned}$$

$$y'' - 4y' + 4y = 8x + 1$$

We have first degree polynomials on both sides of the equation. They are equal if and only if they have the same corresponding coefficients. Matching the coefficients of x and the constants on the left and right we get the pair of equations

$$\begin{array}{rcrcl} 4A & & & = & 8 \\ -4A & + & 4B & = & 1 \end{array}$$

This system of equations has solution $A = 2$, $B = 9/4$. Hence we've found a particular solution

$$\boxed{y_p = 2x + \frac{9}{4}.}$$

It is worth taking a moment to substitute this into our ODE to confirm that it is, in fact, a solution. We do note that it is not the *general* solution as we have not addressed the associated homogeneous equation.

The Method of Undetermined Coefficients

As the previous example suggests, this method entails guessing that the particular solution has the same basic *form* as the right hand side function $g(x)$. We must keep in mind that the idea of *form* should be considered in the most general context. The following *forms* lend themselves to this method:

- n^{th} degree polynomial,
- exponentials e^{rx} for r constant,
- a linear combination of $\sin(kx)$ AND $\cos(kx)$ for some constant k,
- a product of any two or three of the above

When we assume a form for y_p we leave unspecified *coefficients* (hence the name of the method). The values of factors such as r or k inside an exponential, sine, or cosine are fixed by reference to the function g.

Example

Find the general solution of the differential equation
$y'' - 4y' + 4y = 6e^{-3x}$.

First we'll note that the left side is constant coefficient and the right side is an exponential. So this equation satisfies the restrictions for the method of undetermined coefficients. Since we're asked for the general solution, we need both y_c and y_p. Note that the associated homogeneous equation is

$$y'' - 4y' + 4y = 0$$

which we solve using the method of the previous section. We find that $y_c = c_1 e^{2x} + c_2 x e^{2x}$ (repeated root case). Next, we note that the right side $g(x) = 6e^{-3x}$ is an expontial. Let's suppose that y_p has the same form,

$$y_p = Ae^{-3x}.$$

(Since we're going to sum multiples of y_p and its derivatives, we keep the -3 in the exponential.)

$$y'' - 4y' + 4y = 6e^{-3x}$$

To sub, we'll need the first two derivatives,

$$y_p = Ae^{-3x}, \quad y_p' = -3Ae^{-3x}, \quad \text{and} \quad y_p'' = 9Ae^{-3x}.$$

Plugging these in

$$\begin{aligned} 6e^{-3x} &= 9Ae^{-3x} - 4(-3Ae^{-3x}) + 4Ae^{-3x} \\ &= 25Ae^{-3x} \end{aligned}$$

We have like terms on the left and right, and this last equation will be satisfied if $A = \frac{6}{25}$. So we have found a particuar solution

$$y_p = \frac{6}{25}e^{-3x}.$$

Finally,

The general solution to the ODE is $\quad y = c_1 e^{2x} + c_2 x e^{2x} + \frac{6}{25}e^{-3x}.$

Another Example

Before we summarize the basics of the method, let's consider one more example that highlights what might be meant by the *form*[28] of the function g.

Find a particular solution to the ODE $y'' - 4y' + 4y = 16x^2$.

We note that the left and right sides match the restrictions of this section, so using the method is reasonable. We're only looking for a particular solution[29], so we won't bother with y_c now. Let's note that the function $g(x) = 16x^2$ can be thought of in different ways. On the one hand, it's a monomial (constant times x^2). Alternatively, it's a quadratic polynomial. It turns out that one of these classifications is useful, and the other one is not. Let's suppose that we consider g a monomial, and we set

$$y_p = Ax^2.$$

[28] Spoiler alert: I'm going to take an incorrect approach to this problem for a couple of reasons. One, it will show us that if we guess incorrectly, that fact will become obvious. And two, making an error can reveal how to correct it.

[29] We already solved the associated homogeneous equation in the last example anyways.

$$y'' - 4y' + 4y = 16x^2$$

So we're assuming that $y_p = Ax^2$ and hope to find A. We substitute as before.

$$y_p = Ax^2, \quad y_p' = 2Ax, \quad \text{and} \quad y_p'' = 2A.$$

We get

$$\begin{aligned} 16x^2 &= 2A - 4(2Ax) + 4(Ax^2) \\ &= 4Ax^2 - 8Ax + 2A \end{aligned}$$

We know that these two quadratics are equivalent if they have the same coefficients. Let's write $16x^2 = 16x^2 + 0x + 0$, so that we can match like terms.

$$16x^2 + 0x + 0 = 4Ax^2 - 8Ax + 2A.$$

Matching x^2, x and constant terms, we arrive at the system of equations

$$4A = 16, \quad -8A = 0, \quad \text{and} \quad 2A = 0.$$

And now we're stuck! The first equation requires $A = 4$, but the last two require $A = 0$. The process fell apart!

$$y'' - 4y' + 4y = 16x^2$$

Clearly, our assumption that $y_p = Ax^2$ didn't pan out. However, we can see where things went wrong. We're dealing with a differential equation, and when we take derivatives of a term such as x^2, we give rise to other terms such as x and a constant. Fortunately, we can salvage the approach by thinking about $g(x) = 16x^2$ in a more general way. If we view g as a second degree polynomial, then assuming that y_p is also a second degree polynomial, we can start with the form $y_p = Ax^2 + Bx + C$. Let's try again. Taking the derivatives,

$$y_p = Ax^2 + Bx + C, \quad y_p' = 2Ax + B, \quad \text{and} \quad y_p'' = 2A.$$

Substituting

$$\begin{aligned} 16x^2 &= 2A - 4(2Ax + B) + 4(Ax^2 + Bx + C) \\ &= 4Ax^2 + (-8A + 4B)x + (2A - 4B + 4C) \end{aligned}$$

$$y'' - 4y' + 4y = 16x^2$$

Now we can try to match like terms, and we've got three parameters to work with.

$$16x^2 + 0x + 0 = 4Ax^2 + (-8A + 4B)x + (2A - 4B + 4C).$$

Comparing like terms, we require

$$4A = 16, \quad -8A + 4B = 0, \quad \text{and} \quad 2A - 4B + 4C = 0.$$

This system is solvable, and we find

$$A = 4, \quad B = 8, \quad \text{and} \quad C = 2.$$

The correct way to classify g is as a quadratic polynomial. And even though g doesn't have an x or constant term, a general quadratic might. We have found a particular solution

$$\boxed{y_p = 4x^2 + 8x + 2.}$$

Remark: We can generalize here and say that if g is an n^{th} degree polynomial, the correct form of the particular solution will be an n^{th} degree polynomial and we account for every possible power of x up to and including n (this includes the constant, a.k.a. zero power).

General Form: sines and cosines

With polynomials, we see that derivatives give rise to different powers. A similar relationship exists between sines and cosines (because they have each other in their derivatives). Consider the task of finding a particular solution of the ODE

$$y'' - y' = 20\sin(2x)$$

The ode has the right properties for our method with constant coefficient left side and right side $g(x) = 20\sin(2x)$. If we think of g as a constant multiple of $\sin(2x)$, it is tempting to suppose that $y_p = A\sin(2x)$. Upon substitution into the ODE, we get

$$-4A\sin(2x) - 2A\cos(2x) = 20\sin(2x).$$

If we match like terms (sines to sines and cosines to cosines), we end up with the system

$$-4A = 20 \quad \text{and} \quad -2A = 0.$$

But this is impossible as it requires $A = -5$ AND $A = 0$.

General Form: sines and cosines

When we took derivatives, we ended up with cosines, but we didn't account for that in our initial guess. We should think of g as a *linear combination* of $\sin(2x)$ and $\cos(2x)$. That is, we can think of $y''-y'=20\sin(2x)$ as

$$y''-y'=20\sin(2x)+0\cos(2x).$$

The correct format for y_p is

$$y_p=A\sin(2x)+B\cos(2x).$$

With a little effort, it can be shown that $A=-4$ and $B=2$ giving $y_p=-4\sin(2x)+2\cos(2x)$.

Remark: We can generalize here too. When we encounter a term such as $\sin(kx)$ or $\cos(kx)$, the correct form of the particular solution will include both the sine and the cosine terms. Again, we won't change k because this value doesn't change when taking derivatives.

Method Basics: $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$

We are ready to outline some initial method basics.

- Confirm that the left side is constant coefficient and the right side is one of the allowed function types.
- Classify g as a certain *type*, and assume y_p is of this same type[30] with unspecified coefficients, A, B, C, etc.
- Substitute the assumed y_p into the ODE and collect like terms.
- Match like terms on the left and right to get a linear system of equations for the coefficients.
- Solve the resulting system to determine the coefficients for y_p.

[30] We will see shortly that our final conclusion on the format of y_p can depend on y_c.

Some Rules & Caveats

Rules of Thumb

- Polynomials include all powers from constant up to the degree.
- Where sines go, cosines follow and vice versa.
- Constants inside of sines, cosines, and exponentials (e.g., the "2" in e^{2x} or the "π" in $\sin(\pi x)$) are not undetermined. We don't change those.

Caution

- The method is self correcting, meaning if the initial *guess* is wrong, it will become apparant. But it's best to get the set up correct to avoid unnecessary work.
- The form of y_p can depend on y_c, but this hasn't been considered yet. (We'll come back to this shortly.)

Initial Guesses

Let's go through several examples to determine what the form of y_p should be based on the right hand side $g(x)$.

At the moment, we're ignoring y_c. While **we can't do that in general**, we can always use these examples as the starting point for our set up.

On the following slides, we'll look at eleven examples. For each one, assume that we are dealing with an ODE that looks like

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x).$$

Examples of Forms of y_p based on g. If...

(a) $g(x) = 1$ (or really any nonzero constant) This is a degree zero polynomial.

$y_p = A$

(b) $g(x) = x - 7$ This is a degree 1 polynomial.

$y_p = Ax + B$

(c) $g(x) = 5x$ This is a degree 1 polynomial.

$y_p = Ax + B$

(d) $g(x) = 3x^3 - 5$ This is a degree 3 polynomial.

$y_p = Ax^3 + Bx^2 + Cx + D$

(e) $g(x) = xe^{3x}$ A degree1 polynomial times an exponential e^{3x}.

$y_p = (Ax + B)e^{3x}$

(f) $g(x) = \cos(7x)$ A linear combination of $\cos(7x)$ and $\sin(7x)$.

$y_p = A\cos(7x) + B\sin(7x)$

(g) $g(x) = \sin(2x) - \cos(4x)$ A linear combination of $\cos(2x)$ and $\sin(2x)$ plus a linear combination of $\cos(4x)$ and $\sin(4x)$.

$y_p = A\cos(2x) + B\sin(2x) + C\cos(4x) + D\sin(4x)$

(h) $g(x) = x^2\sin(3x)$ A product of a second degree polynomial and a linear combination of sines and cosines of $3x$.

$y_p = (Ax^2 + Bx + C)\cos(3x) + (Dx^2 + Ex + F)\sin(3x)$

Still More Trial Guesses

(i) $g(x) = e^x \cos(2x)$ A product of an exponential e^x and a linear combination of sines and cosines of $2x$.

$$y_p = Ae^x \cos(2x) + Be^x \sin(2x)$$

(j) $g(x) = x^3 e^{8x}$ A degree3 polynomial times an exponential e^{8x}.

$$y_p = (Ax^3 + Bx^2 + Cx + D)e^{8x}$$

(k) $g(x) = xe^{-x} \sin(\pi x)$ A product of a degree 1 polynomial, an exponential e^{-x} and a linear combination of sines and cosines of πx.

$$y_p = (Ax + B)e^{-x} \cos(\pi x) + (Cx + D)e^{-x} \sin(\pi x)$$

Note on the Correct Set Up

The correct form of y_p should include every like term that appears in the forcing function g along with every like term that can arise via differentiation. For example, in the last example on the previous slide, $g(x) = xe^{-x}\sin(\pi x)$. If we list out some derivative of g, no matter how many derivatives we take, exactly four types of like terms will arise:

$$xe^{-x}\sin(\pi x), \quad e^{-x}\sin(\pi x), \quad xe^{-x}\cos(\pi x), \quad \text{and} \quad e^{-x}\cos(\pi x).$$

Our initial guess y_p should include one of each of these like terms.

What are Like Terms?

Two (nonzero) expressions are considered **like terms** if their ratio is constant. This means that like terms have identical dependence on any variables present. For example, $4x\cos(3x)$ and $17x\cos(3x)$ are like terms because $\dfrac{4x\cos(3x)}{17x\cos(3x)} = \dfrac{4}{17}$ is constant, but $4x\cos(3x)$ and $17\cos(3x)$ are not like terms because $\dfrac{4x\cos(3x)}{17\cos(3x)} = \dfrac{4x}{17}$ depends on x.

The Superposition Principle

Suppose our right hand side is the sum of two allowed types of functions. In this case, we use the principle of superposition for nonhomogeneous equations. For example, suppose we wish to find a particular solution to

$$y'' - y' = 20\sin(2x) + 4e^{-5x}.$$

Given the theorem in section 6 regarding superposition for nonhomogeneous equations, we can consider two subproblems

$$y'' - y' = 20\sin(2x), \quad \text{and} \quad y'' - y' = 4e^{-5x}.$$

Calling the particular solutions y_{p_1} and y_{p_2}, respectively, the correct forms to guess would be

$$y_{p_1} = A\cos(2x) + B\sin(2x), \quad \text{and} \quad y_{p_2} = Ce^{-5x}$$

It can be shown (details left to the reader) that $A = 2$, $B = -4$ and $C = 2/15$.

The Superposition Principle

$$y'' - y' = 20\sin(2x) + 4e^{-5x}$$

The superposition principle says that the particular solution to the whole problem is the sum of those for the partial problems. Hence, the particular solution is

$$y = y_{p_1} + y_{p_2} = 2\cos(2x) - 4\sin(2x) + \frac{2e^{-5x}}{15}.$$

The complementary solution can be found using the characteristic equation. If can be shown (details left to the reader) that $y_c = c_1 e^x + c_2$. The general solution to the ODE is

$$y = c_1 e^x + c_2 + 2\cos(2x) - 4\sin(2x) + \frac{2e^{-5x}}{15}.$$

The Superposition Principle Summarized

Consider the nonhomogeneous equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x) + \ldots + g_k(x).$$

The Superposition Principle

The principle of superposition for nonhomogeneous equations tells us that we can find y_p by considering separate problems

$$y_{p_1} \text{ solves } \quad a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x)$$

$$y_{p_2} \text{ solves } \quad a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_2(x),$$

and so forth.

Then $y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_k}$.

A Glitch!

Thus far, we haven't paid enough attention to the complementary solution. We can run into some issue if $g(x)$ has a like term in common with the complementary solution. Unfortunately, it's not always obvious without some effort. Let's see it in an example, then we will see how it can be dealt with. Consider the following constant coefficient ODE with exponential right hand side.

$$y'' - y' = 3e^x$$

Here we note that $g(x) = 3e^x$ is a constant times e^x. So we may guess that our particular solution

$$y_p = Ae^x.$$

When we attempt the substitution, we end up with an unsolvable problem. Note that $y_p' = Ae^x$ and $y_p'' = Ae^x$ giving upon substitution

$$\begin{aligned} 3e^x &= y_p'' - y_p' \\ &= Ae^x - Ae^x \\ &= 0 \end{aligned}$$

This requires $3 = 0$ which is always false (i.e. we can't find a value of A to make a true statement out of this result.)

$$y'' - y' = 3e^x$$

The reason for our failure here comes to light by consideration of the associated homogenous equation

$$y'' - y' = 0$$

with fundamental solution set $y_1 = e^x$, $y_2 = 1$. Our initial guess of Ae^x is a solution to the associated homogeneous equation for every constant A. And for any nonzero A, we've only duplicated part of the complementary solution. Fortunately, there is a fix for this problem. Taking a hint from a previous observation involving reduction of order, we may modify our initial guess by including a factor of x. If we guess

$$y_p = Axe^x$$

we find that this actuall works. It can be shown that (details left to the reader) $A = 3$. So $y_p = 3xe^x$ is a particular solution.

The Possible Glitch Cases

As this example illustrates, if our assumed form for y_p has one or more like terms in common with y_c, the complementary solution of the ODE under consideration, then these terms will vanish when we do the substitution. This causes us to lose coefficients, and we won't be able to match all of the like terms. This means that we will have two cases to consider when we use the Method of Undetermine Coefficients; (1) when we don't have like terms in common with y_c, and (2) when we do.

For what follows, let's assume we have an ODE of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_1(x) + \ldots + g_k(x)$$

and we are focusing on one of the g's, say g_i. So we wish to consider the subproblem

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_i(x).$$

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_i(x)$$

The first thing we do is solve the associated homogeneous equation,

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = 0,$$

for the complementary solution y_c.

Case 1:

We write out our guess for y_{p_i} using the general rules of thumb and principles already discussed (see all the examples we went through). We compare our guess for y_{p_i} to y_c and **there are no like terms in common**.

We have the correct form for y_{p_i} so we start the substitution process and complete finding our particular solution.

Remark: All the examples so far, up to the slide that says "A Glitch!," were Case 1 examples.

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g_i(x)$$

Case 2:

We write out our guess for y_{p_i} using the general rules of thumb and principles already discussed. We compare our guess for y_{p_i} to y_c and **there is one or more like terms in common between y_{p_i} and y_c.**

We have to adjust our form of y_{p_i}. We do this by multiplying the whole function y_{p_i} by a factor of x^n, where n is the smallest positive integer such that our new y_{p_i} has no like terms in common with y_c.

Once we have the correct format for y_{p_i}, we start the substitution process and complete finding our particular solution.

Remark: In practice, we can multiply by x. If the new y_{p_i} still has a like term in common with y_c, multiply by x again. Continue to multiply by x until there are no common like terms left. That is, we don't have to know what n is up front.

Case 2 Example

Find the general solution of the nonhomogeneous equation
$y'' - 2y' + y = -4e^x$.

First, we find y_c. The associated homogeneous equation $y'' - 2y' + y = 0$ has characteristic equation $r^2 - 2r + 1 = 0$ with repeated root $r = 1$. So $y_1 = e^x$ and $y_2 = xe^x$ form a fundamental solution set. This gives us

$$y_c = c_1 e^x + c_2 x e^x.$$

Next, we note that $g(x) = -4e^x$ which is a constant multiple of e^x. Based on our general principles, we start by setting

$$y_p = Ae^x.$$

Now we compare this to y_c, and we see that a contant e^x is part of y_c, so this is not the correct form.

$$y'' - 2y' + y = -4e^x$$

We can update our guess by multiplying by a factor x. This gives us the new assumed form

$$y_p = Axe^x.$$

But again, we compare to y_c and see that a constant times xe^x is a common term. Continuing, we multiply by x again, and try

$$y_p = Ax^2e^x.$$

We again compare this to y_c, but this time it does not duplicate the complementary solution and will work as the correct form (i.e. it is possible to find a value of A such that this function solves the nonhomogeneous ODE). Now we're ready to complete the process. We have

$$y_p = Ax^2e^x, \quad y_p' = Ax^2e^x + 2Axe^x, \quad \text{and} \quad y_p'' = Ax^2e^x + 4Axe^x + 2Ae^x.$$

$$y'' - 2y' + y = -4e^x$$

Subbing it all into the ODE, we get

$$Ax^2e^x + 4Axe^x + 2Ae^x - 2(Ax^2e^x + 2Axe^x) + Ax^2e^x = -4e^x.$$

Collecting like terms (i.e., x^2e^x, xe^x, and e^x) together gives

$$(A - 2A + A)x^2e^x + (4A - 4A)xe^x + 2Ae^x = -4e^x.$$

Notice that most of the terms on the left side cancel leaving the equation

$$2Ae^x = -4e^x \quad \Longrightarrow \quad 2A = -4 \quad \Longrightarrow \quad A = -2.$$

We have determined that $y_p = -2x^2e^x$, and the general solution to the ODE is

$$y = c_1e^x + c_2xe^x - 2x^2e^x.$$

Find the form of the particular soluition

Let's do an example in which we find the form of the particular solution (but we don't complete the whole solution process.)

Find the form of the particular solution of the following ODE using the method of undetermined coefficients.

$$y'' - 4y' + 4y = \sin(4x) + xe^{2x}$$

First we find a fundamental solution set to the associated homogeneous equation.

$$y_1 = e^{2x}, \quad y_2 = xe^{2x}, \quad \text{so} \quad y_c = c_1 e^{2x} + c_2 xe^{2x}.$$

To find y_p, we use the principle of superposition and consider two sub problems

$$y'' - 4y' + 4y = \sin(4x) \quad \text{and} \quad y'' - 4y' + 4y = xe^{2x}.$$

For $g_1(x) = \sin(4x)$, we start with the assumed particular solution

$$y_{p_1} = A\sin(4x) + B\cos(4x).$$

Comparing this form y_{p_1} to y_c, we see that there are no common like terms. Hence y_{p_1} will suffice as written.

$$y'' - 4y' + 4y = \sin(4x) + xe^{2x}$$

Now we work with the second equation with $g_2(x) = xe^{2x}$. Using our basic principles, we start with the assumption that

$$y_{p_2} = (Cx + D)e^{2x}.$$

We compare this to y_c, but we see that both of the terms are like terms with the complementary solution. So this will not work as written. We may attempt including an extra factor of x

$$y_{p_2} = x(Cx + D)e^{2x} = (Cx^2 + Dx)e^{2x}$$

to fix the problem. However, this still contains a term (Dxe^{2x}) that duplicates part of the fundamental solution set. Hence we introduce another factor of x putting

$$y_{p_2} = x^2(Cx + D)e^{2x} = (Cx^3 + Dx^2)e^{2x}.$$

Finally, none of the terms in y_{p_2} are like terms in common with y_c. This is our correct form for y_{p_2}.

$$y'' - 4y' + 4y = \sin(4x) + xe^{2x}$$

Now we apply our superposition principle, $y_p = y_{p_1} + y_{p_2}$. The particular solution for the whole ODE is of the form

$$\boxed{y_p = A\sin(4x) + B\cos(4x) + (Cx^3 + Dx^2)e^{2x}.}$$

Remark 1: It is important to note that this problem involved a Case 1 part and a Case 2 part. It is critical that we consider these set ups separately. If we multiply our guess for y_{p_1} by x^2 **it would be wrong**.

Remark 2: If we want to find A, B, C, and D, we can do the sub-problems separately. That is, we can find A and B by solving $y'' - 4y' + 4y = \sin(4x)$, then find C and D by solving $y'' - 4y' + 4y = xe^{2x}$ by treating these as two separate problems.

Another Example

Find the form of the particular solution of the nonhomogeneous ODE. (Do not bother finding any of the coefficients, A, B, etc.)

$$y'' + 4y = 5\cos(2x) - 3xe^{-2x}$$

First, we find the complementary solution because we know that it can affect the form of the particular solution. The characteristic equation is $r^2 + 4 = 0$ with complex roots $r = \pm 2i$. This gives us

$$y_c = c_1 \cos(2x) + c_2 \sin(2x).$$

Next, we use the principle of superposition to divide our task into two subproblems:

$$\text{find } y_{p_1} \text{ solving} \quad y'' + 4y = 5\cos(2x), \quad \text{and}$$

$$\text{find } y_{p_2} \text{ solving} \quad y'' + 4y = -3xe^{-2x}.$$

$$y'' + 4y = 5\cos(2x) - 3xe^{-2x}$$

$$\boxed{y_c = c_1\cos(2x) + c_2\sin(2x)}$$

We have $g_1(x) = 5\cos(2x)$ which we think of as a linear combination of $\cos(2x)$ and $\sin(2x)$. So our first attempt would be to set

$$y_{p_1} = A\cos(2x) + B\sin(2x).$$

However, upon comparing this to y_c, we see that it shares like terms in common (they're all common in this example). So we modify our guess by including an extra factor of x. Now,

$$y_{p_1} = x(A\cos(2x) + B\sin(2x)) = Ax\cos(2x) + Bx\sin(2x).$$

This shares no like terms in common with y_c, so it is the correct form.

$$y'' + 4y = 5\cos(2x) - 3xe^{-2x}$$

$$\boxed{y_c = c_1\cos(2x) + c_2\sin(2x)}$$

Next we move on to the second subproblem. $g_2(x) = -3xe^{-2x}$ which we classify as a first degree polynomial times e^{-2x}. Using our process, we start by setting up (since A and B are already in use, we'll pick up with C)

$$y_{p_2} = (Cx + D)e^{-2x}.$$

We compare this to our complementary solution, and we see that it doesn't have any like terms in common. Hence it is the correct form. Finally, the correct form of the particular solution is given by using our principle of superposition. Our final answer is

$$\boxed{y_p = Ax\cos(2x) + Bx\sin(2x) + (Cx + D)e^{-2x}.}$$

Let's finish this section by solving an initial value problem.

Final Example

Solve the initial value problem

$$y'' - y = 4e^{-x} \quad y(0) = -1, \quad y'(0) = 1$$

First, let's note that the left side of the ODE is constant coefficient and the right side is one of the types of functions for which the method of undetermined coefficients is applicable. The associated homogeneous equation $y'' - y = 0$ has fundamental solution set $y_1 = e^x$, $y_2 = e^{-x}$. Based on the right side, $g(x) = 4e^{-x}$, we may guess that our particular solution

$$y_p = Ae^{-x},$$

but seeing that this duplicates y_2 we will need to modify our guess as

$$y_p = Axe^{-x}.$$

This doesn't have any like terms in common with y_c, so it is the correct form. We substitute this into the ODE to find A.

$$y'' - y = 4e^{-x} \quad y(0) = -1, \quad y'(0) = 1$$

Substitution into the ODE gives[31] $A = -2$. So our particular solution is $y_p = -2xe^{-x}$ and the general solution of the ODE is

$$y = c_1 e^x + c_2 e^{-x} - 2xe^{-x}.$$

Finally, we apply our initial conditions to the general solution. Note that

$$y = c_1 e^x + c_2 e^{-x} - 2xe^{-x} \quad \Longrightarrow \quad y' = c_1 e^x - c_2 e^{-x} - 2e^{-x} + 2xe^{-x}.$$

So

$$y(0) = c_1 + c_2 = -1 \quad \text{and} \quad y'(0) = c_1 - c_2 - 2 = 1.$$

Solving this system of equations for c_1 and c_2 we find $c_1 = 1$ and $c_2 = -2$. The solution to the IVP is

$$y = e^x - 2e^{-x} - 2xe^{-x}.$$

---

[31] Don't read anything in to the fact that we found $A = -2$ in two different examples. It's just a coincidence.

Section 10: Variation of Parameters

In this section, we continue to consider linear, nonhomogeneous equations. While the method of this section can be extended to ODEs of any order, we'll restrict our attention to 2^{nd} order ODEs of the form

$$a_2(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x). \tag{20}$$

In the previous section, we considered such problems, but we put two critical restrictions on the ODE. We insisted that the coefficients on the left (a_0, a_1, a_2) were constant, and we insisted that g was from certain classes of functions (exponential, polynomial, sine/cosine). In this section, we will relax these conditions.

Our theory still tells us that the solution to (20) will have the from $y = y_c + y_p$ where y_c solves the associated homogeneous equation and y_p is a particular solution.

Section 10: Variation of Parameters

To illustrate, suppose we wish to consider either of the nonhomogeneous equations

$$y'' + y = \tan x \quad \text{or} \quad x^2y'' + xy' - 4y = e^x?$$

Neither of these equations lend themselves to the method of undetermined coefficients for identification of a particular solution.

- The first one fails because $g(x) = \tan x$ does not fall into any of the classes of functions required for the method.
- The second one fails because the left hand side is not a constant coefficient equation.

Dealing with these equations will require some new solution process.

Variation of Parameters

Consider the ODE in standard form, $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = g(x)$, and assume that P, Q, and g are continuous on some interval. Let $\{y_1\ y_2\}$ be a **known** fundamental solution set for the associated homogeneous equation. We have y_c, but to find the general solution, we require y_p.

As we've done before, we will assume that y_p has a specific format and use the ODE to determine the details. We start with the ansatz[32]

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where u_1 and u_2 are functions[33] we will determine (in terms of y_1, y_2 and g).

This method is called **variation of parameters**.

[32] Recall that *ansatz* means assumed form.

[33] Note the similarity to $y_c = c_1y_1 + c_2y_2$. The coefficients u_1 and u_2 are varying, hence the name *variation of parameters*.

Variation of Parameters: Derivation of y_p

Here, we will derive a pair of formulas for the varying parameters u_1 and u_2. We begin with our equation in standard form[34]

$$y'' + P(x)y' + Q(x)y = g(x), \tag{21}$$

and we assume that a fundamental solution set $\{y_1, y_2\}$ of the associated homogeneous equation is known.

Now, we set $y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$, and seek to determine functions u_1 and u_2 so that this is a particular solution of (21).

As we've done before, we will substitute our ansatz into (21) in an effort to obtain conditions on the u's. Before we do so, we need to note that we have two unknowns, u_1 and u_2, but we only have one equation, the ODE (21). We can say that our system of equations is *under-determined*. This actually gives us a little bit of freedom. We can introduce[35] a second equation for the u's. We'll do this in a way that is both simple and that makes the resulting computations a little bit easier.

[34] Recall that *standard form* means that the coefficient of the highest derivative is 1.

[35] Obviously, this means that there could be other ways to derive our solution. We'll make the traditional choice of second condition, but you can experiment with alternatives.

$$y'' + P(x)y' + Q(x)y = g(x)$$

Let's start the substitution. Suppressing the x dependence notation, and assuming that u_1 and u_2 are sufficiently differentiable, we take the first derivative[36] of y_p. We have

$$y_p = u_1 y_1 + u_2 y_2 \quad \Longrightarrow \quad y_p' = u_1 y_1' + u_2 y_2' + u_1' y_1 + u_2' y_2.$$

Now, we will make an assumption. This is going to give us the second equation alluded to earlier. Assume

$$\boxed{u_1' y_1 + u_2' y_2 = 0.} \qquad (22)$$

This means the last two terms of y_p vanish, so now we have

$$y_p = u_1 y_1 + u_2 y_2 \quad \text{and} \quad y_p' = u_1 y_1' + u_2 y_2'.$$

[36] You might notice that I've written the terms in the derivative in an unconventional order, but you can confirm that y_p' is correct.

$$y'' + P(x)y' + Q(x)y = g(x)$$

$$y_p = u_1 y_1 + u_2 y_2 \quad \text{and} \quad y_p' = u_1 y_1' + u_2 y_2'.$$

Continuing, we take the second derivative to get

$$y_p'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''.$$

Next, we substitute y_p, y_p' and y_p'' into our ODE and collect the terms u_1', u_2', u_1 and u_2. We have

$$\begin{aligned} g(x) &= y_p'' + Py_p' + Qy_p \\ &= u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2'' + P(u_1 y_1' + u_2 y_2') + Q(u_1 y_1 + u_2 y_2) \\ &= u_1' y_1' + u_2' y_2' + (y_1'' + Py_1' + Qy_1)u_1 + (y_2'' + Py_2' + Qy_2)u_2 \end{aligned}$$

At this point, it looks like we've arrived at a very complicated equation for our u's. However, this simplifies greatly[37] when we remember that y_1 and y_2 solve the associated homogeneous equation.

[37] This process might remind you of Reduction of Order where we saw something very similar.

$$y'' + P(x)y' + Q(x)y = g(x)$$

Remembering that $\{y_1, y_2\}$ is a fundamental solution set for the associated homogeneous equation, we know that

$$y_i'' + P(x)y_i' + Q(x)y_i = 0, \quad \text{for } i = 1, 2$$

Hence the last two terms in our equation

$$u_1'y_1' + u_2'y_2' + \overbrace{(y_1'' + Py_1' + Qy_1)}^{0}u_1 + \overbrace{(y_2'' + Py_2' + Qy_2)}^{0}u_2 = g(x)$$

vanish. This leave us with the equation for u_1 and u_2

$$\boxed{u_1'y_1' + u_2'y_2' = g.} \tag{23}$$

$$y'' + P(x)y' + Q(x)y = g(x)$$

Now we have a system of equations, (22)–(23), for our u's. Actually, the system is for the derivatives, u_1' and u_2', but if we solve this system, we can simply integrate once to get our pair of functions. Our system of equations is

$$\begin{array}{|ccccc|} u_1'y_1 & + & u_2'y_2 & = & 0 \\ u_1'y_1' & + & u_2'y_2' & = & g \end{array}$$

It is convenient to express this system using a matrix formalism as

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}$$

Look carefully at the matrix. It should be familiar[38]. Since $\{y_1, y_2\}$ is a fundamental solution set, we are guaranteed that the determinant of that matrix is nonzero.

[38] It is the Wronskian matrix.

$$y'' + P(x)y' + Q(x)y = g(x)$$

The system $\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}$ can be solved using any appropriate technique. It is convenient to use Cramer's Rule. Applying Cramer's rule, we arrive at

$$u_1' = \frac{W_1}{W} = \frac{-y_2 g}{W} \quad u_2' = \frac{W_2}{W} = \frac{y_1 g}{W}$$

where

$$W_1 = \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix}$$

and W is the Wronskian of y_1 and y_2. We simply integrate to obtain u_1 and u_2.

Variation of Parameters

$$y'' + P(x)y' + Q(x)y = g(x)$$

If $\{y_1, y_2\}$ is a fundamental solution set for the associated homogeneous equation, then the general solution is

$$y = y_c + y_p \quad \text{where}$$

$$y_c = c_1 y_1(x) + c_2 y_2(x), \quad \text{and} \quad y_p = u_1(x)y_1(x) + u_2(x)y_2(x).$$

Letting W denote the Wronskian of y_1 and y_2, the functions u_1 and u_2 are given by the formulas

$$u_1 = \int \frac{-y_2 g}{W}\,dx, \quad \text{and} \quad u_2 = \int \frac{y_1 g}{W}\,dx.$$

Example

Find the general solution of the ODE $y'' + y = \tan x$.

The associated homogeneous equation is $y'' + y = 0$. It's characteristic equation is $r^2 + 1 = 0$ with complex roots $r = \pm i$. Hence a fundamental solution[39] set is given by $y_1 = \cos x$ and $y_2 = \sin x$. The complementary solution

$$y_c = c_1 \cos x + c_2 \sin x.$$

Turning our attention to y_p, we note that the method of undetermined coefficients is not an option because the tangent is not one of the function types for which the method applies. So we will use **variation of parameters.**

The equation is already in standard form, so we can identify g. We also compute the Wronskian of y_1 and y_2. We have

$$g(x) = \tan x, \quad \text{and} \quad W = 1.$$

[39] How we number the functions in the fundamental solution set is completely arbitrary. However, the designations are important for finding our u's and constructing our y_p. So we pick an ordering at the beginning and stick with it.

Example Continued...$y'' + y = \tan x$

$$y_1 = \cos x, \quad y_2 = \sin x, \quad W = 1, \quad \text{and} \quad g(x) = \tan x.$$

We have all of the necessary terms to apply the formulas for our u's. We have[40]

$$u_1 = \int \frac{-y_2 g}{W}\,dx = \int -\frac{\sin x \tan x}{1}\,dx = \sin x - \ln|\sec x + \tan x|$$

$$u_2 = \int \frac{y_1 g}{W}\,dx = \int \frac{\cos x \tan x}{1}\,dx = -\cos x.$$

That is,

$$u_1 = \sin x - \ln|\sec x + \tan x| \quad \text{and} \quad u_2 = -\cos x.$$

Because we will add the complementary solution to the particular solution, we can take these intermediate integration constants to be anything we like. I've taken them to be zero to keep things simple.

[40] The first integral requires some trig IDs, but I'm omitting the details.

Example Continued...$y'' + y = \tan x$

$$y_1 = \cos x, \quad y_2 = \sin x, \quad u_1 = \sin x - \ln|\sec x + \tan x| \quad \text{and} \quad u_2 = -\cos x.$$

We construct our particular solution

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 \\ &= (\sin x - \ln|\sec x + \tan x|)\cos x - \cos x \sin x \\ &= \cos x \sin x - \cos x \ln|\sec x + \tan x| - \cos x \sin x \\ &= -\cos x \ln|\sec x + \tan x|. \end{aligned}$$

Finally, the general solution to the ODE is

$$y = c_1 \cos x + c_2 \sin x - \cos x \ln|\sec x + \tan x|.$$

Remark:

It's worth noting here that, unlike what we saw with the method of undetermined coefficients, the particular solution $y_p = -\cos x \ln|\sec x + \tan x|$ is not related to $g(x) = \tan x$ in any obvious way. It doesn't seem realistic that we could simply *guess* that y_p should be a constant multiple of the product $\cos x \ln|\sec x + \tan x|$ based on the right side of our ODE. Fortunately, this method doesn't require such a guess.

Example

Solve the initial value problem

$$x^2y'' + xy' - 4y = 8x^2, \quad y(1) = 0, \quad y'(1) = 0.$$

The complementary solution of the ODE is $y_c = c_1x^2 + c_2x^{-2}$.

First, we look for the general solution to the ODE. We'll note that the ODE is not constant coefficient, so variation of parameters is the only viable method for finding a particular solution. The complementary solution is already given, and from it we see that a fundamental solution set is

$$y_1 = x^2 \quad \text{and} \quad y_2 = x^{-2}.$$

We proceed to find a particular solution. We need $g(x)$ and the Wronskian. The equation is not in standard form, so we rewrite it to find

$$y'' + \frac{1}{x}y' - \frac{4}{x^2}y = 8 \quad \Longrightarrow \quad g(x) = 8.$$

The Wronskian

$$W = x^2(-2x^{-3}) - 2x(x^{-2}) = -\frac{4}{x}.$$

$$x^2y'' + xy' - 4y = 8x^2, \quad y(1) = 0, \quad y'(1) = 0$$

$$y_1 = x^2, \quad y_2 = x^{-2}, \quad W = -\frac{4}{x}, \quad \text{and} \quad g(x) = 8$$

Next, we apply the formulas to find u_1 and u_2 for our particular solution $y_p = u_1 y_1 + u_2 y_2$. We have

$$u_1 = \int \frac{-y_2 g}{W}\,dx = \int -\frac{8x^{-2}}{-\frac{4}{x}}\,dx = \int \frac{2}{x}\,dx = 2\ln|x|.$$

$$u_2 = \int \frac{y_1 g}{W}\,dx = \int \frac{8x^2}{-\frac{4}{x}}\,dx = \int -2x^3\,dx = -\frac{x^4}{2}.$$

Since the initial conditions are given at $x = 1$, we can assume that $x > 0$ and forego the absolute value bars. We have particular solution

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 \\ &= 2x^2 \ln x - \frac{x^2}{2} \end{aligned}$$

$$x^2y'' + xy' - 4y = 8x^2, \quad y(1) = 0, \quad y'(1) = 0$$

$$y_c = c_1x^2 + c_2x^{-2}, \quad \text{and} \quad y_p = 2x^2\ln x - \frac{x^2}{2}$$

Before we apply the initial conditions, let's notice that the term $-\frac{x^2}{2}$ appearing in y_p can be combined with the term c_1x^2 in y_c. That is, we can simplify[41] our general soluion.

$$y = c_1x^2 + c_2x^{-2} + 2x^2\ln x - \frac{x^2}{2} \quad \Longrightarrow \quad y = k_1x^2 + k_2x^{-2} + 2x^2\ln x,$$

where $k_1 = c_1 - \frac{1}{2}$ and $k_2 = c_2$. Finally, we apply the initial conditions and complete the solution to the IVP.

[41] It is not required that we simplify our general solution, but it will make the process of applying the initial conditions easier to carry out.

$$x^2y'' + xy' - 4y = 8x^2, \quad y(1) = 0, \quad y'(1) = 0$$

$$y = k_1x^2 + k_2x^{-2} + 2x^2 \ln x$$

We'll need the derivative of our general solution.

$$y' = 2k_1x - 2k_2x^{-3} + 4x \ln x + 2x.$$

We have

$$y(1) = k_1(1)^2 + k_2(1)^{-2} + 2(1)^2 \ln 1 = 0 \quad \Longrightarrow \quad k_1 + k_2 = 0$$

$$y'(1) = 2k_1(1) - 2k_2(1)^{-3} + 4(1) \ln 1 + 2(1) = 0 \quad \Longrightarrow \quad 2k_1 - 2k_2 = -2$$

The solution of this system is $k_1 = -\frac{1}{2}$, $k_2 = \frac{1}{2}$. We conclude that

the solution to the IVP is $\quad y = -\dfrac{x^2}{2} + \dfrac{1}{2x^2} + 2x^2 \ln x.$

Remark

This example illustrates a general point. If the particular solution y_p contains a term (an addend) that is already part of the complementary solution, this term can be dropped and the resulting y_p is also a correct particular solution.

Section 11: Linear Mechanical Equations

In this section, we will consider a physical model that gives rise to a second order, linear ODE. We'll approach the topic of linear mechanical systems in three stages.

First, we'll consider what is known as simple harmonic motion. The classic example is the motion of a mass suspended from a spring in the absence of any additional forces (no damping, no friction, no outside anything). Once we have the model, we'll add in a damping force (e.g., friction, shock absorber, dashpot). Then finally, we'll allow for an external force.

Simple Harmonic Motion

We consider a flexible spring from which a mass is suspended. In the absence of any damping forces (e.g. friction, a dash pot, etc.), and free of any external driving forces, any initial displacement or velocity imparted will result in **free, undamped motion**–a.k.a. **simple harmonic motion**.

If you have an active internet connection, you can click on the button below to see simple harmonic motion in action. Note that the motion is uniform and unchanging.

Harmonic Motion gif

To proceed, we need to identify the independent and dependent variables, the physical principles, and some basic terms and concepts.

Building an Equation: Hooke's Law

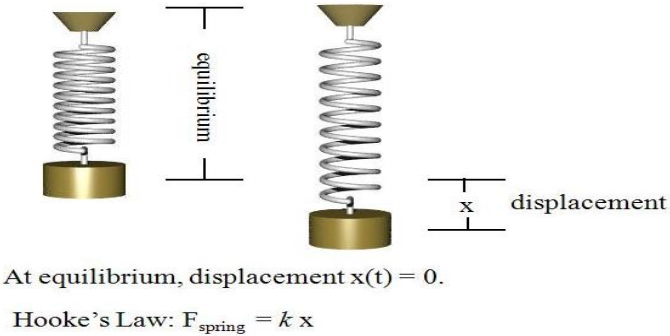


Figure: In the absence of movement, the spring with mass will have a natural length called an equilibrium length.

Let $x(t)$ be the displacement of the mass from the equilibrium position, $x = 0$. Our dependent variable will be x and the independent variable is time t. **We'll use the convention that $x > 0$ above equilibrium, and $x < 0$ below equilibrium.**

Building an Equation: Hooke's Law

To build an equation for the displacement x, we refer to two physical laws, Newton's second and Hooke's laws.

Newton's Second Law

states that the force on an object is the product of mass and acceleration.

$$F = ma$$

If m is the mass of the object, and $x(t)$ is the displacement at time t, then the acceleration $a = \frac{d^2x}{dt^2}$. Hence the force on the object

$$F = m\frac{d^2x}{dt^2}.$$

Building an Equation: Hooke's Law

Hooke's Law

states that the force required to displace the object attached to a flexible spring a distance x from equilibrium is proportional to x.

$$F_{spring} = kx,$$

where k is a contant that depends on the spring (and can be determined experimentally). The constant k is called the *spring constant*.

The equation of motion is constructed by equating the forces on the object.

Building an Equation: Hooke's Law

Observe that the force imparted by the spring is in the direction opposite of the motion. Equating forces

$$m\frac{d^2x}{dt^2} = -kx \quad \Longrightarrow \quad m\frac{d^2x}{dt^2} + kx = 0,$$

we obtain a second order, linear, homogeneous ODE for the displacement.

Equation for Simple Harmonic Motion

The displacement, $x(t)$, from the equilibrium position of an object of mass m subject to a spring with spring constant k satisfies the second order linear homogeneous differential equation

$$x'' + \omega^2 x = 0$$

where $\omega = \sqrt{\frac{k}{m}}$ is called the *circular frequency*.

Displacment in Equilibrium

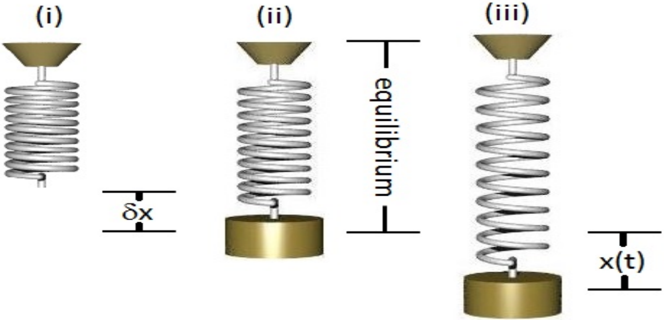


Figure: Spring only, versus spring-mass equilibrium, and spring-mass (nonzero) displacement

To determine the spring constant, we can measure the difference, δx, between the spring only natural length and the spring+mass equilibrium length. This length is called the *displacement in equilibrium*. Note that the spring+mass equilibrium length (image (ii)) is the $x = 0$ position.

Units and Constants

We tend to encounter problems in two systems of measurements, US customary and SI[42]. We will use agreed upon basic units of measure in each system.

If an object with weight W pounds stretches a spring δx feet from it's length with no mass attached, the by Hooke's law we compute the spring constant via the equation

$$W = k\delta x \quad \Longrightarrow \quad k = \frac{W}{\delta x}.$$

The units for k in this system of measure are lb/ft.

---

[42] S.I. stands for Système Internationale or international system.

Units and Constants

US Customary Units: Note also that Weight = mass $\times$ acceleration due to gravity. Hence if we know the weight of an object, we can obtain the mass via

$$W = mg \implies m = \frac{W}{g}.$$

We typically take the approximation[43] $g = 32$ ft/sec^2. The units for mass are lb sec^2/ft which are called slugs.

The basic units in the US Customary system are

- Force $\sim$ pounds,
- Length $\sim$ feet,
- Mass $\sim$ slugs, and
- Time $\sim$ seconds.

---

[43] Assuming the motion is taking place on the Earth near the equator, or unless otherwise indicated.

Units and Constants

SI Units: In SI units, the weight W would be expressed in Newtons (N). The appropriate units for displacement would be meters (m). As before,

$$W = k\delta x \quad \Longrightarrow \quad k = \frac{W}{\delta x}.$$

In these units, the spring constant would have units of N/m. Objects are usually described by their mass in kilograms. The weight can be determined via the equation

$$W = mg$$

where we can take the approximation[44] $g = 9.8$ m/sec^2.

The basic units in the SI system are

- Force $\sim$ Newtons,
- Length $\sim$ meters,
- Mass $\sim$ kilograms, and
- Time $\sim$ seconds.

---

[44] Assuming the motion is taking place on the Earth near the equator, or unless otherwise indicated.

Displacment in Equilibrium

If we have enough information to determine the mass m and the spring constant k for a given system, then we can determine the circular frequency via the equation

$$\omega^2 = \frac{k}{m}.$$

It may be possible to determine ω^2 if we know the displacement in equilibrium, even in the absence of enough information to deduce m and k.

If an object stretches a spring δx units from it's length (with no object attached), we may say that it stretches the spring δx units *in equilibrium*. Applying Hooke's law with the weight as force, we have

$$mg = k\delta x.$$

If we divide both sides of this equation by $m\delta x$, we arrive at a representation

$$\omega^2 = \frac{k}{m} = \frac{g}{\delta x}.$$

With g and δx given in appropriate units, ω will be in units of per second.

Simple Harmonic Motion

If we are given the initial displacement, $x(0) = x_0$, and intial velocity, $x'(0) = x_1$, of an object subject to simple harmonic motion, then the displacement satisfies the initial value problem

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1 \tag{24}$$

This second order, linear, homogeneous equation with constant coefficients is readily solved (using a characteristic equation) to obtain the displacement

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) \tag{25}$$

called the **equation of motion**.

Some Remarks

Language Caution

The phrase *equation of motion* is used differently by different authors. Some use this phrase to refer the IVP (24); others use it to refer to the **solution** (25). I'm using it in the latter sense.

Sign Convention

We'll use the arbitrary convention that $x > 0$ is above equilibrium (up) and $x < 0$ is below equilibrium (down).

Special Initial Conditions (*from equilibrium/from rest*)

The equilibrium position is the reference for the displacement. Hence saying something like *an object starts from/at equilibrium* means that $x(0) = 0$. When the initial velocity is zero, $x'(0) = 0$, we say *the object starts from rest*.

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1$$

We can identify several characteristics of the motion. These include

- the period $T = \frac{2\pi}{\omega}$,
- the frequency[45] $f = \frac{1}{T} = \frac{\omega}{2\pi}$
- the circular (or angular) frequency ω, and
- the amplitude or maximum displacement $A = \sqrt{x_0^2 + (x_1/\omega)^2}$

To see how the amplitude corresponds to its usual definition (as seen in studying trigonometry), it is helpful to look at how x can be expressed in terms of a single sine or cosine using the sum or difference of angles formulas. This requires us to introduce a **phase shift**.

[45] While f is accurately called a *linear* frequency, some authors call f the *natural* frequency. The term *natural frequency* is sometimes used to refer to ω.

Amplitude and Phase Shift

We can formulate the solution in terms of a single sine (or cosine) function. Letting

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) = A \sin(\omega t + \phi)$$

requires

$$A = \sqrt{x_0^2 + (x_1/\omega)^2},$$

and the **phase shift** ϕ must be defined by

$$\sin\phi = \frac{x_0}{A}, \quad \text{with} \quad \cos\phi = \frac{x_1}{\omega A}.$$

(Alternatively, we can let $x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) = A \cos(\omega t - \hat{\phi})$ in which case $\hat{\phi}$ is defined by

$$\cos\hat{\phi} = \frac{x_0}{A}, \quad \text{with} \quad \sin\hat{\phi} = \frac{x_1}{\omega A}.$$

The phase shift defined above $\phi = \frac{\pi}{2} - \hat{\phi}$, up to an added multiple of 2π.)

Example

An object stretches a spring 6 inches in equilibrium. Assuming no driving force and no damping, set up the differential equation describing this system.

Letting the displacement after t seconds be $x(t)$ feet, we have

$$mx'' + kx = 0 \quad \Longrightarrow \quad x'' + \omega^2 x = 0$$

where $\omega = \sqrt{\frac{k}{m}}$. We seek the value of ω, but we do not have the mass of the object to calculate the weight. Since the displacement is described as displacment in equilibrium, we can use the alternative (displacement in equilibrium) formula to calculate

$$\omega = \sqrt{\frac{g}{\delta x}}.$$

Example Continued...

We're given the displacement in equilibrium as $\delta x = 6$ inches. This must be converted to feet, since the basic unit of length is feet. Using the appropriate value for g, the one in US customary units, we have

$$\omega = \sqrt{\frac{g}{\delta x}} = \sqrt{\frac{32\,\text{ft/sec}^2}{\frac{1}{2}\,\text{ft}}} = 8\,\frac{1}{\text{sec}}.$$

The differential equation is therefore

$$x'' + 64x = 0.$$

Example

A 4 pound weight stretches a spring 6 inches. The mass is released from a position 4 feet above equilibrium with an initial downward velocity of 24 ft/sec. Find the equation of motion, the period, amplitude, phase shift, and frequency of the motion. (Take $g = 32$ ft/sec^2.)

Note that since the displacement in equilibrium is 6 inches, we know from the previous example that $\omega^2 = 64$. However, since we have enough information to find both m and k, let's find both of these values and confirm this. We can calculate the spring constant and the mass from the given information. Converting inches to feet, we have

$$W = k\delta x \quad \implies \quad 4\,\text{lb} = \left(\frac{1}{2}\,\text{ft}\right) k \quad \implies \quad k = 8\,\text{lb/ft} \quad \text{and}$$

$$W = mg \quad \implies \quad 4\,\text{lb} = m\left(32\,\text{ft/sec}^2\right) \quad \implies \quad m = \frac{4\,\text{lb}}{32\,\text{ft/sec}^2} = \frac{1}{8}\,\text{slugs}.$$

Example Continued...

The value of ω is therefore

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{8}{\frac{1}{8}}}\,\frac{1}{\text{sec}} = 8\,\frac{1}{\text{sec}}.$$

As expected, $\omega^2 = 64$. To determine the initial conditions, we use the convention that up is positive and down is negative. With this in mind, the object starting 4 feet above equilibrium gives $x(0) = 4$. Note that feet are the appropriate units, so the correct numerical value is “4”. The starting velocity of 24 ft/sec downward gives the condition $x'(0) = -24$. The units of ft/sec are the basic units making “24” the correct numerical value, and the sign is in keeping with our direction convention. Hence the displacement x satisfies the initial value problem

$$x'' + 64x = 0 \quad x(0) = 4, \quad x'(0) = -24.$$

Example Continued...

$$x'' + 64x = 0 \quad x(0) = 4, \quad x'(0) = -24.$$

This can be solved using the techniques from this class to obtain the equation of motion

$$x(t) = 4\cos(8t) - 3\sin(8t).$$

The period and frequency are

$$T = \frac{2\pi}{8} = \frac{\pi}{4}\,\text{sec} \quad \text{and} \quad f = \frac{1}{T} = \frac{4}{\pi}\,\frac{1}{\text{sec}}.$$

The amplitude

$$A = \sqrt{4^2 + (-3)^2} = 5\,\text{ft}.$$

Example Continued...

To determine the phase shift, we express x in terms of a sine function

$$x(t) = 5\sin(8t+\phi)$$

where the phase shift ϕ satisfies the equations

$$\sin\phi = \frac{4}{5} \quad \text{and} \quad \cos\phi = -\frac{3}{5}.$$

We note that $\sin\phi > 0$ and $\cos\phi < 0$ indicating that ϕ is a quadrant II angle (in standard position). Taking the smallest possible positive value[46], we have

$$\phi \approx 2.21 \quad \text{(roughly } 127^\circ\text{)}.$$

We are ready to add in the next level of complexity, damping.

[46] This value can be computed using a calculator as $\cos^{-1}\left(-\frac{3}{5}\right)$. Note that the inverse sign function cannot be used here without a correction to give a quadrant II angle. That is, to use the inverse sine, we would have to compute $\phi = \pi - \sin^{-1}\left(\frac{4}{5}\right)$.

Free Damped Motion

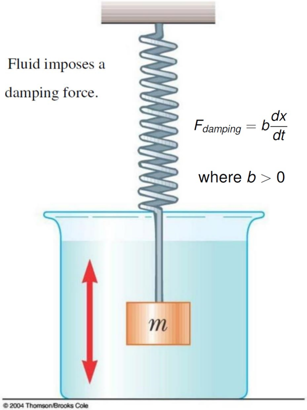


Figure: If a damping force is added, we'll assume that this force is proportional to the instantaneous velocity. This is a common assumption sometimes referred to as a *linear damping* model.

Free Damped Motion

Now we wish to consider an added force corresponding to damping—e.g., due to friction, a dashpot, air resistance. Now we have two forces acting on the object, that of the spring $F_{spring} = kx$ and that of damping $F_{damping} = b\frac{dx}{dt}$. We start by adding all forces.

$$\text{Total Force} = \text{Force of damping} + \text{Force of spring}$$

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx \quad \Longrightarrow \quad m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0.$$

We have a second order, linear, homogeneous equation. We introduce some convenient notation. Let

$$2\lambda = \frac{b}{m} \quad \text{and} \quad \omega = \sqrt{\frac{k}{m}}.$$

The factor of 2 is simply for convenience, and the parameter ω is the same one we encountered with simple harmonic motion.

Free Damped Motion

The displacement of an object undergoing damped unforced motion is governed by the equation

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0.$$

Unlike the undamped case, this equation gives rise to three qualitatively different solutions depending on the nature of the roots of the characteristic equation. Using the parameter r for the characteristic equation (so as not to confuse it with mass m), the characteristic equation is

$$r^2 + 2\lambda r + \omega^2 = 0 \quad \text{with roots} \quad r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}.$$

Remark

The three cases are the ones we are familiar with, two real roots, one real root, complex conjugate roots. We have names for these based on the quality of the damping. Let's look at each case.

Case 1: $\lambda^2 > \omega^2$ Overdamped

If the characteristic equation has two distinct real roots, the system is said to be **overdamped**. In this case, the solution

$$x(t) = c_1 e^{-\lambda t + t\sqrt{\lambda^2 - \omega^2}} + c_2 e^{-\lambda t - t\sqrt{\lambda^2 - \omega^2}}$$

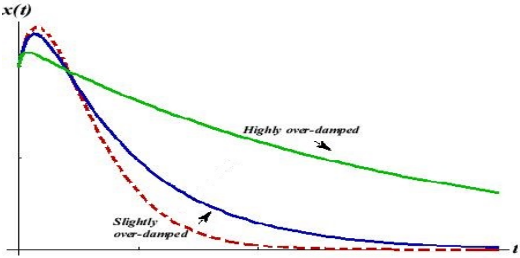


Figure: An overdamped system does not exhibit oscillations. The displacement tends to equilibrium. The return to equilibrium can be slow (highly overdamped) or somewhat fast (slightly overdamped).

The red dashed curve is the critical damping case. It is shown for reference.

Case 2: $\lambda^2 = \omega^2$ Critically Damped

If the characteristic equation has one repeated real root, the system is said to be **critically damped**. In this case, the solution

$$x(t) = c_1 e^{-\lambda t} + c_2 t e^{-\lambda t}$$

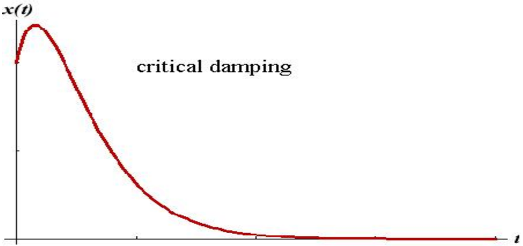


Figure: A critically damped system does not exhibit oscillations. The system returns to equilibrium. This is the fastest approach to equilibrium.

Case 3: $\lambda^2 < \omega^2$ Underdamped

If the characteristic equation has complex conjugate roots, the system is said to be **underdamped**. In this case, the solution

$$x(t) = c_1 e^{-\lambda t}\cos(\omega_1 t) + c_2 e^{-\lambda t}\sin(\omega_1 t), \quad \omega_1 = \sqrt{\omega^2 - \lambda^2}$$

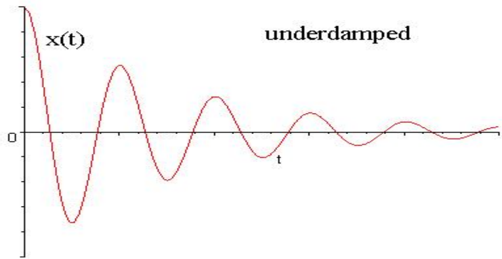


Figure: An underdamped system decays to equilibrium and exhibits oscillations about the equilibrium position.

Remark

There is no need to memorize the solutions in any of these cases. They are obtained in the usual way for constant coefficient, homogeneous differential equations.

Damping Ratio

Engineers may refer to the *damping ratio* when determining which of the three types of damping a system exhibits. Simply put, the damping ratio is the ratio of the system damping to the critical damping for the given mass and spring constant. Calling this damping ratio ζ,

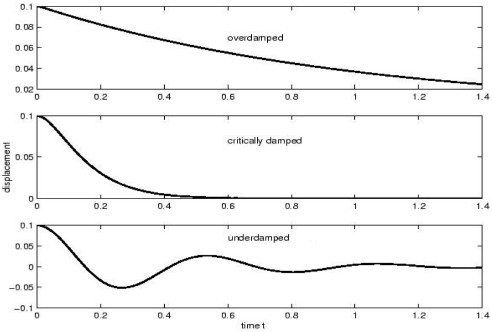
$$\zeta = \frac{\text{damping coefficient}}{\text{critical damping}} = \frac{b}{2\sqrt{mk}} = \frac{\lambda}{\omega}$$

Relative to this ratio, the damping cases are given by

$\zeta < 1$	under damped
$\zeta = 1$	critically damped
$\zeta > 1$	over damped

This criterion leads to identical definitions of the damping types discussed here. That is, if $\zeta < 1$, the characteristic equation has complex roots; if $\zeta = 1$ it has one real root, and if $\zeta > 1$ is has two real roots.

Comparison of Damping



Remark

We appear to have lots of formulas and conditions here. But we don't have to use complicated tests. We just need to make the associations between damping types and characteristic roots. Negative exponentials decay; sines and cosines oscillate. The familiar functions match the observed behavior!

Example

A 2 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 10 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped.

The basic model is $mx'' + bx' + kx = 0$. Here, $m = 2$ kg, $k = 12$ N/m, and[47] $b = 10$. So the ODE is

$$2x'' + 10x' + 12x = 0 \quad \implies \quad x'' + 5x' + 6x = 0.$$

We don't need any special formulas involving λ and ω to determine the damping. The characteristic equation is

$$r^2 + 5r + 6 = 0 \quad \implies \quad (r + 2)(r + 3) = 0$$

We have two distinct real roots, -2 and -3. Hence the system is **overdamped**.

---

[47] The parameter b is the proportionality constant for the damping, so saying damping is 10 *times the instantaneous velocity* means that $b = 10$. The units of b would be Newtons per meter per second.

Example

Although we have answered the question, it doesn't hurt to confirm that this corresponds to the case $\lambda^2 > \omega^2$. From the standard form of the ODE

$$x'' + 2\lambda x' + \omega^2 x = 0 \quad \Longrightarrow \quad x'' + 5x' + 6x = 0$$

we see that

$$\lambda = \frac{5}{2} \quad \text{and} \quad \omega^2 = 6.$$

Note that

$$\lambda^2 - \omega^2 = \frac{25}{4} - 6 = \frac{1}{4} > 0 \quad \Longrightarrow \quad \lambda^2 > \omega^2$$

and the system is overdamped (as we saw before).Similarly, we can confirm that

$$\zeta = \frac{\lambda}{\omega} = \frac{5}{2\sqrt{6}} = \sqrt{\frac{25}{24}} > 1.$$

Important Observation

Because the damping coefficient must be positive, the real part of the roots of the characteristic equation **must be negative**. This is true for all three types of damping.

Example

A 3 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 12 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped. If the mass is released from the equilibrium position with an upward velocity of 1 m/sec, solve the resulting initial value problem.

From the description, we have $m = 3$, $b = 12$ and $k = 12$. To say that the object starts at equilibrium is to say that $x(0) = 0$. An upward initial velocity of 1 m/sec means that $x'(0) = 1$. The displacement satisfies the IVP

$$3x'' + 12x' + 12x = 0, \quad x(0) = 0, \quad x'(0) = 1.$$

Example

We can rewrite the ODE in standard form and identify the characteristic equation.

$$x'' + 4x' + 4x = 0 \implies r^2 + 4r + 4 = 0 \implies (r+2)^2 = 0.$$

There is one, repeated real root $r = -2$. Hence the system is **critically damped**. This result also gives us the general solution to the ODE

$$x(t) = c_1 e^{-2t} + c_2 t e^{-2t}.$$

Finally, we apply the initial conditions to find that $c_1 = 0$ and $c_2 = 1$. The displacement

$$x(t) = te^{-2t}.$$

Remark 1:

Note that our real root $-2 < 0$. Conservation of energy implies that the real root(s) or real part of complex roots must be negative.

Remark 2:

Note that there's no new techniques or special process here. We're just applying the process from section 8 for constant coefficient equations. We found one real root, so the two solutions are $x_1 = e^{-2t}$ and $x_2 = te^{-2t}$.

Driven Motion

Our final model extension is to include the application of an external driving force, i.e., a force that is independent of position. Assume a time dependent force $f(t)$ is applied to the system. As before, we sum all of the forces acting on the object and arrive at the ODE

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx + f(t), \quad b \geq 0.$$

Note that allowing $b = 0$ or $b > 0$ means that we can consider a forced system with or without damping. Divide out m and let $F(t) = f(t)/m$ to obtain the nonhomogeneous equation

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)$$

Remark: We often refer to the right hand side of a nonhomogeneous equation like this as a **forcing function**. This model shows why that terminology makes sense.

Linear Mechanical System

The most general model that we have for displacement in a linear mechanical system (anything that can reasonably be modeled as a spring-mass-damper) is

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = f(t)$$

where m, b, and k represent the mass, damping coefficient, and spring constant, respectively; and f represents an external force.

If

- $f(t) = 0$, the system is **unforced**
- $b = 0$, the system is **undamped**.

If the system is unforced and undamped, the equation describes **simple harmonic motion**.

We can use the methods already covered to find the solution, the complementary part and particular solution in the forced case. Before we leave this section, let's consider an interesting case in which an undamped system is driven by a simple oscillator.

Forced Undamped Motion and Resonance

$$x'' + \omega^2 x = F(t) \tag{26}$$

Let's consider an undamped forced system (26) in which the forcing function is a simple oscillator. That is, we'll assume that

$$F(t) = F_0 \cos(\gamma t) \quad \text{or} \quad F(t) = F_0 \sin(\gamma t)$$

where F_0 and γ are some real numbers. The behavior of the solution comes in two distinct flavors, so we have to consider two cases

$$(1) \quad \gamma \neq \omega, \quad \text{and} \quad (2) \quad \gamma = \omega.$$

Here, we'll take the case of a sine function, but similar results can be found by taking F to be a cosine.

In what follows, we will examine the nonhomogeneous equation

$$x'' + \omega^2 x = F_0 \sin(\gamma t).$$

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

Note that no matter what the values of F_0, γ and ω are, the complementary solution is always

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

So suppose we are looking for x_p. Using the method of undetermined coefficients, the **first guess** to the particular solution is

$$x_p = A\cos(\gamma t) + B\sin(\gamma t).$$

Case 1: $\gamma \neq \omega$

If $\omega \neq \gamma$, then this form does not duplicate the solution to the associated homogeneous equation. Hence it is the correct form for the particular solution. The general solution will end up looking like

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) + A\cos(\gamma t) + B\sin(\gamma t).$$

This is a sum of oscillating functions and will have some maximum displacement.

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Suppose $\gamma = \omega$. Again, using the method of undetermined coefficients, the initial set up for x_p is still

$$x_p = A\cos(\gamma t) + B\sin(\gamma t) = A\cos(\omega t) + B\sin(\omega t) \quad \text{(since } \gamma = \omega\text{)}.$$

Case 2: $\gamma = \omega$

If $\omega = \gamma$, then this form DOES duplicate the terms in x_c. In this case, the correct form for x_p is

$$x_p = At\cos(\omega t) + Bt\sin(\omega t).$$

The general solution in this case will look like

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) + At\cos(\omega t) + Bt\sin(\omega t).$$

Note that the amplitude of the latter terms depends on t and is unbounded (i.e., tends to ∞ as $t \to \infty$)!

Forced Undamped Motion and Resonance

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

When the frequency, γ, of the forcing oscillator matches the natural frequency, ω, i.e., when

$$\gamma = \omega.$$

the system exhibits **Pure Resonance**. While pure resonance is an idealization[48], even excitation close to the natural frequency ($\gamma \approx \omega$) can cause extreme amplification of vibrations resulting in damage to mechanical structures.

Let's look at the solution to an IVP in both cases. We'll couple the ODE with the condition that the object starts from rest at equilibrium—i.e., $x(0) = 0$ and $x'(0) = 0$.

[48] Physical systems tend to include at least some damping, and real life frequencies are only known up to measurement tolerances.

Forced Undamped Motion and Resonance

For $F(t) = F_0 \sin(\gamma t)$ starting from rest at equilibrium:

$$\text{Case (1):} \quad x'' + \omega^2 x = F_0 \sin(\gamma t), \quad x(0) = 0, \quad x'(0) = 0$$

The solution to the IVP is

$$x(t) = \frac{F_0}{\omega^2 - \gamma^2} \left(\sin(\gamma t) - \frac{\gamma}{\omega} \sin(\omega t) \right).$$

Note that in the case of *near resonance*, $\gamma \approx \omega$, the denominator of the coefficients will be very small. This means that the amplitude of the motion will be very **large**! This could result in mechanically destructive vibrations.

Pure Resonance

$$\text{Case (2):} \quad x'' + \omega^2 x = F_0 \sin(\omega t), \quad x(0) = 0, \quad x'(0) = 0$$

The solution to this IVP is

$$x(t) = \frac{F_0}{2\omega^2}\sin(\omega t) - \frac{F_0}{2\omega} t\cos(\omega t).$$

Note that the amplitude of the second term is $\frac{F_0 t}{2\omega}$, which grows without bound!

If you have an active internet connection, the button below will take you to an applet simulating forced oscillations. Choose "elongation diagram" to see a plot of displacement of the mass and the exciter. My ω is ω_0 in the applet, and my γ is ω in the applet. You can adjust the exciter frequency close to ω_0 and see the effects of resonance.

▸ Forced Motion and Resonance Applet

Section 12: LRC Series Circuits

Now that we have solution techniques for second order, linear equations, we return our attention to linear circuits. We can track the charge q on the capacitor, or the current i in an LRC circuit.

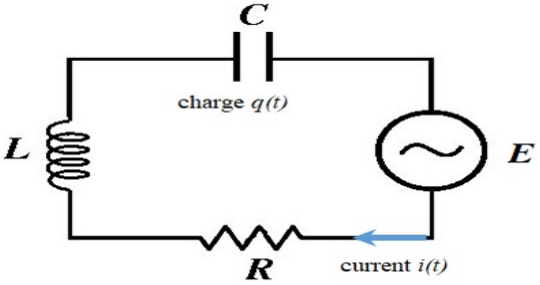


Figure: Simple circuit with inductance L, resistance R, capacitance C, and implied voltage E. The current $i(t) = \frac{dq}{dt}$ where q is the charge on the capacitor at time t in seconds.

Potential Drop Across Each Element

We will recall the voltage drop across each element in terms of charge or current.

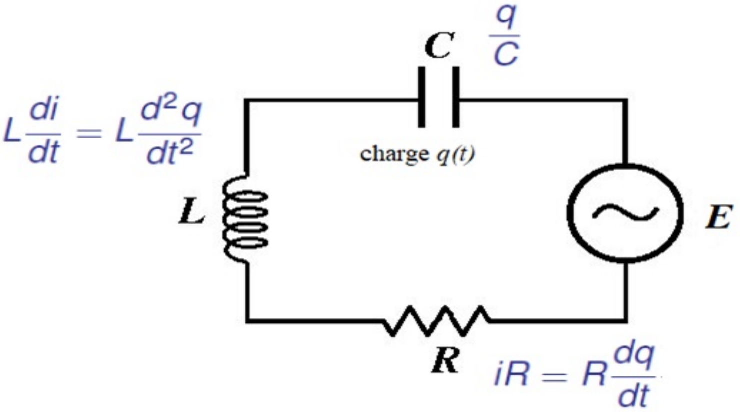


Figure: The potential drop across the capacitor is q/C, across the resistor is iR, and across the inductor is $L\frac{di}{dt}$.

Kirchhoff's Voltage Law

Kirchhoff's Law

The sum of the potential drops around a the loop of a closed circuit is zero, i.e., the sum of the potential drops across the passive elements in the circuit must equal the implied electromotive force.

LRC Differential Equation

Mathematically, the charge on the capacitor satisfies the second order, linear initial value problem

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=E(t),\quad q(0)=q_0,\quad q'(0)=i_0$$

where q_0 and i_0 are the initial charge and current, respectively.

Note that taking the derivative of the ODE produces a second order linear equation for the current i.

$$L\frac{d^2i}{dt^2}+R\frac{di}{dt}+\frac{1}{C}i=E'(t)$$

LRC Series Circuit (Free Electrical Vibrations)

If the implied voltage is zero, $E = 0$, the circuit exhibits *free electrical vibrations*. The charge can be characterized by a type of damping with language borrowed from the linear mechanical setting.

Free Electrical Vibrations

If we consider the equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0,$$

the free electrical vibrations are called

overdamped if $R^2 - 4L/C > 0$,
critically damped if $R^2 - 4L/C = 0$,
underdamped if $R^2 - 4L/C < 0$.

Note that this is the same condition we saw before. Overdamped = two real roots, critically damped = one real root, underdamped = complex roots.

Steady and Transient States

Given a nonzero applied voltage $E(t)$, we obtain an IVP with nonhomogeneous ODE for the charge q.

$$Lq'' + Rq' + \frac{1}{C}q = E(t), \quad q(0) = q_0, \quad q'(0) = i_0.$$

From our basic theory of linear equations we know that the solution will take the form

$$q(t) = q_c(t) + q_p(t).$$

Transient State Charge

The function of q_c is influenced by the initial state (q_0 and i_0) and will decay exponentially as $t \to \infty$. Hence q_c is called the **transient state charge** of the system. The **transient state current** in the circuit $i_c = \frac{dq_c}{dt}$.

Steady and Transient States

$$Lq'' + Rq' + \frac{1}{C}q = E(t), \quad q(0) = q_0, \quad q'(0) = i_0.$$

$$q(t) = q_c(t) + q_p(t).$$

Steady State Charge

The function q_p is independent of the initial state but depends on the characteristics of the circuit (L, R, and C) and the applied voltage E. q_p is called the **steady state charge** of the system. The **steady state current** in the circuit $i_p = \frac{dq_p}{dt}$.

Example

An LRC series circuit has inductance 0.5 h, resistance 10 ohms, and capacitance $4 \cdot 10^{-3}$ f. Find the steady state current of the system if the applied force is $E(t) = 5\cos(10t)$.

Note that we are asked for the steady state **current**. We do not have initial conditions, but since we are only interested in a particular solution, we can answer the question. With $L = 0.5$, $R = 10$ and $C = 4 \cdot 10^{-3}$, the equation for the charge is

$$0.5q'' + 10q' + \frac{1}{4 \cdot 10^{-3}}q = 5\cos(10t) \implies q'' + 20q' + 500q = 10\cos(10t).$$

The characteristic equation $r^2 + 20r + 500 = 0$ has roots $r = -10 \pm 20i$. To determine q_p we can assume

$$q_p = A\cos(10t) + B\sin(10t)$$

which does not duplicate solutions of the homogeneous equation (such duplication would only occur if the roots above were $r = \pm 10i$).

Example Continued...

Working through the details, we find that $A = 1/50$ and $B = 1/100$. The steady state charge is therefore

$$q_p = \frac{1}{50}\cos(10t) + \frac{1}{100}\sin(10t).$$

The steady state current

$$i_p = \frac{dq_p}{dt} = -\frac{1}{5}\sin(10t) + \frac{1}{10}\cos(10t).$$

Remark: An alternative approach is to take the derivative of the original ODE to obtain an equation for the current.

$$q'' + 20q' + 500q = 10\cos(10t) \quad \Longrightarrow \quad i'' + 20i' + 500i = -100\sin(10t).$$

We would find the same particular solution $i_p = -\frac{1}{5}\sin(10t) + \frac{1}{10}\cos(10t)$.

Section 13: The Laplace Transform

In this and the next few sections, we will consider a mathematical tool called the **Laplace transform**. The Laplace transform is a type of mapping that assigns to a function (e.g., a dependent variable for an ODE) a new function. We will use this transform to express certain initial value problems into a new format. Our *differential* equation can become an *algebraic* equation. We solve this new problem, and translate back to the original dependent variable.

We'll see that properties of or operations on the original dependent variable will have a known affect on the new Laplace transform variable (and vice versa). This will allow us to solve initial value problems, including those with piecewise defined forcing functions. The Laplace transform can be used to analyze problems arising in Engineering.

In this section, we will define the Laplace transform and become familiar with the transforms for some basic functions. First, a quick look at integration involving more than one variable and a working definition of an *integral transform*.

Integration with Two Variables

A quick word about functions of 2-variables:

Suppose $G(s,t)$ is a function of two independent variables (s and t) defined over some rectangle in the plane $a \le t \le b$, $c \le s \le d$. If we compute an integral with respect to one of these variables, say t,

$$\int_{\alpha}^{\beta} G(s,t)\,dt$$

- the result is a function of the remaining variable s, and
- the variable s is treated as a constant while integrating with respect to t.

For Example...

Assume that $s \neq 0$ and $b > 0$. Compute the integral

$$\int_0^b e^{-st}\,dt =$$

$$\begin{aligned}\int_0^b e^{-st}\,dt &= \frac{1}{-s}e^{-st}\Big|_0^b \\ &= -\frac{1}{s}e^{-bs} + \frac{1}{s}e^0 \\ &= \frac{1}{s}\left(1 - e^{-bs}\right)\end{aligned}$$

Remark This is a function of the variable s as expected

Integral Transform

An **integral transform** is a mapping that assigns to a function $f(t)$ another function $F(s)$ via an integral of the form

$$\int_a^b K(s,t)f(t)\,dt.$$

- The function K is called the **kernel** of the transformation.
- The limits a and b may be finite or infinite.
- The integral may be improper so that convergence/divergence must be considered.
- This transform is **linear** in the sense that

$$\int_a^b K(s,t)(\alpha f(t)+\beta g(t))\,dt = \alpha\int_a^b K(s,t)f(t)\,dt + \beta\int_a^b K(s,t)g(t)\,dt.$$

The Laplace Transform

Definition:

Let $f(t)$ be piecewise continuous on $[0, \infty)$. The Laplace transform of f, denoted $\mathscr{L}\{f(t)\}$ is given by.

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

We will often use the upper case/lower case convention that $\mathscr{L}\{f(t)\}$ will be represented by $F(s)$. The domain of the transformation $F(s)$ is the set of all s such that the integral is convergent.

Remark 1: The **kernel** for the Laplace transform is $K(s,t) = e^{-st}$.

Remark 2: In general, s is considered a complex variable. We will generally take s to be real, but this will not restrict our use of the Laplace transform.

Limits at Infinity e^{-st}

In what follows, we will encounter the limit $\lim_{t\to\infty} e^{-st}$ which depends on the value of s.

If $s > 0$, evaluate $\lim_{t\to\infty} e^{-st}$.
We note that if $s > 0$, then $-st \to -\infty$ as $t \to \infty$. Hence

$$\lim_{t\to\infty} e^{-st} = 0.$$

If $s < 0$, evaluate $\lim_{t\to\infty} e^{-st}$.
Now we that if $s < 0$, then $-st \to +\infty$ as $t \to \infty$. In this case,

$$\lim_{t\to\infty} e^{-st} = \infty.$$

We will keep these results in mind.

Find[49] the Laplace transform of $f(t) = 1$.

We apply the defintion of the Laplace transform.

$$\mathscr{L}\{1\} = \int_0^\infty e^{-st} \cdot 1\, dt.$$

Let's observe right away that this integral is improper. Moreover, convergence will depend on the value of s.

We need to consider $s = 0$ and $s \neq 0$ separately, as these require different integration formulas. It is readily seen that if $s = 0$, the integral

$$\int_0^\infty e^{-0t}\, dt = \int_0^\infty dt$$

is divergent. This tells us that $s = 0$ is not in the domain of $\mathscr{L}\{1\}$. For $s \neq 0$, we can evaluate the improper integral using the antiderivative of an expontial. Let's note that the correct approach is to use

$$\int_0^\infty e^{-st}\, dt = \lim_{b\to\infty} \int_0^b e^{-st}\, dt$$

[49]Unless stated otherwise, the domain for each example is $[0, \infty)$.

Find the Laplace transform of $f(t) = 1$.

$$\mathscr{L}\{1\} = \lim_{b\to\infty}\int_0^b e^{-st}\,dt = \lim_{b\to\infty} -\frac{1}{s}e^{-st}\Big|_0^b = \lim_{b\to\infty} -\frac{1}{s}\left(e^{-bs}-1\right)$$

Recalling the previous limits, we see that convergence in the limit $t\to\infty$ requires $s>0$. In this case, we have

$$\mathscr{L}\{1\} = -\frac{1}{s}(0-1) = \frac{1}{s}.$$

$\mathscr{L}\{f(t)\}$ for $f(t)=1$, $t\geq 0$

So we have the transform along with its domain

$$\mathscr{L}\{1\} = \frac{1}{s}, \quad s>0.$$

Find the Laplace transform of $f(t) = t$, $t \geq 0$

Remark:

Going forward, we will suppress the limit process in replacing the infinite limit of integration. That is, we will *write* the upper limit ∞ as though it were finite with the understanding that we are treating the improper integral appropriately.

We again turn to the definition.

$$\mathscr{L}\{t\} = \int_0^\infty e^{-st} \cdot t\, dt.$$

Again, we have to consider $s = 0$ and $s \neq 0$ separately. It is straightforward to see that the integral is divergent when $s = 0$. For $s \neq 0$, we can integrate by parts to obtain

$$\mathscr{L}\{t\} = \int_0^\infty e^{-st} \cdot t\, dt = -\frac{1}{s} te^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\, dt$$

Find the Laplace transform of $f(t) = t$, $t \geq 0$

It can be shown that convergence requires $s > 0$. With $s > 0$, the first term vanishes leaving

$$\mathscr{L}\{t\} = \int_0^\infty e^{-st} \cdot t\, dt = \frac{1}{s}\int_0^\infty e^{-st}\, dt$$

We can recognize that last factor as $\mathscr{L}\{1\}$, which we just computed. That is,

$$\mathscr{L}\{t\} = \int_0^\infty e^{-st} \cdot t\, dt = \frac{1}{s}\int_0^\infty e^{-st}\, dt = \frac{1}{s}\mathscr{L}\{1\} = \frac{1}{s^2}.$$

$\mathscr{L}\{f(t)\}$ for $f(t) = t$, $t \geq 0$

We have the transform along with its domain

$$\mathscr{L}\{t\} = \frac{1}{s^2}, \quad s > 0.$$

A piecewise defined function

Find the Laplace transform of f defined by

$$f(t) = \begin{cases} 2t, & 0 \le t < 10 \\ 0, & t \ge 10 \end{cases}$$

By definition,

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^{10} 2te^{-st}\,dt + \int_{10}^\infty 0 \cdot e^{-st}\,dt$$

For $s \neq 0$, integration by parts gives

$$\mathscr{L}\{f(t)\} = \frac{2}{s^2} - \frac{2e^{-10s}}{s^2} - \frac{20e^{-10s}}{s}.$$

When $s = 0$, the value $\mathscr{L}\{f(t)\}|_{s=10} = 100$ can be computed by evaluating the integral or by taking the limit of the above as $s \to 0$.

Evaluating Laplace Transforms

We've seen that we can evaluate Laplace transforms by using the definition directly. In practice, Laplace transforms are rarely evaluated by actually integrating. Instead, the Laplace transforms for common functions have been extensively cataloged, and tables are used. An internet search for *tables of Laplace transforms* will garner thousands of easily accessible tables.

In general, we will take Laplace transforms by looking up functions in a table. As we build up what we know about Laplace transforms, we will be able to take transforms of broader classes of functions. While this will eliminate the need to do a lot of tedious integration, we often require some amount of algebra or function identities combined with our table.

Tables commonly use $f(t)$ and $F(s)$ to represent functions and their Laplace transforms, respectively. We'll follow this upper/lower case convention.

The Laplace Transform is a Linear Transformation

A beginning table of Laplace transforms may include:

- $\mathscr{L}\{\alpha f(t)+\beta g(t)\}=\alpha F(s)+\beta G(s)$
- $\mathscr{L}\{1\}=\frac{1}{s},\quad s>0$
- $\mathscr{L}\{t^n\}=\frac{n!}{s^{n+1}},\quad s>0$ for $n=1,2,\dots$
- $\mathscr{L}\{e^{at}\}=\frac{1}{s-a},\quad s>a$
- $\mathscr{L}\{\cos kt\}=\frac{s}{s^2+k^2},\quad s>0$
- $\mathscr{L}\{\sin kt\}=\frac{k}{s^2+k^2},\quad s>0$

Examples: Evaluate the Laplace Transform of

(a) $f(t) = 5t^3 - 7\sin(\pi t)$

We'll use the table. First note that the linearity property says that

$$\mathscr{L}\left\{f(t)\right\} = \mathscr{L}\left\{5t^3 - 7\sin(\pi t)\right\} = 5\mathscr{L}\left\{t^3\right\} - 7\mathscr{L}\left\{\sin(\pi t)\right\}$$

We have an entry for t^n, which we'll use with $n = 3$, and we have an entry for $\sin(kt)$, which we'll use with $k = \pi$. This gives

$$\begin{aligned} \mathscr{L}\left\{f(t)\right\} &= 5\left(\frac{3!}{s^{3+1}}\right) - 7\left(\frac{\pi}{s^2+\pi^2}\right) \\ &= \frac{30}{s^4} - \frac{7\pi}{s^2+\pi^2} \end{aligned}$$

Caution: The value of $5(3!)$ is 30 because $3! = 1 \cdot 2 \cdot 3 = 6$. It's perfectly acceptable to leave this as $5(3!)$. However, it would be wrong to write something like "15!." Factorials don't work like that!

Examples: Evaluate the Laplace Transform of

(b) $f(t) = (2 - t)^2$

We'll use the table. First, we expand the square.

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{4 - 4t + t^2\} = 4\mathscr{L}\{1\} - 4\mathscr{L}\{t\} + \mathscr{L}\{t^2\}$$

$$= \frac{4}{s} - \frac{4}{s^2} + \frac{2}{s^3}$$

Examples: Evaluate the Laplace Transform of

(c) $f(t) = \sin^2 5t$

We don't have a table entry for the sine squared. However, we can use the power reducing identity $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$. We have

$$\mathscr{L}\{f(t)\} = \mathscr{L}\left\{\frac{1}{2} - \frac{1}{2}\cos(10t)\right\} = \frac{1}{2}\mathscr{L}\{1\} - \frac{1}{2}\mathscr{L}\{\cos(10t)\}$$

$$= \frac{\frac{1}{2}}{s} - \frac{\frac{1}{2}s}{s^2+100}$$

Remark: We'll find that it's not typically useful to *simplify* the Laplace transform since we want to trace the function of s back to a function of t. So while this answer can be written as a single fraction

$$\mathscr{L}\{\sin^2(5t)\} = \frac{100}{2s(s^2+100)}$$

there's no motivation to bother. We leave it as written above.

Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

It is reasonable to ask whether every function defined in the interval $[0,\infty)$ has a Laplace transform. To finish this section, we state a theorem on the existence of the Laplace transform. First, two definitions.

Definition: Exponential Order

Let $c > 0$. A function f defined on $[0,\infty)$ is said to be of *exponential order c* provided there exists positive constants M and T such that $|f(t)| < Me^{ct}$ for all $t > T$.

Definition: Piecewise Continuous

A function f is said to be *piecewise continuous* on an interval $[a,b]$ if f has at most finitely many jump discontinuities on $[a,b]$ and is continuous between each such jump.

Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

Theorem

If f is piecewise continuous on $[0, \infty)$ and of exponential order c for some $c > 0$, then f has a Laplace transform for $s > c$.

An example of a function that is NOT of exponential order for any c is $f(t) = e^{t^2}$. Note that

$$f(t) = e^{t^2} = \left(e^t\right)^t \quad \Longrightarrow \quad |f(t)| > e^{ct} \quad \text{whenever} \quad t > c.$$

This is a function that doesn't have a Laplace transform. We won't be dealing with this type of function here.

Section 14: Inverse Laplace Transforms

The Laplace transform provides a tool that can be used to solve and sometimes analyze certain IVPs. As was alluded to in the introduction of the previous section, we'll use the Laplace transform to convert an IVP (in the variable t) into an algebraic equation (in the variable s), solve that equation, and then revert back. To do this, we'll need to be able to go backwards. Given a function $F(s)$, can we identify a function $f(t)$ having F as its Laplace transform?

There is a formula for this backward transform, and it's even an integral[50]. Unfortunately (or perhaps fortunately since it means we won't be computing integrals), the *interval* for the integration is along a line in the complex plane. This requires some mathematics beyond the scope of this course. The good news is that the same tables used to take Laplace transforms can be used for this backward process. This is what is typically done in practice, and it's the approach we'll take here.

[50] The formula involves something called a *Bromwich* integral, and involves integrating along a contour in the complex plane.

Inverse Laplace Transforms

Inverse Laplace Transform

Let $F(s)$ be a function. An **inverse Laplace transform** of F is a piecewise continuous function $f(t)$ provided $\mathscr{L}\{f(t)\} = F(s)$. We will use the notation

$$\mathscr{L}^{-1}\{F(s)\} = f(t) \quad \text{if} \quad \mathscr{L}\{f(t)\} = F(s).$$

Helpful Hint:

In the same way that fluency with taking derivatives is useful for evaluating integrals, being comfortable using a table to take Laplace transforms will make it easier to go backwards. So taking the time to do plenty of practice taking Laplace transforms will help you develop a good eye for going in reverse.

A Table of Inverse Laplace Transforms

We don't need a separate table of inverse Laplace transforms. One table can be used to go both directions. If we restated our earlier table in terms of the inverse, we could write:

- $\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$
- $\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$, for $n = 1, 2, \dots$
- $\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
- $\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$
- $\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$

The inverse Laplace transform is also linear so that

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha f(t) + \beta g(t)$$

A Note on Using a Table of Laplace Transforms

We often have to use some algebra or function identities when taking an inverse Laplace transform. This is because our expression must match the table entry **exactly** (close isn't good enough). For example,

$$\mathscr{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}} \quad \text{implies} \quad \mathscr{L}^{-1}\left\{\frac{3!}{s^4}\right\} = t^3.$$

Note that since the power on s is 4, the expression must have 3! in the numerator to align perfectly with the result from the table.

If we wanted to evaluate something like $\mathscr{L}^{-1}\left\{\frac{1}{s^4}\right\}$, we must make the appropriate adjustment for the missing factor of 3!. Fortunately, we have a few tricks of the trade that we use frequently. Let's look at some examples to see them.

Find the Inverse Laplace Transform

(a) $\mathscr{L}^{-1}\left\{\frac{1}{s^7}\right\}$

We notice that the expression looks sort of like $\frac{n!}{s^{n+1}}$ with $n = 6$ (since $n + 1 = 7$). However, we don't have an exact match with the table entry because we're missing the required 6! in the numerator. We can fix this by multiplying our argument by a strategic representation of 1 (which is always legitimate). Note that we can multiply and divide by 6! to find that

$$\frac{1}{s^7} = \left(\frac{6!}{6!}\right)\left(\frac{1}{s^7}\right) = \frac{1}{6!}\left(\frac{6!}{s^7}\right).$$

The factor $\frac{1}{6!}$ is simply a constant coefficient that we can factor out when taking the inverse transform. We have

$$\mathscr{L}^{-1}\left\{\frac{1}{s^7}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{6!}\frac{6!}{s^7}\right\} = \frac{1}{6!}\mathscr{L}^{-1}\left\{\frac{6!}{s^7}\right\} = \frac{1}{6!}t^6$$

Example: Evaluate

(b) $\mathscr{L}^{-1}\left\{\dfrac{s+1}{s^2+9}\right\}$

We have table entries that match up nicely with $\frac{s}{s^2+3^3}$ and $\frac{3}{s^2+3^2}$. If we first write our argument as the sum of two terms, we can use these.

$$\mathscr{L}^{-1}\left\{\frac{s+1}{s^2+9}\right\}=\mathscr{L}^{-1}\left\{\frac{s}{s^2+9}+\frac{1}{s^2+9}\right\}=\mathscr{L}^{-1}\left\{\frac{s}{s^2+9}\right\}+\mathscr{L}^{-1}\left\{\frac{1}{s^2+9}\right\}$$

The first term matches nicely with the transform for $\cos(3t)$. The second one looks like the transform of $\sin(3t)$ except that it's missing the factor of 3 in the numerator. We can apply the same approach from the previous example and multiply by 1 in the form of $\frac{3}{3}$. Finally, we have

$$\mathscr{L}^{-1}\left\{\frac{s+1}{s^2+9}\right\}=\mathscr{L}^{-1}\left\{\frac{s}{s^2+9}\right\}+\frac{1}{3}\mathscr{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$=\cos(3t)+\frac{1}{3}\sin(3t)$$

Example: Evaluate

(c) $\mathscr{L}^{-1}\left\{\dfrac{s-8}{s^2-2s}\right\}$

This problem will require us to use a process frequently needed when working with Laplace transforms. That is **partial fraction decomposition** (PFD). The Laplace transforms for the most common elementary functions we work with (polynomials, exponentials, sine/cosine) are rational functions. In solving IVPs, we'll find that we often end up with a rational expression in s, and so decomposing it into more basic rational functions is usually required. For the problem at hand, we first perform a PFD. The correct form for the decomposition is

$$\frac{s-8}{s^2-2s} = \frac{s-8}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}.$$

With some effort, we determine that $A = 4$ and $B = -3$.

$$\mathscr{L}^{-1}\left\{\frac{s-8}{s^2-2s}\right\}$$

Now using the table to evaluate our inverse transform, we have

$$\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{s-8}{s^2-2s}\right\} &= \mathscr{L}^{-1}\left\{\frac{4}{s}-\frac{3}{s-2}\right\} \\
&= 4\mathscr{L}^{-1}\left\{\frac{1}{s}\right\}-3\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\} \\
&= 4-3e^{2t}
\end{aligned}$$

Convolutions

The process of using a partial fraction decomposition, as illustrated in the last example, is typical when using Laplace transforms as a tool for solving initial value problems[51].

As an integral, it is clear that the transform or inverse transform of a product is **NOT** the product of the transforms. That is

$$\mathscr{L}\{f(t)g(t)\} \neq \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\}$$

and similarly

$$\mathscr{L}^{-1}\{F(s)G(s)\} \neq \mathscr{L}^{-1}\{F(s)\}\mathscr{L}^{-1}\{G(s)\}$$

There is a special type of *product* of functions that can be used to evaluate an inverse transform of the form $\mathscr{L}^{-1}\{F(s)G(s)\}$. The special product is called a **convolution**

[51] If you find that you are never using partial fraction decomp when solving IVPs, almost surely you are not solving them correctly

Convolution

Definition

Let f and g be piecewise continuous on $[0, \infty)$ and of exponential order c for some $c \geq 0$. The **convolution** of f and g is denoted by $f * g$ and is defined by

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

Remark: In a more general setting in which functions of interest are defined on $(-\infty, \infty)$, the convolution is typically defined as

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,d\tau$$

If the functions $f(t)$ and $g(t)$ are assigned to take the value of zero for $t < 0$, this definition reduces to the one given here.

Example

Compute the convolution of $f(t) = e^{-3t}$ and $g(t) = e^{-5t}$.

Note that

$$f(\tau)g(t-\tau) = e^{-3\tau}e^{-5(t-\tau)} = e^{-3\tau}e^{-5t}e^{5\tau} = e^{-5t}e^{2\tau}.$$

The factor e^{-5t} does not depend on the dummy variable of integration, so we can factor it out of the integral.

$$\begin{aligned}
(f * g)(t) &= \int_0^t e^{-3\tau}e^{-5(t-\tau)}\,d\tau. \\
&= e^{-5t}\int_0^t e^{2\tau}\,d\tau \\
&= e^{-5t}\left[\frac{1}{2}e^{2\tau}\right|_0^t \\
&= e^{-5t}\left(\frac{1}{2}e^{2t} - \frac{1}{2}e^{0}\right) \\
&= \frac{1}{2}e^{-3t} - \frac{1}{2}e^{-5t}.
\end{aligned}$$

Laplace Transforms & Convolutions

The Laplace transform of a convolution is related to the product of Laplace transforms.

Theorem

Suppose $\mathscr{L}\{f(t)\} = F(s)$ and $\mathscr{L}\{g(t)\} = G(s)$. Then

$$\mathscr{L}\{f * g\} = F(s)G(s)$$

Theorem

Suppose $\mathscr{L}^{-1}\{F(s)\} = f(t)$ and $\mathscr{L}^{-1}\{G(s)\} = g(t)$. Then

$$\mathscr{L}^{-1}\{F(s)G(s)\} = (f * g)(t)$$

Remark: This is the same theorem stated first from the perspective of a Laplace transform and then from the perspective of an inverse Laplace transform.

Example

Use the convolution to evaluate[52]

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+8s+15}\right\} = \mathscr{L}^{-1}\left\{\left(\frac{1}{s+3}\right)\left(\frac{1}{s+5}\right)\right\}$$

Let $F(s) = \dfrac{1}{s+3}$ and $G(s) = \dfrac{1}{s+5}$, and let f and g be their inverse Laplace transforms, respectively. Then

$$f(t) = \mathscr{L}^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t} \quad \text{and} \quad g(t) = \mathscr{L}^{-1}\left\{\frac{1}{s+5}\right\} = e^{-5t}$$

Accoring to our theorem, $\mathscr{L}^{-1}\{F(s)G(s)\} = (f * g)(t)$. Having just computed this convolution in the last example, we have

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+8s+15}\right\} = \mathscr{L}^{-1}\left\{\left(\frac{1}{s+3}\right)\left(\frac{1}{s+5}\right)\right\} = \frac{1}{2}e^{-3t} - \frac{1}{2}e^{-5t}.$$

[52] It is a worthwhile exercise to perform a PFD on the argument and confirm that the inverse Laplace transform obtained via that PDF is the same as the one obtained using the convolution.

Example

Evaluate $\mathscr{L}\left\{\int_0^t \tau^6 e^{-4(t-\tau)}\,d\tau\right\}$

Note that the argument of the transform seems to be a convolution. If $f(t) = t^6$ and $g(t) = e^{-4t}$, then

$$(f * g)(t) = \int_0^t \tau^6 e^{-4(t-\tau)}\,d\tau.$$

According to our theorem, the Laplace transform should be the product $F(s)G(s)$ where

$$F(s) = \mathscr{L}\{t^6\} = \frac{6!}{s^7} \quad \text{and} \quad G(s) = \mathscr{L}\{e^{-4t}\} = \frac{1}{s+4}.$$

Hence

$$\mathscr{L}\left\{\int_0^t \tau^6 e^{-4(t-\tau)}\,d\tau\right\} = \left(\frac{6!}{s^7}\right)\left(\frac{1}{s+4}\right) = \frac{6!}{s^7(s+4)}.$$

Example

Evaluate the inverse Laplace transform in two ways, using a partial fraction decomposition and using a convolution.

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2(s+1)}\right\}$$

First, let's use a partial fraction decomposition to write the rational expression as a sum of simpler expressions. The proper set up is

$$\frac{1}{s^2(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1}.$$

After some algebra, we find that $A = -1$, $B = 1$ and $C = 1$.

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2(s+1)}\right\}$$

So the inverse transform

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1}{s^2(s+1)}\right\} &= \mathscr{L}^{-1}\left\{-\frac{1}{s}+\frac{1}{s^2}+\frac{1}{s+1}\right\}\\ &= -\mathscr{L}^{-1}\left\{\frac{1}{s}\right\}+\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\}+\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\}\\ &= -1+t+e^{-t}\end{aligned}$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2(s+1)}\right\}$$

To compute this using a convolultion, we want to express the argument as a product $F(s)G(s)$. While there can be multiiple ways to do this, we want to choose F and G so that we can easily evaluate $\mathscr{L}^{-1}\{F(s)\}$ and $\mathscr{L}^{-1}\{G(s)\}$ from our table. One such choice is

$$F(s)=\frac{1}{s^2}, \quad \text{and} \quad G(s)=\frac{1}{s+1}.$$

Note then that

$$f(t)=\mathscr{L}^{-1}\{F(s)\}=t, \quad \text{and} \quad g(t)=\mathscr{L}^{-1}\{G(s)\}=e^{-t}$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2(s+1)}\right\}$$

For $f(t) = t$ and $g(t) = e^{-t}$, we have

$$\begin{aligned}
(f * g)(t) &= \int_0^t \tau e^{-(t-\tau)}\,d\tau \\
&= e^{-t}\int_0^t \tau e^{\tau}\,d\tau \\
&= e^{-t}\left[\tau e^{\tau} - e^{\tau}\right|_0^t \\
&= e^{-t}\left(te^t - e^t - (0 - e^0)\right) \\
&= t - 1 + e^t
\end{aligned}$$

Both approaches yield the same result (as they should). In practice, we can choose whichever approach seems to require the least amount of effort for a given problem.

A Look Ahead: Solving IVPs

In this section, we introduce two important results on Laplace transforms that will allow us to take the transform and inverse transforms of broader classes of function, including piecewise defined functions. This is especially valuable when dealing with certain circuits or mechanical systems in which switches may be used to result in discontinuous forcing.

Before we get to these new results, let's take a peek at the Laplace transform in action as a means of solving an IVP. We will look at derivatives in more detail in the next section. For now, we will need the following theorem.

Theorem

If $f(t)$ is defined on $[0, \infty)$, is differentiable, and has Laplace transform $F(s) = \mathscr{L}\left\{f(t)\right\}$, then

$$\mathscr{L}\left\{f'(t)\right\} = sF(s) - f(0)$$

Let's use this, along with what we know about Laplace transforms to solve the IVP

$$y'(t) + 2y(t) = 4, \quad y(0) = 1$$

$$y'(t) + 2y(t) = 4, \quad y(0) = 1$$

We will assume that our solution $y(t)$ has a Laplace transform and we will set $Y(s) = \mathscr{L}\{y(t)\}$. The first step is to take the Laplace transform of both sides of the differential equation. Making use of the linearity property of the transform, we have

$$\mathscr{L}\{y'(t) + 2y(t)\} = \mathscr{L}\{4\} \quad \Longrightarrow \quad \mathscr{L}\{y'(t)\} + 2\mathscr{L}\{y(t)\} = 4\mathscr{L}\{1\}.$$

The second term on the left is simply $2Y(s)$. For the left most term, we use the theorem from the previous page

$$\mathscr{L}\{y'(t)\} = sY(s) - y(0).$$

From the table, we can evaluate $\mathscr{L}\{1\}$ on the right side. This gives us an equation in the new function Y

$$sY(s) - y(0) + 2Y(s) = \frac{4}{s}.$$

$$sY(s) - y(0) + 2Y(s) = \frac{4}{s}$$

Note that we now have an algebraic equation in the unknown function Y. If we can isolate Y, we can take the inverse transform and obtain our desired solution $y(t)$. Let's note that part of our IVP is the condition $y(0) = 1$. We can substitute this into our equation and do whatever algebra is needed to determine $Y(s)$. We have

$$sY(s) - 1 + 2Y(s) = \frac{4}{s} \quad \Longrightarrow \quad (s+2)Y(s) = \frac{4}{s} + 1 \quad \Longrightarrow \quad (s+2)Y(s) = \frac{4+s}{s}.$$

Dividing through by $s+2$, we arrive at

$$Y(s) = \frac{4+s}{s(s+2)}.$$

This is the Laplace transform of the solution to our IVP.

$$Y(s) = \frac{4+s}{s(s+2)}$$

Lastly, we will compute our solution $y(t) = \mathscr{L}^{-1}\{Y(s)\}$. As is typical, we require a partial fraction decomposition (or we can use a convolution). Note that (omitting the details of the PFD)

$$Y(s) = \frac{4+s}{s(s+2)} = \frac{2}{s} - \frac{1}{s+2}.$$

Hence

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}\left\{\frac{2}{s} - \frac{1}{s+2}\right\} \\ &= \mathscr{L}^{-1}\left\{\frac{2}{s}\right\} - \mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\} \\ &= 2 - e^{-2t} \end{aligned}$$

The solution to the IVP is $\quad y(t) = 2 - e^{-2t}.$

(This can be confirmed by substituting y back into the original IVP if desired.)

Section 15: Shift Theorems

In this section, we will add two major results to our catalog of properties of Laplace transforms. These results are often called *shift* or *translation* theorems. What they sort of boil down to is this:

- if our function, $f(t)$, is multiplied by an exponential e^{at}, then the Laplace transform undergoes a horizontal shift in s, and
- if our if we translate/shift our function $f(t)$ in an appropriate way, its Laplace transform obtains an exponential factor e^{-as}.

These results will permit us to use a table to take Laplace transforms and inverse transforms of more types of functions. We'll start with the shift in s and see why the shift and exponential factors are related. Let's start with an example.

Shift in s

We know that $\mathscr{L}\{t^2\} = \dfrac{2}{s^3}$. So suppose we wish to evalute the inverse Laplace transform $\mathscr{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\}$. Does it help to notice that

$$\text{If} \quad F(s) = \frac{2}{s^2}, \quad \text{then} \quad \mathscr{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\} = \mathscr{L}^{-1}\{F(s-1)\}?$$

More Generally: What is the relationship between a Laplace transform $F(s)$ and a translated version $F(s-a)$ for real number a?

Shift in s

Let's notice that

$$\frac{2}{s^3} = \int_0^\infty e^{-st} t^2 \, dt \quad \Longrightarrow \quad \frac{2}{(s-1)^3} = \int_0^\infty e^{-(s-1)t} t^2 \, dt.$$

If we look at that integrand and makes use of properties of exponentials, we have

$$e^{-(s-1)t} t^2 = e^{-st+1t} t^2 = e^{-st} e^{1t} t^2 = e^{-st}\left(e^{1t} t^2\right).$$

It follows that

$$\frac{2}{(s-1)^2} = \int_0^\infty e^{-st}\left(e^t t^2\right) \, d = \mathscr{L}\{e^t t^2\}.$$

We can see that replacing the 1 with some other number a will result in the analogous result

$$\mathscr{L}^{-1}\left\{\frac{2}{(s-a)^2}\right\} = e^{at} t^2.$$

Theorem (translation in s)

Theorem: Shift in s

Suppose $\mathscr{L}\left\{f(t)\right\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

This result can also be stated by thinking about it in terms of the inverse Laplace transform.

Theorem: Shift in s

Suppose $\mathscr{L}^{-1}\left\{F(s)\right\} = f(t)$, then for any real number a

$$\mathscr{L}^{-1}\left\{F(s-a)\right\} = e^{at}\mathscr{L}^{-1}\{F(s)\}, \text{ i.e., } \mathscr{L}^{-1}\left\{F(s-a)\right\} = e^{at}f(t).$$

Remark: These are the same statement, but we'll need to be able to recognize and use it taking both forward and inverse transforms.

Example

Suppose $f(t)$ is some function defined on $[0,\infty)$ with the property that

$$\mathscr{L}\{f(t)\} = \frac{\ln(s)+\gamma}{s}$$

where γ is a fixed[53] constant. Evaluate $\mathscr{L}\{e^{3t}f(t)\}$.

Using our shift theorem, let's call $\mathscr{L}\{f(t)\} = F(s)$. Then

$$\mathscr{L}\{e^{3t}f(t)\} = F(s-3) = \frac{\ln(s-3)+\gamma}{s-3}.$$

Remark: This doesn't require us to compute anything, we're simply applying the theorem. Let's consider a few more forward transforms and then look at some inverse transform examples.

[53] The number $\gamma \approx 0.577216$ is called the *Euler Mascheroni constant*. The actual function here $f(t) = \ln(1/t)$, but we don't even need to know that to answer the question.

Evaluate each Laplace Transform

(a) $\mathscr{L}\left\{t^8 e^{4t}\right\}$

If we ignore the exponential for a moment, we need $\mathscr{L}\{t^8\} = \frac{8!}{s^9}$. Calling this $F(s)$, since our exponential factor has a 4 in it, our solution will be $F(s-4)$. Hence

$$\mathscr{L}\left\{t^8 e^{4t}\right\} = \frac{8!}{(s-4)^9}$$

Evaluate each Laplace Transform

(b) $\mathscr{L}\left\{e^{-t}\cos(t)\right\}$

Again, we ignore the exponential initially to identify the $F(s)$ part of the transform. Letting $F(s)=\mathscr{L}\{\cos(t)\}=\frac{s}{s^2+1^2}$, and noting that our exponential has factor $a=-1$, the transform we're after will be $F(s-(-1))=F(s+1)$. So

$$\mathscr{L}\left\{e^{-t}\cos(t)\right\}=\frac{s+1}{(s+1)^2+1}.$$

(c) $\mathscr{L}\left\{e^{-t}\sin(t)\right\}$

Same game with only slightly different players. For this example, $F(s)=\mathscr{L}\{\sin(t)\}=\frac{1}{s^2+1^2}$. The factor in the exponential is the same $a=-1$, giving us

$$\mathscr{L}\left\{e^{-t}\sin(t)\right\}=\frac{1}{(s+1)^2+1}.$$

Inverse Laplace Transforms (s-shift)

Evaluate $\mathscr{L}^{-1}\left\{\frac{1}{(s+2)^5}\right\}$

The key here is to recognize the argument as something familiar but translated. If we *ignore* the $+2$, we can see that we're working with the expression $F(s)=\frac{1}{s^5}$. The argument of our transform is $F(s+2)=F(s-(-2))$. So the approach is to determine $\mathscr{L}^{-1}\{F(s)\}$, then our desired result with be $e^{-2t}\mathscr{L}^{-1}\{F(s)\}$. From the table, with a bit of algebra,

$$\mathscr{L}^{-1}\left\{\frac{1}{s^5}\right\}=\frac{1}{4!}\mathscr{L}^{-1}\left\{\frac{4!}{s^5}\right\}=\frac{1}{4!}t^4.$$

Finally,

$$\mathscr{L}^{-1}\left\{\frac{1}{(s+2)^5}\right\}=\frac{1}{4!}t^4e^{-2t}.$$

Inverse Laplace Transforms (s-shift)

The s-shift result comes into play with rational functions of s in which we are required to complete the square or when the denominator contains a repeated linear factor. Let's look at an example of each.

Evaluate

(a) $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+2s+2}\right\}$

The initial thought, when looking at the argument is that a PFD might be needed. However, the denominator s^2+2s+2 is irreducible. In such a case, we need to complete the square. Note that $s^2+2s+2=(s+1)^2+1$. So our argument

$$\frac{s}{s^2+2s+2}=\frac{s}{(s+1)^2+1}.$$

This looks similar to the transform for $\cos(t)$ with s replaced with $s+1$, but that's not quite right—see our recent example.

Inverse Laplace Transforms (s-shift)

An expression like $F(s+1)$ would require $s+1$ to appear **everywhere**. We can deal with the numerator by using the fact that $s = s+1-1$. Then we can write our expression as a sum of two ratios.

$$\frac{s}{s^2+2s+2} = \frac{s}{(s+1)^2+1} = \frac{s+1-1}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}.$$

Now, we're ready to take the inverse transform (again see our recent example)

$$\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{s}{s^2+2s+2}\right\} &= \mathscr{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right\} \\
&= \mathscr{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} - \mathscr{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} \\
&= e^{-t}\cos t - e^{-t}\sin t.
\end{aligned}$$

Remark:

The maneuver used here is to add zero, i.e., replace s with $s-a+a$ with strategically chosen value of a. It's similar to multiplying by 1 in the form $\frac{a}{a}$ when needed. This is another commonly used trick-of-the-trade when relying on a table for computations.

Inverse Laplace Transforms (repeat linear factors)

(b) $\mathscr{L}^{-1}\left\{\dfrac{1+3s-s^2}{s(s-1)^2}\right\}$

We can do a PFD on this argument. Omitting the details, the argument decomposes as

$$\frac{1+3s-s^2}{s(s-1)^2}=\frac{1}{s}-\frac{2}{s-1}+\frac{3}{(s-1)^2}.$$

So

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1+3s-s^2}{s(s-1)^2}\right\} &= \mathscr{L}^{-1}\left\{\frac{1}{s}-\frac{2}{s-1}+\frac{3}{(s-1)^2}\right\}\\ &= \mathscr{L}^{-1}\left\{\frac{1}{s}\right\}-2\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\}+3\mathscr{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}\end{aligned}$$

Inverse Laplace Transforms (repeat linear factors)

The first two terms are straightforward. The third term will require the s-shift result. Note that if we ignore the -1, we have

$$F(s) = \frac{1}{s^2} = \mathscr{L}\{t\}.$$

This means that

$$\mathscr{L}^{-1}\{F(s-1)\} = e^{1t}\mathscr{L}^{-1}\{F(s)\} = e^t t.$$

Pulling it all together

$$\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{1+3s-s^2}{s(s-1)^2}\right\} &= \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} + 3\mathscr{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} \\
&= 1 - 2e^t + 3te^t
\end{aligned}$$

Translation in t

The second major result of this section deals with translating the function $f(t)$ and allows us to use a table to evaluate the Laplace transform of a piecewise defined function. (Without this result, we must resort to using the integral definition and splitting the integration as needed.) However, the Laplace transform requires the input function f to be defined on the interval $[0, \infty)$. If f is defined on $[0, \infty)$, and $a > 0$, then $f(t - a)$ is defined on $[a, \infty)$. Unless we somehow define this new translated function on the interval $[0, a)$, we're no longer able to consider a Laplace transform. This is a technical detail that has to be addressed.

In practice, we would like to be able to state piecewise defined functions in a way that facilitates taking Laplace transforms and inverse transforms using a table. We first introduce a simple, piecewise constant function called a **unit step function**. We'll define a unit step function defined on $[0, \infty)$. An analogous function can readily be defined on $(-\infty, \infty)$, and it often called the **Heaviside step function**[54].

---

[54] In honor of physicist and mathematician Oliver Heaviside.

The Unit Step Function

Definition: Unit Step Function

Let $a > 0$. The unit step function *centered at a* is denoted $\mathscr{U}(t-a)$. It is defined by

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

The name *unit step function* is derived from its graph which looks like a stair step of height 1.

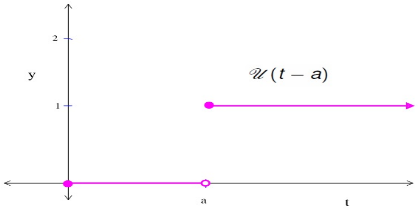


Figure: A graph of $\mathscr{U}(t-a)$ which jumps from zero to one at $t = a$.

The Unit Step Function

There are various notations used for the unit step function. Some of the more common include

$$\mathscr{U}(t-a), \quad \mathbf{u}(t-a), \quad \mathbf{u}_a(t), \quad \theta(t-a), \quad \text{and} \quad H(t-a).$$

There are also variations[55] in how it's defined at the point of discontinuity. In the definition given here, we are taking $\mathscr{U}(0) = 1$, which results in the function being continuous from the right but not continuous from the left at $t = a$. Since we're interested in Laplace transforms, and changing an integrand at a single point won't affect the integral, these discrepancies don't really cause us any trouble. We'll take $\mathscr{U}(t)$ (i.e., the case when $a = 0$) to be

$$\mathscr{U}(t) = 1, \quad \text{for all} \quad t \geq 0.$$

[55] The value $\mathscr{U}(0)$ is typically taken to be one of 1, 0, or $\frac{1}{2}$.

Piecewise Defined Functions

We usually see piecewise functions defined by a vertical list of expressions, along with where they're valid. The list is preceded by a single, right facing curly bracket "$\{$."

The unit step function provides an alternative format for writing a piecewise defined function in which we can use factors of $\mathscr{U}$ centered at the points where our function changes form. As an illustration, let's show that

$$f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases} \quad = \; g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a)$$

We want to show that the expression on the right is equivalent to $f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases}$. This means that we have to show that on the interval $[0, a)$, it is equal to $g(t)$, and on the interval $[a, \infty)$, it is equal to $h(t)$.

$$f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases} = g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a)$$

First, let's consider the case that $0 \le t < a$. In this case (see the definition of $\mathscr{U}$) $\mathscr{U}(t-a) = 0$. So for $0 \le t < a$,

$$g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a) = g(t) - g(t)\cdot 0 + h(t)\cdot 0 = g(t).$$

If $t \ge a$, then $\mathscr{U}(t-a) = 1$. So for $t \ge a$.

$$g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a) = g(t) - g(t)\cdot 1 + h(t)\cdot 1 = h(t).$$

We see that for all $t \ge 0$, the expressions $f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases}$ and $g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a)$ are equivalent.

Remark:

We can play this game with piecewise defined function consisting of more than two pieces. We simply add in the pieces, multiplied by the unit step centered at the point the piece is supposed to start, and then subtract off the piece multiplied by a unit step function centered where the piece is supposed to end.

Example

Verify that

$$f(t) = \begin{cases} t, & 0 \le t < 2 \\ \cos(t), & 2 \le t < 10 \\ 3t^2, & 10 \le t \end{cases}$$

is equivalent to

$$f(t) = t - t\mathscr{U}(t-2) + \cos(t)\mathscr{U}(t-2) - \cos(t)\mathscr{U}(t-10) + 3t^2\mathscr{U}(t-10).$$

Exercise left to the reader.

To lead into our Laplace transform result, suppose $f(t)$ is defined on the interval $[0, \infty)$, and $a > 0$. We will determine the Laplace transform of the function obtained by translating f to the right, $f(t-a)$, and setting our result to zero for $0 \le t < a$. This new function is $f(t-a)\mathscr{U}(t-a)$.

Translation in t

Suppose $f(t)$ is defined for $t \geq 0$ and $a > 0$. Then

$$f(t-a)\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ f(t-a), & t \geq a \end{cases}.$$

Graphically, $f(t-a)\mathscr{U}(t-a)$ is the curve f translated to the right a units and taking the value of zero on the interval $[0, a)$.

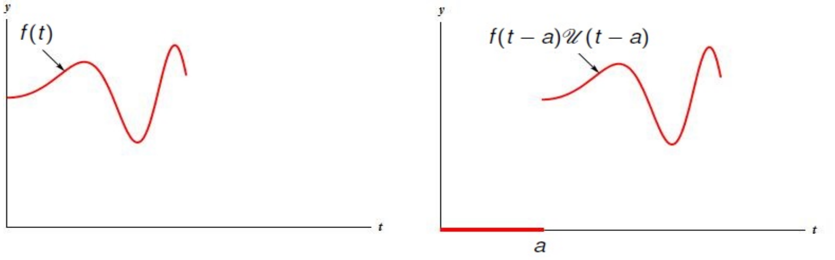


Figure: The graph of $f(t)$ on the left shown with the graph of $f(t-a)\mathscr{U}(t-a)$ on the right.

Translation in t

Theorem: Shift in t

If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{f(t)\} = e^{-as}F(s).$$

We can state this same result from the perspective of the inverse Laplace transform.

Theorem: Shift in t

If $f(t) = \mathscr{L}^{-1}\{F(s))\}$ and $a > 0$, then

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a).$$

These are exactly the same statement. As with other results on Laplace transforms, we want to be able to use this to take both forward and inverse transforms.

$$\mathscr{L}\{\mathscr{U}(t-a)\} = \frac{e^{-as}}{s}, \quad s > 0$$

Suppose $a > 0$. Let's show this using the definition of the Laplace transform, and then see how it follows from the theorem just stated. By definition,

$$\mathscr{L}\{\mathscr{U}(t-a)\} = \int_0^\infty e^{-st}\mathscr{U}(t-a)\,dt = \int_0^a e^{-st}\cdot 0\,dt + \int_a^\infty e^{-st}\cdot 1\,dt.$$

The integral from zero to a is zero, and it remains to evaluate the right most term. Convergence of the integral requires $s > 0$. When $s > 0$,

$$\mathscr{L}\{\mathscr{U}(t-a)\} = -\frac{1}{s}e^{-st}\Big|_a^\infty = -\frac{1}{s}\left(0 - e^{-as}\right) = \frac{e^{-as}}{s}$$

To see how this follows from our theorem, we can write $\mathscr{U}(t-a) = 1\cdot\mathscr{U}(t-a)$. If $f(t) = 1$ for t in $[0,\infty)$, then for any $a > 0$, $f(t-a) = 1$. Recalling that $\mathscr{L}\{1\} = \frac{1}{s}$, we have

$$\mathscr{L}\{\mathscr{U}(t-a)\} = \mathscr{L}\{1\cdot\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{1\} = e^{-as}\left(\frac{1}{s}\right) = \frac{e^{-as}}{s}.$$

Example

Find the Laplace transform $\mathscr{L}\left\{f(t)\right\}$ where

$$f(t) = \left\{\begin{array}{ll} 1, & 0 \le t < 1 \\ t, & t \ge 1 \end{array}\right.$$

First, let's rewrite f using unit step factors and collect some like terms.

$$f(t) = 1 - 1\mathscr{U}(t-1) + t\mathscr{U}(t-1) = 1 + (t-1)\mathscr{U}(t-1).$$

Applying our theorem, we have

$$\begin{aligned} \mathscr{L}\{f(t)\} &= \mathscr{L}\{1\} + \mathscr{L}\{(t-1)\mathscr{U}(t-1)\}^{56} \\ &= \frac{1}{s} + e^{-1s}\mathscr{L}\{t\} \\ &= \frac{1}{s} + e^{-1s}\left(\frac{1}{s^2}\right) \\ &= \frac{1}{s} + \frac{e^{-s}}{s^2}. \end{aligned}$$

---

[56] Note that if $f(t) = t$, then $f(t-1) = t-1$. This is why $\mathscr{L}\left\{(t-1)\mathscr{U}(t-1)\right\}$ becomes $e^{-1s}\mathscr{L}\left\{t\right\}$. We have to look at the expression $t-1$ and identify $f(t) = t$ from this.

Example

Evaluate the inverse Laplace transform $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}}{s(s+1)}\right\}$.

We'll use the fact that $\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a)$, which requires us to identify $f(t) = \mathscr{L}^{-1}\{F(s)\}$. Ignoring the exponential factor for a moment (a strategy you'll recall from the first shift theorem), we need

$$f(t) = \mathscr{L}^{-1}\left\{\frac{1}{s(s+1)}\right\}.$$

This requires a PFD. Omitting the details,

$$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1},$$

giving

$$f(t) = \mathscr{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - \mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} = 1 - e^{-t}.$$

Example $\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s(s+1)}\right\}$

Because we have the factor e^{-2s}, we will need to evaluate $f(t-2)$.

$$f(t) = 1 - e^{-t} \quad \Longrightarrow \quad f(t-2) = 1 - e^{-(t-2)}.$$

Finally,

$$\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s(s+1)}\right\} = \left(1 - e^{-(t-2)}\right)\mathscr{U}(t-2).$$

This can also be written as

$$\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s(s+1)}\right\} = \mathscr{U}(t-2) - e^{-(t-2)}\mathscr{U}(t-2).$$

Remark:

As this example illustrates, when working with $e^{-as}F(s)$, we may have to do a PFD or complete the square or otherwise manipulate the $F(s)$ factor. In doing so, we do not include the exponential. For example, here we performed a PFD on $\frac{1}{s(s+1)}$. We didn't try to decompose $\frac{e^{-2s}}{s(s+1)}$, which is **NOT** a rational expression.

Alternative Form for Translation in t

It is often the case that we wish to take the transform of a product of the form

$$g(t)\mathscr{U}(t-a)$$

in which the function g is not translated. We can reformulate our translation theorem.

Theorem: Shift in t

If $a > 0$, then

$$\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}.$$

Remark 1: Note that this is based on the observation that $g(t) = g((t+a)-a)$.

Remark 2: In practice, this tends to be the most useful form of this result for taking forward transforms, whereas the way it was originally stated (see the shift in t theorem) tends to be most useful for taking inverse transforms.

The Unit Impulse

Before moving on to derivatives and using the Laplace transform to solve initial value problems, we introduce a mathematical object known as the *Dirac delta function*. While it's not a function in the traditional sense, it is used to model a force that is highly localized in time (occurs instantaneously). We start with a motivating example.

Consider an RC circuit with resistance R and capacitance C subjected to a constant V_0 volts applied over a fixed time interval from $t = t_0$ to $t = t_1$ second. We can represent such an applied force in terms of unit step functions.

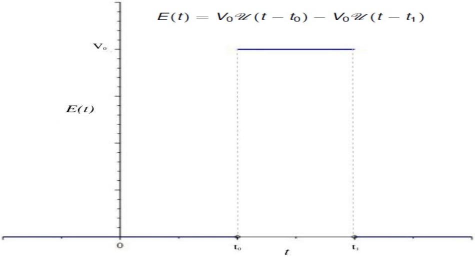


Figure: The charge on the capacitor satisfies the differential equation $R\dfrac{dq}{dt} + \dfrac{1}{C}q = V_0\mathscr{U}(t - t_0) - V_0\mathscr{U}(t - t_1)$

The Unit Impulse

In engineering applications, it is useful to have a model of a force (or signal) that is applied over an infinitesimal time interval.

For example, we would like to model the voltage applied in the previous RC circuit example in the limit $t_1 \to t_0$ while keeping the total *impulse* (a measure of its effect over time) of the applied voltage [57] fixed.

Question: How can we model a force that is applied for only an instant yet has a prescribed impulse?

We'll start with some simple functions, and then take an appropriately constructed limit.

[57] If f is a force applied over the time interval $[t_0, t_1]$, then the *impulse*, $\int_{t_0}^{t_1} f(t)\,dt$, is the change in momentum over this time interval.

The Unit Impulse

In order to build up to the definition of our unit impulse, we introduce the family of piecewise constant, rectangular functions $R_\epsilon(t) = \begin{cases} \frac{1}{2\epsilon}, & |t| < \epsilon \\ 0, & |t| > \epsilon \end{cases}$.

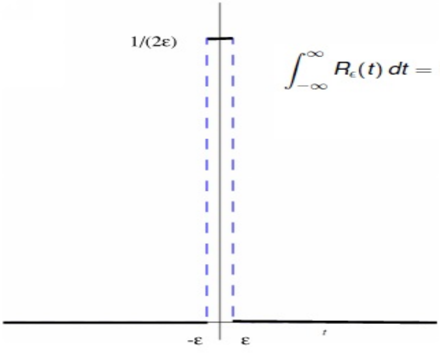


Figure: The value of ϵ determines the height and width of the rectangle. But for every $\epsilon > 0$, the integral of R_ϵ over the real line is 1.

Unit Impulse

We can place the peak of our rectangular functions at $t = a$ by considering $R_\epsilon(t-a)$. We see that at the value ϵ gets smaller, the height of the rectangle increases while the width decreases in such a way that the area remains constant at 1.

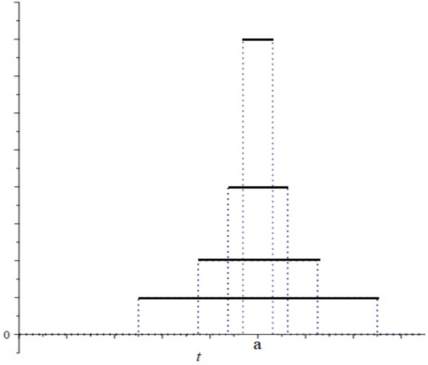


Figure: Plots of $R_\epsilon(t-a) = \begin{cases} \frac{1}{2\epsilon}, & |t-a| < \epsilon \\ 0, & |t-a| > \epsilon \end{cases}$ for a few different values of ϵ.

Unit Impulse

$$R_\epsilon(t-a) = \begin{cases} \frac{1}{2\epsilon}, & |t-a| < \epsilon \\ 0, & |t-a| > \epsilon \end{cases}$$

An important feature of our family of functions is that they all integrate to 1. For $a > 0$, note that

$$\int_0^\infty R_\epsilon(t-a)\,dt = \int_{a-\epsilon}^{a+\epsilon} \frac{1}{2\epsilon}\,dt = \left.\frac{t}{2\epsilon}\right|_{a-\epsilon}^{a+\epsilon} = \frac{a+\epsilon-(a-\epsilon)}{2\epsilon} = \frac{2\epsilon}{2\epsilon} = 1.$$

It is tempting to try to define our unit impulse, which we'll denote as δ (*delta*), using a limit such as $\delta(t-a) = \lim_{\epsilon\to 0} R_\epsilon(t-a)$. This is often taken as an informal definition. It leads to an interesting observation about this delta *function*. Namely, it's not really a function in the traditional real-numbers-in-real-numbers-out sense. In fact, pointwise this leads to

$$\delta(t-a) = \begin{cases} 0, & t \neq a \\ +\infty, & t = a \end{cases}.$$

Defining the Dirac delta

Recall the Mean Value Theorem (for integrals):

MVT

If g is continuous on $[a, b]$, then there exists t_0 in (a, b) such that

$$\int_a^b g(t)\,dt = g(t_0)(b-a).$$

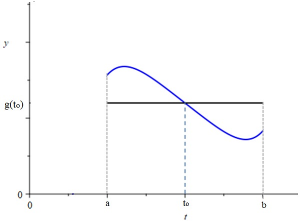


Figure: In the case that $g(t) > 0$, this says that there is some value t_0 such that the rectangle with height $g(t_0)$ and width $b-a$ has the same area as that enclosed by the curve.

Defining the Dirac delta

Now, suppose that g is any continuous function on $[0, \infty)$ and let $a > 0$. For any $\epsilon > 0$, the MVT provides a value $t_{0,\epsilon}$ such that

$$\int_{a-\epsilon}^{a+\epsilon} g(t)\,dt = 2\epsilon g(t_{0,\epsilon}).$$

Then making use of the MVT, we can take the limit

$$\lim_{\epsilon\to 0}\int_0^\infty g(t)R_\epsilon(t-a)\,dt = \lim_{\epsilon\to 0}\int_{a-\epsilon}^{a+\epsilon}\frac{g(t)}{2\epsilon}\,dt = \lim_{\epsilon\to 0} g(t_{0,\epsilon}) = g(a).$$

This is the limit we'll use to define the unit impulse δ. When a continuous function is weigthed by $\delta(t-a)$ and integrated, it will produce the value $g(a)$.

Unit Impulse $\delta(t-a)$

The Dirac delta function a.k.a. Unit Impulse

We will define δ as the limit of the functions R_ϵ in the sense that for each function g that is continuous on $[0,\infty)$ and for any $a \geq 0$,

$$\int_0^\infty g(t)\delta(t-a)\,dt = g(a).$$

Remark 1: The Dirac delta is an example of a *generalized function*, sometimes call a *functional* or a *distribution*.

Remark 2: The property shown here, in which the integral produces the value of g at a, is called a *sifting property*.

Remark 3: Observe that taking $g(t)=e^{-st}$, the statement above provides the Laplace transform for $a \geq 0$

$$\mathscr{L}\left\{\delta(t-a)\right\} = e^{-as}, \quad \text{in particular} \quad \mathscr{L}\left\{\delta(t)\right\} = 1.$$

Delta as a Model of a Unit Impulse

The *function* $\delta(t)$ is used as a model of a force of magnitude 1 applied instantaneously at time $t=0$. Hence a function $f(t)=f_0\delta(t-a)$ can be used to model a force of magnitude f_0 applied instantaneously at the time $t=a$.

For example, suppose our RC series circuit has zero applied voltage for $t\neq t_0$. A switch is closed and opened immediately to apply a voltage V_0 at $t=t_0$. The differential equation modeling the charge on the capacitor is given by

$$R\frac{dq}{dt}+\frac{1}{C}q=V_0\delta(t-t_0).$$

Remark

We can't work with the Dirac delta function the way we might work with other forcing functions (e.g., exponentials or sines and cosines). But we do know what the Laplace transform of $\delta(t-t_0)$ is, so we will be able to solve IVPs that involve differential equations of the form shown here.

Section 16: Laplace Transforms of Derivatives and IVPs

We are ready to use the Laplace transform as a tool for obtaining the solution to an initial value problem. We will consider both single IVPs as well as systems of IVPs. There are additional properties of Laplace transforms that are not under consideration at present. So the limits we will place on its use here can be lifted in other settings.

We will use the Laplace transform for linear IVPs and linear systems of IVPs in which

- the equation(s) are constant coefficient, and
- the initial conditions are prescribed at $t = 0$.

First, let's derive the relationship between the Laplace transform of a function f and the Laplace transforms of its derivatives, f', f'', and so forth.

Transforms of Derivatives

Suppose f has a Laplace transform and that f is differentiable on $[0,\infty)$. Let's obtain an expression for the Laplace tranform of $f'(t)$.

By definition $\quad \mathscr{L}\left\{f'(t)\right\} = \int_0^\infty e^{-st} f'(t)\, dt$

Let us assume that f is of exponential order c for some real c and take[58] $s > c$. Integrate by parts to obtain

$$\begin{aligned} \mathscr{L}\left\{f'(t)\right\} &= \int_0^\infty e^{-st} f'(t)\, dt = f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\, dt \\ &= 0 - f(0) + s\mathscr{L}\left\{f(t)\right\} \\ &= s\mathscr{L}\left\{f(t)\right\} - f(0) \end{aligned}$$

If we let $F(s) = \mathscr{L}\{f(t)\}$, then we can write $\mathscr{L}\{f'(t)\} = sF(s) - f(0)$.

[58] This ensures that $\lim_{t\to\infty} f(t)e^{-st} = 0$.

Transforms of Derivatives

If $\mathscr{L}\{f(t)\} = F(s)$, we have $\mathscr{L}\{f'(t)\} = sF(s) - f(0)$. We can use this relationship recursively to obtain Laplace transforms for higher derivatives of f.

For example

$$\begin{aligned}
\mathscr{L}\{f''(t)\} &= s\mathscr{L}\{f'(t)\} - f'(0) \\
&= s\left(sF(s) - f(0)\right) - f'(0) \\
&= s^2F(s) - sf(0) - f'(0)
\end{aligned}$$

We have $\mathscr{L}\{f'(t)\} = sF(s) - f(0)$ and $\mathscr{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$. Note that the operation of differentiation where the variable t lives corresponds to an algebraic operation, *multiply by some power of s and add a polynomial*, where s lives.

The Laplace Transform of Derivatives

For $y = y(t)$ defined on $[0, \infty)$ having derivatives y', y'' and so forth, if

$$\mathscr{L}\{y(t)\} = Y(s), \quad \text{then}$$

$$\begin{aligned}
\mathscr{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0) \\
\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0) \\
\mathscr{L}\left\{\frac{d^3y}{dt^3}\right\} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \\
&\vdots \\
\mathscr{L}\left\{\frac{d^ny}{dt^n}\right\} &= s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0).
\end{aligned}$$

Notation

Be careful here! Case matters. The characters Y and y do not represent the same thing and can't be used interchangeably. An expression such as $y'(0)$ means the value of the function $y'(t)$ when the input $t = 0$.

Solving an IVP

Let's look at a generic second order, linear IVP with constant coefficients. Suppose a, b, c, y_0 and y_1 are constants with $a \neq 0$. Assume g is some function that has a Laplace transform. Let's obtain the solution of the IVP in terms of a Laplace transform.

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1$$

Let $Y(s) = \mathscr{L}\left\{y(t)\right\}$ and $G(s) = \mathscr{L}\left\{g(t)\right\}$. Take the Laplace transform of both sides of the ODE and apply the linearity property of the transform.

$$\mathscr{L}\left\{ay'' + by' + cy\right\} = \mathscr{L}\left\{g(t)\right\} \quad \Longrightarrow$$

$$a\mathscr{L}\left\{y''\right\} + b\mathscr{L}\left\{y'\right\} + c\mathscr{L}\left\{y\right\} = G(s)$$

Now we use the results for derivatives, $\mathscr{L}\{y'\} = sY(s) - y(0)$ and $\mathscr{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$.

$$a(s^2Y(s) - sy(0) - y'(0)) + b(sY(s) - y(0)) + cY(s) = G(s).$$

Solving an IVP

The original problem includes the conditions $y(0) = y_0$ and $y'(0) = y_1$. We can substitute these into the equation[59].

$$a(s^2Y(s) - sy_0 - y_1) + b(sY(s) - y_0) + cY(s) = G(s).$$

Now we have an algebraic equation for our unknown $Y(s)$. Using whatever algebra is necessary, we isolate the function $Y(s)$.

$$(as^2 + bs + c)Y(s) - asy_0 - ay_1 - by_0 = G(s) \quad \Longrightarrow$$

$$Y(s) = \frac{ay_0s + ay_1 + by_0}{as^2 + bs + c} + \frac{G(s)}{as^2 + bs + c}.$$

Recall that $Y(s) = \mathscr{L}\{y(t)\}$, the Laplace transform of the solution to the original IVP. Applying the inverse Laplace transform, the solution to the IVP

$$y(t) = \mathscr{L}^{-1}\{Y(s)\}.$$

---

[59] Unlike previous methods, the initial conditions are included into the process at this step. Once we get to the end, we have solved the differential equation and applied the initial conditions.

Solving IVPs

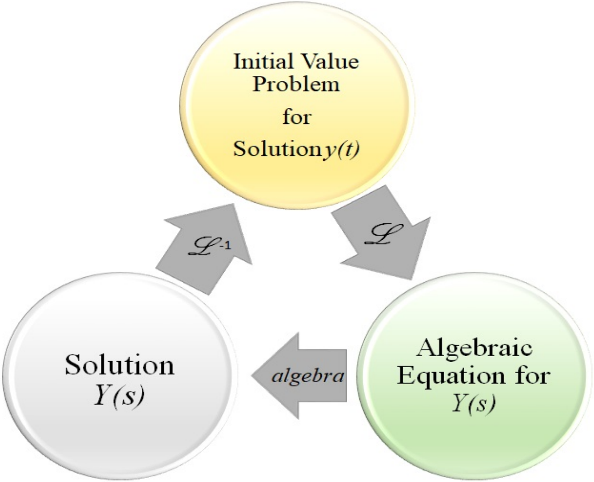


Figure: The Laplace transform converts the IVP into an algebraic equation which is then solved and the result fed into the inverse Laplace transform.

Input & State Responses

Note that our solution has the basic format

$$Y(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)}$$

where Q is a polynomial with coefficients determined by the initial conditions, G is the Laplace transform of $g(t)$ and P is the **characteristic polynomial** of the original equation. The solution y consists of two corresponding terms.

Zero Input Response

$$\mathscr{L}^{-1}\left\{\frac{Q(s)}{P(s)}\right\}$$ is called the **zero input response**,

and

Zero State Response

$$\mathscr{L}^{-1}\left\{\frac{G(s)}{P(s)}\right\}$$ is called the **zero state response**.

Input & State Responses

The zero input and zero state responses are related to the complementary and particular solutions, but they are not quite the same since they are related to initial value problems as opposed to simply the differential equation.

Zero Input Response

The **zero input response** satisfies the initial value problem with homogeneous differential equation and nonhomogeneous initial conditions,

$$ay'' + by' + cy = 0, \quad y(0) = y_0, \quad y'(0) = y_1.$$

Zero State Response

The **zero state response** satisfies the initial value problem with nonhomogeneous differential equation and homogeneous initial conditions,

$$ay'' + by' + cy = g(t), \quad y(0) = 0, \quad y'(0) = 0.$$

Solve the IVP using the Laplace Transform

(a) $\dfrac{dy}{dt} + 3y = 2t \quad y(0) = 2$

Apply the Laplace transform and use the initial condition. Let $Y(s) = \mathscr{L}\{y\}$, and take the Laplace transform of the ODE.

$$\mathscr{L}\{y' + 3y\} = \mathscr{L}\{2t\}$$

Apply rules from the table (including the transform $\mathscr{L}\{y'\} = sY(s) - y(0)$)

$$sY(s) - y(0) + 3Y(s) = \frac{2}{s^2}$$

Substitute in the known value of $y(0)$ given as part of the IVP, and isolate Y.

$$(s+3)Y(s) - 2 = \frac{2}{s^2}$$

$$Y(s) = \frac{2}{s^2(s+3)} + \frac{2}{s+3} = \frac{2s^2+2}{s^2(s+3)}$$

Example Continued...

We use a partial fraction decomposition to facilitate taking the inverse transform.

$$Y(s) = \frac{-\frac{2}{9}}{s} + \frac{\frac{2}{3}}{s^2} + \frac{\frac{20}{9}}{s+3}$$

Take the inverse Laplace transform to get the solution to the IVP.

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = -\frac{2}{9} + \frac{2}{3}t + \frac{20}{9}e^{-3t}.$$

Helpful Suggestion

One can check to see that this satisfies both the ODE $y' + 3y = 2t$ and the initial condition $y(0) = 2$. In practice, it might be time consuming to do this verification, but at least checking the initial condition is a fast way of confirming that the solution is plausible. For this example, we see that $y(0) = -2/9 + 20/9 = 2$ which is encouraging!

Solve the IVP using the Laplace Transform

(b) $y''+4y'+4y = te^{-2t} \quad y(0) = 1, \quad y'(0) = 0$

Sticking with convention, let $Y(s) = \mathscr{L}\{y(t)\}$. Apply the Laplace transform to the equation and use the table and properties. Note that the right side will require the s-shift theorem, $F(s+2)$ for $F(s) = \mathscr{L}\{t\} = \frac{1}{s^2}$.

$$\mathscr{L}\{y'' + 4y' + 4y\} = \mathscr{L}\{te^{-2t}\}$$

$$s^2Y(s) - sy(0) - y'(0) + 4sY(s) - 4y(0) + 4Y(s) = \frac{1}{(s+2)^2}$$

Isolate $Y(s)$ using whatever algebra is necessary[60].

$$(s^2 + 4s + 4)Y(s) - s - 4 = \frac{1}{(s+2)^2}$$

$$Y(s) = \frac{1}{(s+2)^4} + \frac{s+4}{(s+2)^2}.$$

---

[60] Note that the coefficient of Y before the last division step is the characteristic polynomial. That has to be the case.

Example Continued...

The first term is managable, but we can perform a partial fraction decomposition (or other clever algebra) on the last ratio.

$$Y(s) = \frac{1}{(s+2)^4} + \frac{1}{s+2} + \frac{2}{(s+2)^2}$$

The first and last terms will require our s-shift theorem. For example,

$$\mathscr{L}^{-1}\left\{\frac{1}{(s+2)^4}\right\} = e^{-2t}\mathscr{L}^{-1}\left\{\frac{1}{s^4}\right\} = \frac{1}{3!}e^{-2t}\mathscr{L}^{-1}\left\{\frac{3!}{s^4}\right\} = \frac{1}{3!}e^{-2t}t^3.$$

Finally, the solution to the IVP

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = \frac{1}{3!}t^3e^{-2t} + e^{-2t} + 2te^{-2t}.$$

Quick Check

A quick check, not requiring any computations, shows that $y(0) = 1$. With a little more time and effort, we can verify that our solution satisfies the ODE and the $y'(0) = 0$ condition as well.

An IVP with Piecewise Input

One of the most powerful uses of the Laplace transform is in applications that involve a piecewise defined forcing function. Let's look at an example.

An LR-series circuit has inductance $L = 1$h, resistance $R = 10\Omega$, and implied voltage $E(t)$ whose graph is given below. If the initial current $i(0) = 0$, find the current $i(t)$ in the circuit.

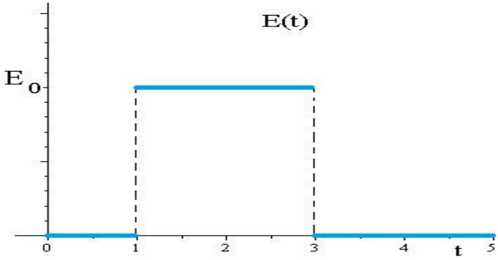
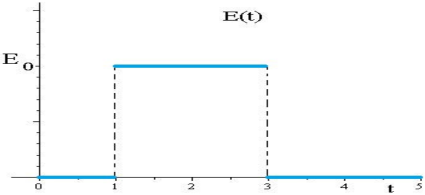


Figure: A switch is closed for two seconds from $t = 1$ until $t = 3$ during which a constant E_0 volts is applied.

An IVP with Piecewise Input



We can express the voltage in stacked notation and in terms of the unit step function[61].

$$E(t) = \begin{cases} 0, & 0 \leq t < 1 \\ E_0, & 1 \leq t < 3 \\ 0, & 3 \leq 3 \end{cases} = 0 - 0\mathscr{U}(t-1) + E_0\mathscr{U}(t-1) - E_0\mathscr{U}(t-3) + 0\mathscr{U}(t-3)$$

Which simplifies to

$$E(t) = E_0\mathscr{U}(t-1) - E_0\mathscr{U}(t-3).$$

[61] It's not really clear from the figure which value the function takes at $t = 1$ and $t = 3$. We'll assume continuity from the right and will not be overly concerned with the behavior at these single points.

LR Circuit Example

Recalling that the basic model is $L\frac{di}{dt} + Ri = E$, the IVP can be stated as

$$\frac{di}{dt} + 10i = E_0\mathscr{U}(t-1) - E_0\mathscr{U}(t-3), \quad i(0) = 0.$$

Letting $I(s) = \mathscr{L}\{i(t)\}$, we apply the Laplace transform to obtain

$$\mathscr{L}\{i' + 10i\} = \mathscr{L}\{E_0\mathscr{U}(t-1) - E_0\mathscr{U}(t-3)\}$$

$$sI(s) - i(0) + 10I(s) = \frac{E_0e^{-s}}{s} - \frac{E_0e^{-3s}}{s}$$

Solving for I, we have the Laplace transform of the current,

$$I(s) = \frac{E_0}{s(s+10)}\left(e^{-s} - e^{-3s}\right).$$

Example Continued...

We can perform a partial fraction decomposition on the rational factor and recover the current i. Omitting the details, we find that

$$\frac{E_0}{s(s+10)} = \frac{\frac{E_0}{10}}{s} - \frac{\frac{E_0}{10}}{s+10}.$$

Hence,

$$\begin{aligned} I(s) &= \left[\frac{\frac{E_0}{10}}{s} - \frac{\frac{E_0}{10}}{s+10}\right](e^{-s} - e^{-3s}) \\ &= \frac{E_0}{10}\frac{e^{-s}}{s} - \frac{E_0}{10}\frac{e^{-s}}{s+10} - \frac{E_0}{10}\frac{e^{-3s}}{s} + \frac{E_0}{10}\frac{e^{-3s}}{s+10} \end{aligned}$$

And finally

$$\begin{aligned} i(t) &= \mathscr{L}^{-1}\{I(s)\} \\ &= \frac{E_0}{10}\left(1 - e^{-10(t-1)}\right)\mathscr{U}(t-1) - \frac{E_0}{10}\left(1 - e^{-10(t-3)}\right)\mathscr{U}(t-3). \end{aligned}$$

Alternative Approach using Convolution

Note that in the last example, an alternative to using a partial fraction decomposition would be use of a convolution. We needed to determine

$$\mathscr{L}^{-1}\left\{\frac{E_0}{s(s+10)}\right\} = \mathscr{L}^{-1}\left\{\frac{E_0}{s}\left(\frac{1}{s+10}\right)\right\}$$

Letting

$$F(s) = \frac{E_0}{s} = \mathscr{L}\{E_0\} \quad \text{and} \quad G(s) = \frac{1}{s+10} = \mathscr{L}\{e^{-10t}\}.$$

We can compute the inverse transform as $(f * g)(t)$ with $f(t) = E_0$ and $g(t) = e^{-10t}$.

Note that

$$\begin{aligned}(f*g)(t) &= \int_0^t E_0 e^{-10(t-\tau)}\,d\tau\\ &= E_0 e^{-10t}\int_0^t e^{10\tau}\,d\tau\\ &= E_0 e^{-10t}\left[\frac{1}{10}e^{10\tau}\right|_0^t\\ &= E_0 e^{-10t}\left(\frac{1}{10}e^{10t}-\frac{1}{10}\right)\\ &= \frac{E_0}{10}-\frac{E_0}{10}e^{-10t}\end{aligned}$$

Letting $\hat{f}(t)=(f*g)(t)$, we have

$$\begin{aligned}i(t) &= \hat{f}(t-1)\mathscr{U}(t-1)-\hat{f}(t-3)\mathscr{U}(t-3)\\ &= \frac{E_0}{10}\left(1-e^{-10(t-1)}\right)\mathscr{U}(t-1)-\frac{E_0}{10}\left(1-e^{-10(t-3)}\right)\mathscr{U}(t-3),\end{aligned}$$

which matches the solution we found using partial fractions.

Representation of the Solution

We can express our piecewise defined solution in the traditional, stacked notation. To do so, we have to consider t in the intervals $[0,1)$, $[1,3)$ and $[3,\infty)$. Note that when $0 \le t < 1$, both $\mathscr{U}(t-1) = 0$ and $\mathscr{U}(t-3) = 0$. Hence $i(t) = 0$ when $0 \le t < 1$. When $1 \le t < 3$, we have $\mathscr{U}(t-1) = 1$ while $\mathscr{U}(t-3) = 0$. Hence

$$i(t) = \frac{E_0}{10}\left(1 - e^{-10(t-1)}\right), \quad \text{when} \quad 1 \le t < 3.$$

Finally, when $t \ge 3$, both $\mathscr{U}(t-1) = 1$ and $\mathscr{U}(t-3) = 1$. Thus

$$i(t) = \frac{E_0}{10}\left(1 - e^{-10(t-1)}\right) - \frac{E_0}{10}\left(1 - e^{-10(t-3)}\right) = \frac{E_0}{10}\left(e^{-10(t-3)} - e^{-10(t-1)}\right), \text{ when } t \ge 3.$$

The current is therefore

$$i(t) = \begin{cases} 0, & 0 \le t < 1 \\ \frac{E_0}{10}\left(1 - e^{-10(t-1)}\right), & 1 \le t < 3 \\ \frac{E_0}{10}\left(e^{-10(t-3)} - e^{-10(t-1)}\right), & t \ge 3 \end{cases}$$

Exercise: Show that i is continuous on $[0,\infty)$. (Left to the reader.)

IVPs Involving the Unit Impulse

The Laplace transform can be used to determine the behavior of a system subjected to an instantaneous force. Recall that such a force is modeled by the Dirac delta function. Let's look at an example.

A 1 kg mass is suspended from a spring with spring constant 10 N/m. A damper induces damping of 6 N per m/sec of velocity. The object starts from rest from a position 10 cm above equilibrium. At time $t = 1$ second, a unit impulse force is applied to the object. Determine the object's position for $t > 0$.

Recall that the Dirac delta function $\delta(t-a)$ models a unit impulse force applied at time $t = a$. The IVP for the situation[62] described is

$$x'' + 6x' + 10x = \delta(t-1), \quad x(0) = 0.1, \quad x'(0) = 0$$

We can solve this IVP using the Laplace transform making use of the fact that $\mathscr{L}\{\delta(t-a)\} = e^{-as}$.

[62] The basic model of a forced, spring-mass-damper is $mx'' + bx' + kx = f(t)$.

$$x'' + 6x' + 10x = \delta(t-1), \quad x(0) = 0.1, \quad x'(0) = 0$$

Let $X(s) = \mathscr{L}\{x(t)\}$. Taking the transform of the ODE

$$\mathscr{L}\{x'' + 6x' + 10x\} = \mathscr{L}\{\delta(t-1)\}$$

$$s^2X(s) - sx(0) - x'(0) + 6\left(sX(s) - x(0)\right) + 10X(s) = e^{-1s}$$

Applying the initial conditions and solving for X, we find

$$X(s) = \frac{e^{-s}}{s^2+6s+10} + \frac{0.1s+0.6}{s^2+6s+10}.$$

To take the inverse transform, we complete the square on the denominator (as it doesn't factor).

$$s^2 + 6s + 10 = (s+3)^2 + 1$$

$$x'' + 6x' + 10x = \delta(t-1), \quad x(0) = 0.1, \quad x'(0) = 0$$

Hence

$$X(s) = \frac{e^{-s}}{(s+3)^2+1} + \frac{0.1s+0.6}{(s+3)^2+1}.$$

We see that there is an s-shift result, but this requires $s+3$ in place of every s. The numerator for the second term can be written in terms of $s+3$ using

$$0.1s + 0.6 = 0.1(s+3-3) + 0.6 = 0.1(s+3) + 0.3.$$

We finally have X in a format from which we can take the inverse transform.

$$X(s) = \frac{e^{-s}}{(s+3)^2+1} + 0.1\frac{s+3}{(s+3)^2+1} + 0.3\frac{1}{(s+3)^2+1}$$

$$x'' + 6x' + 10x = \delta(t-1), \quad x(0) = 0.1, \quad x'(0) = 0$$

Note that using the s-shift result

$$\mathscr{L}^{-1}\left\{\frac{s+3}{(s+3)^2+1}\right\} = e^{-3t}\mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = e^{-3t}\cos(t),$$

and similarly

$$\mathscr{L}^{-1}\left\{\frac{1}{(s+3)^2+1}\right\} = e^{-3t}\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = e^{-3t}\sin(t).$$

Using the shift in t theorem with $f(t) = e^{-3t}\sin(t)$, we have

$$\mathscr{L}^{-1}\left\{\frac{e^{-s}}{(s+3)^2+1}\right\} = f(t-1)\mathscr{U}(t-1) = e^{-3(t-1)}\sin(t-1)\mathscr{U}(t-1).$$

$$x'' + 6x' + 10x = \delta(t-1), \quad x(0) = 0.1, \quad x'(0) = 0$$

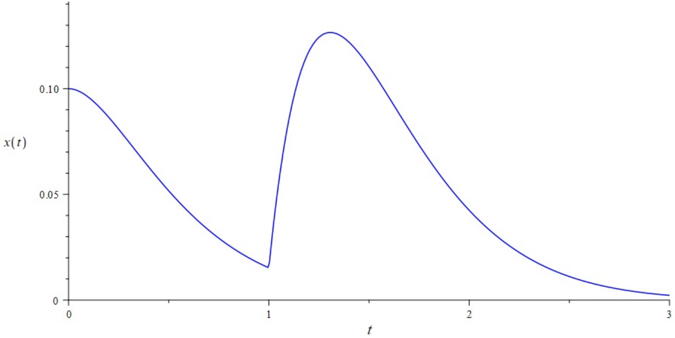
Pulling all of this together, the displacement

$$\begin{aligned} x(t) &= \mathscr{L}^{-1}\{X(s)\} \\ &= \mathscr{L}^{-1}\left\{\frac{e^{-s}}{(s+3)^2+1} + 0.1\frac{s+3}{(s+3)^2+1} + 0.3\frac{1}{(s+3)^2+1}\right\} \\ &= e^{-3(t-1)}\sin(t-1)\mathscr{U}(t-1) + 0.1e^{-3t}\cos(t) + 0.3e^{-3t}\sin(t). \end{aligned}$$

The displacement $x(t)$ for all $t > 0$ is given by

$$x(t) = e^{-3(t-1)}\sin(t-1)\mathscr{U}(t-1) + 0.1e^{-3t}\cos(t) + 0.3e^{-3t}\sin(t).$$

$$x'' + 6x' + 10x = \delta(t-1), \quad x(0) = 0.1, \quad x'(0) = 0$$



$$x(t) = e^{-3(t-1)}\sin(t-1)\mathscr{U}(t-1) + 0.1e^{-3t}\cos(t) + 0.3e^{-3t}\sin(t)$$

Figure: Graph of the solution to the IVP with unit impulse external force at $t = 1$.

To get a visual idea of the behavior, the software Maple™was used to produce a plot of x. The effect of the force is clearly seen at $t = 1$ and appears to result in a point at which differentiability of the solution breaks down (a corner). Note that the initial conditions, $x(0) = 0.1$ and $x'(0) = 0$ are clearly visible in this plot.

Solving a System

Let's turn our attention to systems of initial value problems that are

- linear,
- constant coefficient, and
- have initial conditions at $t = 0$.

As with a single DE, the Laplace transform will convert a system of differential equations to an algebraic system of equations. We will solve that system using what ever algebra is necessary and then apply the inverse transform. Here, we will have at least two dependent variables.

Let's see it in action (i.e. with a couple of examples).

Example

Solve the system of equations

$$\begin{aligned}\frac{dx}{dt} &= -2x-2y+60, \quad x(0)=0\\ \frac{dy}{dt} &= -2x-5y+60, \quad y(0)=0\end{aligned}$$

We'll use the Laplace transforms. We have two dependent variables, but we will stick with our usual upper case/lower case convention for Laplace transforms. That is, we set

$$X(s)=\mathscr{L}\{x(t)\}, \quad \text{and} \quad Y(s)=\mathscr{L}\{y(t)\}.$$

Example Continued...

Applying the transform to both sides of both equations, we obtain an algebraic system of equations for X and Y.

$$\begin{aligned} sX(s) - x(0) &= -2X(s) - 2Y(s) + \frac{60}{s} \\ sY(s) - y(0) &= -2X(s) - 5Y(s) + \frac{60}{s} \end{aligned}$$

We can subsitute in the initial conditions, and rearrange the equations to get our system in a convenient form

$$\begin{aligned} (s+2)X(s) + 2Y(s) &= \frac{60}{s} \\ 2X(s) + (s+5)Y(s) &= \frac{60}{s} \end{aligned}$$

Remark:

Typically, it's advantageous to write the system for our two unknowns in the form $\begin{array}{ccccc} aX(s) & + & bY(s) & = & e \\ cX(s) & + & dY(s) & = & f \end{array}$ where the a, b, c, d, e, f may depend on s but don't depend on X or Y in any way.

Example Continued...

$$\begin{aligned}(s+2)X(s)+2Y(s) &= \frac{60}{s}\\ 2X(s)+(s+5)Y(s) &= \frac{60}{s}\end{aligned}$$

We can solve this system in any number of ways. For those familiar with it, Cramer's Rule is probably the easiest approach. Elimination will work just as well. We find

$$\begin{aligned}X(s) &= \frac{60(s+3)}{s(s+1)(s+6)}\\ Y(s) &= \frac{60}{(s+1)(s+6)}\end{aligned}$$

As is usually the case, a partial fraction decomposition will give us a form from which we can take the inverse transform using the table.

Example Continued...

Performing a PFD, and omitting the details, we find that

$$\begin{aligned} X(s) &= \frac{30}{s} - \frac{24}{s+1} - \frac{6}{s+6} \\ Y(s) &= \frac{12}{s+1} - \frac{12}{s+6} \end{aligned}$$

Finally, we take the inverse transform to obtain the solution to the system.

$$\begin{aligned} \mathscr{L}^{-1}\{X(s)\} &= 30 - 24e^{-t} - 6e^{-6t} \\ \mathscr{L}^{-1}\{Y(s)\} &= 12e^{-t} - 12e^{-6t} \end{aligned}$$

The solution to the system of initial value problems is

$$\begin{aligned} x(t) &= 30 - 24e^{-t} - 6e^{-6t} \\ y(t) &= 12e^{-t} - 12e^{-6t} \end{aligned}$$

A quick check that can be done without computation is to notice that $x(0) = y(0) = 0$ which does match the initial conditions. With more effort, one can verify that this pair does satisfy both differential equations.

Example

Use the Laplace transform to solve the system of equations

$$\begin{aligned} x''(t) &= y, \quad x(0) = 1, \quad x'(0) = 0 \\ y'(t) &= x, \quad y(0) = 1 \end{aligned}$$

This system is second order. Note that the number of initial conditions for each variable matches the highest order derivative for that variable. Sticking with the upper-lowercase convention, we take the Laplace transform of both equations to obtain

$$\begin{aligned} s^2X(s) - sx(0) - x'(0) &= Y(s) \\ sY(s) - y(0) &= X(s) \end{aligned}$$

As before, we substitute in the given initial conditions and rearrange the equations into a convenient format.

$$\begin{aligned} s^2X(s) - Y(s) &= s \\ -X(s) + sY(s) &= 1 \end{aligned}$$

Example Continued...

Using some method to solve for X and Y (e.g., Cramer's Rule), we obtain

$$X(s) = \frac{s^2+1}{s^3-1}, \quad \text{and} \quad Y(s) = \frac{s^2+s}{s^3-1}.$$

It will be necessary to perform a PFD on each ratio, and it is useful to recall the difference of cubes factorization

$$s^3 - 1 = (s-1)(s^2+s+1).$$

We have

$$X(s) = \frac{s^2+1}{(s-1)(s^2+s+1)}, \quad \text{and} \quad Y(s) = \frac{s(s+1)}{(s-1)(s^2+s+1)}.$$

The decomposition is a bit tedious. The quadratic factor is irreducible, so both decompositions will have the general form

$$\frac{A}{s-1} + \frac{Bs+C}{s^2+s+1}.$$

Example Continued...

With a little effort, we obtain the decomposition

$$X(s) = \frac{2/3}{s-1} + \frac{1/3(s-1)}{s^2+s+1}, \quad \text{and} \quad Y(s) = \frac{2/3}{s-1} + \frac{1/3(s+2)}{s^2+s+1}.$$

Completing the square on the quadratic factor

$$s^2 + s + 1 = \left(s + \frac{1}{2}\right)^2 + \frac{3}{4},$$

shows that there will be an s-shift result of the form $F\left(s + \frac{1}{2}\right)$. To obtain $s + \frac{1}{2}$ in place of each incidence of s in the terms with the quadratics, we write

$$s - 1 = s + \frac{1}{2} - \frac{3}{2}, \quad \text{and} \quad s + 2 = s + \frac{1}{2} + \frac{3}{2}.$$

It's useful to note that

$$\left(s + \frac{1}{2}\right)^2 + \frac{3}{4} = \left(s + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2.$$

Example Continued...

Finally, we can write out X and Y with all terms in a form that facilitates the inverse Laplace transform.

$$X(s) = \frac{\frac{2}{3}}{s-1} + \frac{1}{3}\frac{\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} - \frac{1}{\sqrt{3}}\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$Y(s) = \frac{\frac{2}{3}}{s-1} + \frac{1}{3}\frac{\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} + \frac{1}{\sqrt{3}}\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$$

We've multiplied by $\frac{\sqrt{3}}{\sqrt{3}}$ on the end to obtain ratios that look like $\frac{k}{s^2+k^2}$. We'll use the observations

$$\mathscr{L}^{-1}\left\{\frac{\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\} = e^{-t/2}\mathscr{L}^{-1}\left\{\frac{s}{s^2+\frac{3}{4}}\right\} = e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right), \quad \text{and}$$

$$\mathscr{L}^{-1}\left\{\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\} = e^{-t/2}\mathscr{L}^{-1}\left\{\frac{\frac{\sqrt{3}}{2}}{s^2+\frac{3}{4}}\right\} = e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right).$$

Example Continued...

Finally, the solution to the initial value problem is

$$\begin{aligned} x(t) &= \frac{2}{3}e^{t}+\frac{1}{3}e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right)-\frac{1}{\sqrt{3}}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right) \\ y(t) &= \frac{2}{3}e^{t}+\frac{1}{3}e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right)+\frac{1}{\sqrt{3}}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right) \end{aligned}$$

A full verification of the accuracy of this solution can be done with some time and effort. But we can see without tedious computations that $x(0)=1$ and $y(0)=1$, so our answer does satisfy at least that part of the initial conditions.

To complete our treatment of Laplace transforms, we close by discussing two mathematical objects commonly used in signal processing and control theory to characterize the relationship between a system input and output. These are the *transfer function* and the *impulse response*. We'll consider the case of a second order equation, but this can be extended to higher order equations.

Transfer Function & Impulse Response

$$ay'' + by' + cy = g(t), \tag{27}$$

Definition

The function $H(s) = \dfrac{1}{as^2 + bs + c}$ is called the **transfer function** for the differential equation (27). Note that H has the characteristic polynomial as its denominator.

Definition

The **impulse response** function, $h(t)$, for the differential equation (27) is the inverse Laplace transform of the transfer function

$$h(t) = \mathscr{L}^{-1}\{H(s)\} = \mathscr{L}^{-1}\left\{\frac{1}{as^2 + bs + c}\right\}.$$

Transfer Function & Impulse Response

$$ay'' + by' + cy = g(t)$$

Remark 1:

The transfer function is the Laplace transform of the solution to the IVP

$$ay'' + by' + cy = \delta(t), \quad y(0) = 0, \quad y'(0) = 0.$$

This is the **zero state response** when the forcing function is a Dirac delta function at time zero.

Remark 2:

The impulse response is the solution to the IVP

$$ay'' + by' + cy = \delta(t), \quad y(0) = 0, \quad y'(0) = 0.$$

Convolution & The Zero State Response

Consider

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1$$

Recall the **zero state response** is the inverse transform $\mathscr{L}^{-1}\left\{\frac{G(s)}{as^2+bs+c}\right\}$. Note that we can write the ratio as the product

$$\frac{G(s)}{as^2+bs+c} = G(s)H(s)$$

where H is the transfer function. If the impulse response is $h(t)$, then the zero state response, for any choice of forcing function g, can be written in terms of a convolution as

$$\mathscr{L}^{-1}\left\{G(s)H(s)\right\} = \int_0^t g(\tau)h(t-\tau)\,d\tau.$$

Example

Express the zero state response of the following system in terms of a convolution integral.

$$y'' + 2y' + y = g(t).$$

Let's find the transfer function, H, and impulse response, $h(t)$. We have

$$H(s) = \frac{1}{s^2 + 2s + 1} = \frac{1}{(s+1)^2}.$$

So the impulse response

$$h(t) = \mathscr{L}^{-1}\left\{H(s)\right\} = \mathscr{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = e^{-t}\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} = e^{-t}t.$$

Note then that

$$h(t-\tau) = (t-\tau)e^{-(t-\tau)}.$$

We can express the zero state response as

$$\int_0^t g(\tau)(t-\tau)e^{-(t-\tau)}\,d\tau.$$

Section 17: Fourier Series: Trigonometric Series

In these last two sections, we will introduce a tool that can be used to represent a periodic function as a series. Various applications in science and engineering involve periodic forcing or complex signals that can be considered sums of more elementary parts (e.g. harmonics).

- Signal processing (decomposing/reconstructing sound waves or voltage inputs)
- Control theory (qualitative assessment/control of dynamics)
- Approximation of forces or solutions of differential equations

A variety of interesting waveforms (periodic curves) arise in applications and can be expressed by series representations. Some of the most common examples are shown below.

Common Models of Periodic Sources (e.g. Voltage)

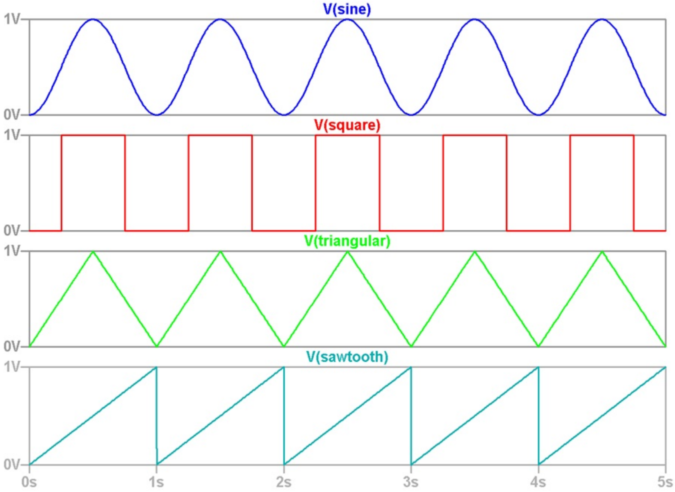


Figure: We'd like to solve, or at least approximate solutions, to ODEs and PDEs with periodic *right hand sides*.

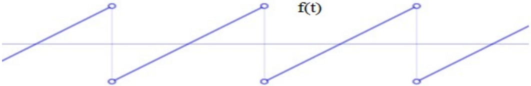
Motivating Example

An undamped spring mass system has a mass of 2 kg attached to a spring with spring constant 128 N/m. The mass is driven by an external force $f(t) = 2t$ for[a] $-1 < t < 1$ that is 2-periodic so that $f(t+2) = f(t)$ for all $t > 0$. The displacement satisfies the ODE

$$2\frac{d^2x}{dt^2} + 128x = f(t),$$

where f is shown graphically (a *sawtooth* wave) below.

[a] It might be more reasonable to define f on $(0, 1)$ and describe it as being extended to be odd.



Question: How can we deal with a right hand side with infinitely many pieces? (Laplace transforms are an option here, but we'll come back and approach this problem using a Fourier series.)

Series Representations for Functions

The goal is to represent a function by a series

$$f(x) = \sum (\text{some } \textit{simple} \text{ functions})$$

In calculus, you saw power series $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ where the simple functions were powers $(x-c)^n$.

Here, you will see how select functions can be written as series of trigonometric functions, for example something like

$$f(x) = \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)\,.$$

Remark 1: We'll move the $n = 0$ term to the front before the rest of the sum for convenience.

Remark 2: We can also consider slight variations on this theme replacing the argument of the sine/cosine with something like $n\pi x$, or $\frac{n\pi x}{2}$, or more generally $\frac{n\pi x}{p}$ for fixed number p.

Some Preliminary Concepts

Definition: Inner Product

Suppose two functions f and g are integrable on the interval $[a, b]$. We define the **inner product** of f and g on $[a, b]$ as[a]

$$\langle f, g \rangle = \int_a^b f(x)g(x)\,dx.$$

[a] The triangular brackets, $\langle \cdot, \cdot \rangle$ are a common notation for this type of product.

Definition: Orthogonal

We say that f and g are **orthogonal** on $[a, b]$ if

$$\langle f, g \rangle = 0.$$

Note that an interval must be specified, and the property of being *orthogonal* is specific to the interval.

Properties of an Inner Product

Remark

More generally, and inner product is a map that assigns a *number* (scalar) to a pair of inputs (called vectors). You may be familiar with other types of inner products (e.g., dot product on vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$). To be called an **inner product**, this map must satisfy four properties. These are stated below in reference to the inner product we are considering here. These properties are the defining factors for any real valued inner product.

Properties of the Inner Product

Let f, g, and h be integrable functions on the interval $[a, b]$ and let c be any real number. Then

(i) $\langle f, g\rangle = \langle g, f\rangle$

(ii) $\langle f, g + h\rangle = \langle f, g\rangle + \langle f, h\rangle$

(iii) $\langle cf, g\rangle = c\langle f, g\rangle$

(iv) $\langle f, f\rangle \geq 0$ and $\langle f, f\rangle = 0$ if and only if $f = 0$

Orthogonal Set

We have defined what it means to say that a pair of functions is orthogonal. Now, we want to define what it means for a **set** of functions to be orthogonal.

Definition: Orthogonal Set

Suppose the set $S = \{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ consists of functions that are integrable on the interval $[a, b]$. The set S is said to be an **orthogonal set** if

$$\langle \phi_m, \phi_n \rangle = \int_a^b \phi_m(x)\phi_n(x)\, dx = 0 \quad \text{whenever} \quad m \neq n.$$

Note, the set is an orthogonal set if each pair of distinct functions is orthogonal.

Norm

The fourth property of the inner product provides a way to define a **norm** on a set of functions. A norm is geometric sort of notion, an analog of the distance between a number and zero (absolute value or modulus) or the length (magnitude) of a vector in $\mathbb{R}^2$ or $\mathbb{R}^n$.

Any function $\phi(x)$ that is not identically zero[63] on $[a, b]$ will satisfy

$$\langle \phi, \phi \rangle = \int_a^b \phi^2(x)\, dx > 0.$$

We define the **square norm** of ϕ (on $[a, b]$) to be

$$\|\phi\| = \sqrt{\int_a^b \phi^2(x)\, dx}.$$

That is $\|\phi\| = \sqrt{\langle \phi, \phi \rangle}$ i.e., $\|\phi\|^2 = \langle \phi, \phi \rangle$.

[63] To be precise, we should say *is not zero except possibly on a set of measure zero*, but this is getting beyond the scope of the discussion.

An Orthogonal Set of Functions

Consider the following set of functions on the interval $[-\pi, \pi]$.

$$\{1, \cos x, \cos 2x, \cos 3x, \dots, \sin x, \sin 2x, \sin 3x, \dots\}$$

It is readily shown that

$$\int_{-\pi}^{\pi} \cos nx\, dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \sin mx\, dx = 0 \quad \text{for all} \quad n, m \geq 1.$$

Note what this says. This says that

$$\boxed{\langle 1, \cos(nx)\rangle = 0}, \quad \text{and} \quad \boxed{\langle 1, \sin(mx)\rangle = 0}$$

for any choices of the integer n or m. In other words,

> The functions 1 and $\cos(nx)$ are orthogonal on $[-\pi, \pi]$ for all integers $n \geq 1$, and similarly 1 and $\sin(mx)$ are orthogonal on $[-\pi, \pi]$ for all integers m. Sometimes this property of being orthogonal to the constant function $\phi(x) = 1$ is called being *orthononal to unity*.

An Orthogonal Set of Functions

Still considering the set of functions on $[-\pi, \pi]$.

$$\{1, \cos x, \cos 2x, \cos 3x, \ldots, \sin x, \sin 2x, \sin 3x, \ldots\}$$

It can be shown that for any choice of the integers m and n,

$$\int_{-\pi}^{\pi} \cos nx \, \sin mx \, dx = 0, \quad \text{i.e.,} \quad \boxed{\langle \cos(nx), \sin(mx)\rangle = 0}.$$

If we consider taking two cosines, or two sines, we have

$$\int_{-\pi}^{\pi} \cos nx \, \cos mx \, dx = \int_{-\pi}^{\pi} \sin nx \, \sin mx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & n = m \end{cases}.$$

Expressed in terms of the inner product and the norm, we have for $n, m \geq 1$

$$\boxed{\langle \cos(nx), \cos(mx)\rangle = 0, \quad n \neq m} \quad \text{and} \quad \boxed{\| \cos(nx)\|^2 = \pi}.$$

Similarly

$$\boxed{\langle \sin(nx), \sin(mx)\rangle = 0, \quad n \neq m} \quad \text{and} \quad \boxed{\| \sin(nx)\|^2 = \pi}.$$

An Orthogonal Set of Functions

An Orthogonal Set on $[-\pi, \pi]$

These results indicate that the set of functions

$$\{1, \cos x, \cos 2x, \cos 3x, \ldots, \sin x, \sin 2x, \sin 3x, \ldots \mid -\pi \le x \le \pi\}$$

is an orthogonal set. Moreover,

$$\|\cos(nx)\|^2 = \pi, \quad \|\sin(nx)\|^2 = \pi, \quad \text{and} \quad \|1\|^2 = 2\pi.$$

Remark:

Note that taking $1 = \cos(0x)$, these are the *simple functions* that make up the terms in the series alluded to before, namely

$$f(x) = \sum_{n=0}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right).$$

Fourier Series

Fourier Series Representation

Suppose $f(x)$ is defined on the interval $[-\pi, \pi]$ (or perhaps $(-\pi, \pi)$) and is either not defined outside of this interval or is 2π-periodic. We would like to find a series representation for f of the form[a]

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right). \tag{28}$$

We will call this series the **Fourier series representation of f.**

[a] The constant term is moved to the front and the factor $\frac{1}{2}$ is included for convenience.

Task: Determine the cofficients, the numbers $a_0, a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ so that $f(x)$ is *equal to*[64] the series shown in equation (28).

[64] The phrase "equal to" is being used imprecisely here. The construction given here will not guarantee the series converges to $f(x)$ at each x.

Fourier Series Representation

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right).$$

Remark 1: The question of convergence naturally arises when we wish to work with infinite series. To highlight convergence considerations, some authors prefer not to use the equal sign when expressing a Fourier series and instead write

$$f(x) \sim \frac{a_0}{2} + \cdots$$

Herein, we'll use the equal sign with the understanding that equality may not hold at each point, and a statement on convergence will be made later.

Remark 2: The factor $\frac{1}{2}$ is included here so that the formula for a_0 is similar to the formulas for a_n and b_n for $n \geq 1$. While this is very common, it is not universal, and you may find sources that write the general series format as $a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$. You'll note that such sources will give a slightly different formula for a_0 than the one given here.

Finding an Example Coefficient

To motivate the formulas for the coefficients of our Fourier series, we'll go through a formal derivation of one such coefficient. Suppose f is some known function defined on $[-\pi, \pi]$ and that f has a Fourier series representation given by (28).

We will find the coefficient b_4, the coefficient of $\sin(4x)$. To begin, we mulitply both sides of equation (28) by $\sin 4x$ and distribute on the right to obtain

$$f(x)\sin 4x = \frac{a_0}{2}\sin 4x + \sum_{n=1}^{\infty}\left(a_n \cos nx \sin 4x + b_n \sin nx \sin 4x\right). \qquad (29)$$

Now, we wish to integrate both sides of (29) from $-\pi$ to π. We will assume that it is valid to swap the order[65] of summation and integration to obtain

$$\int_{-\pi}^{\pi} f(x)\sin 4x\,dx = \int_{-\pi}^{\pi}\frac{a_0}{2}\sin 4x\,dx + \sum_{n=1}^{\infty}\left(\int_{-\pi}^{\pi} a_n \cos nx \sin 4x\,dx + \int_{-\pi}^{\pi} b_n \sin nx \sin 4x\,dx\right).$$

[65] It is not universally valid to interchange the order of summation and integration of an infinite series. However, we will assume that it is valid in this case.

The constants, a_0, a_n, b_n, can be factored out of the integrals

$$\int_{-\pi}^{\pi} f(x)\sin 4x\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi} \sin 4x\,dx + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi} \cos nx\sin 4x\,dx + b_n\int_{-\pi}^{\pi} \sin nx\sin 4x\,dx\right). \tag{30}$$

Recalling that the set of functions $\{1, \cos(x), \cos(2x), \ldots, \sin(x), \sin(2x), \ldots\}$ is orthogonal on $[-\pi, \pi]$, most of the terms on the right vanish. In particular, all of the integrals with coefficients a_0 and a_n are zero.

$$\frac{a_0}{2}\underbrace{\int_{-\pi}^{\pi}\sin 4x\,dx}_{\langle 1,\sin(4x)\rangle=0} + \sum_{n=1}^{\infty}\left(a_n\underbrace{\int_{-\pi}^{\pi} \cos nx\sin 4x\,dx}_{\langle \cos(nx),\sin(4x)\rangle=0} + b_n\int_{-\pi}^{\pi} \sin nx\sin 4x\,dx\right).$$

Hence (30) reduces to

$$\int_{-\pi}^{\pi} f(x)\sin 4x\,dx = \sum_{n=1}^{\infty}\left(b_n\int_{-\pi}^{\pi} \sin nx\sin 4x\,dx\right). \tag{31}$$

Next, recall that $\langle \sin(nx), \sin(4x) \rangle = 0$ whenever $n \neq 4$, and $\langle \sin(4x), \sin(4x) \rangle = \pi$. We can write (31) as

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)\sin 4x\,dx &= \sum_{n=1}^{\infty} b_n \langle \sin nx \sin 4x \rangle \\
&= b_1 \langle \sin x \sin 4x \rangle + \cdots + b_4 \langle \sin 4x \sin 4x \rangle + b_5 \langle \sin 5x \sin 4x \rangle + \cdots \\
&= b_1 \underbrace{\langle \sin x \sin 4x \rangle}_{=0} + \underbrace{\cdots}_{=0} + b_4 \underbrace{\langle \sin 4x \sin 4x \rangle}_{=\pi} + b_5 \underbrace{\langle \sin 5x \sin 4x \rangle}_{=0} + \underbrace{\cdots}_{=0}
\end{aligned}$$

Hence (31) reduces to

$$\int_{-\pi}^{\pi} f(x)\sin 4x\,dx = \pi b_4$$

from which we determine

$$\boxed{b_4 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin 4x\,dx.}$$

There was nothing special about seeking the 4^{th} sine coefficient b_4. We could have just as easily sought b_m for any positive integer m. We would simply start by introducing the factor $\sin(mx)$.

Moreover, using the same orthogonality property, we could pick on the a's by starting with the factor $\cos(mx)$—including the constant term since $\cos(0 \cdot x) = 1$. Care must be taken when considering a_0 as compared to a_m for $m \geq 1$ because

$$\int_{-\pi}^{\pi} \cos^2(mx)\,dx = \begin{cases} 2\pi, & m = 0 \\ \pi, & m \geq 1 \end{cases}$$

This difference leads to the common practice of writing the constant term as $\frac{a_0}{2}$ as opposed to a_0. We can state the results for the Fourier Series representation for a function f on the interval $(-\pi, \pi)$.

Fourier Series Representation on $(-\pi, \pi)$

The Fourier series representation for the function $f(x)$ defined on $(-\pi, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right),$$

where

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx, \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx, \quad \text{and} \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx \end{aligned}$$

If the series is written as $a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)$, then the formulas for a_n and b_n are unchanged, and the formula for a_0 is replaced with $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx$.

Fourier Coefficients

The coefficients, a_0, a_n, b_n, are called the **Fourier coefficients** of f. We can restate the formulas using inner product notation[66]. For example,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{\langle f(x), \cos(nx)\rangle}{\langle \cos(nx), \cos(nx)\rangle} = \frac{\langle f(x), \cos(nx)\rangle}{\|\cos(nx)\|^2}.$$

We'll look at an example and then extend the result for functions defined on an alternative interval.

[66] If we use this notation, the formula for a_0 becomes awkward when the factor of $\frac{1}{2}$ is included in the series instead of the formula for the coefficient.

Example

Find the Fourier series of the piecewise defined function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 \le x < \pi \end{cases}$$

We can find the coefficients by using the integral formulas given.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\,dx = \frac{\pi}{2}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos(nx)\,dx = \frac{\cos(n\pi)-1}{\pi n^2}.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\sin(nx)\,dx = -\frac{\cos(n\pi)}{n}.$$

Example Continued...

It is convenient to use the relation

$$\cos(n\pi) = (-1)^n.$$

This comes up frequently in computing Fourier series. Using the determined coefficients we have

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \\ &= \frac{\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{(-1)^n - 1}{\pi n^2} \cos nx + \frac{(-1)^{n+1}}{n} \sin nx \right) \end{aligned}$$

Helpful Hint:

We frequenty encounter the expressions $\sin(n\pi)$ and $\cos(n\pi)$ when computing Fourier coefficients. It's worth recalling that

$$\sin(n\pi) = 0 \quad \text{and} \quad \cos(n\pi) = (-1)^n$$

for all integers n (positive, negative, and zero).

Fourier Series on $(-p, p)$

Suppose we wish to obtain a Fourier series represenation for a function f defined on the interval $(-p, p)$ for some $p > 0$. If we set $t = \frac{\pi x}{p}$, then the interval $-p < x < p$ is mapped to $-\pi < t < \pi$, and we can use our results to obtain a Fourier series. In fact, we can use this change of variables to obtain the related set of functions that are orthogonal on $[-p, p]$

$$\left\{1, \cos\frac{\pi x}{p}, \cos\frac{2\pi x}{p}, \cos\frac{3\pi x}{p}\ldots, \sin\frac{\pi x}{p}, \sin\frac{2\pi x}{p}, \sin\frac{3\pi x}{p}, \ldots\right\}.$$

By the same process used on $(-\pi, \pi)$, we can derive a similar set of formulas for the Fourier series representation of a function on an interval $(-p, p)$.

It is possible to consider an arbitrary bounded interval (a, b), rather than insisting that the interval is symmetric about zero, by using the transformation $t = \dfrac{2\pi}{b-a}\left(x - \frac{b+a}{2}\right)$.

Fourier Series Representation on $(-p, p)$

The Fourier series representation for the function $f(x)$ defined on $(-p, p)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right) \right), \tag{32}$$

where

$$a_0 = \frac{1}{p} \int_{-p}^{p} f(x)\, dx, \tag{33}$$

$$a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos\left(\frac{n\pi x}{p}\right) dx, \quad \text{and} \tag{34}$$

$$b_n = \frac{1}{p} \int_{-p}^{p} f(x) \sin\left(\frac{n\pi x}{p}\right) dx \tag{35}$$

First, note that this result gives the previous statement in the case $p = \pi$. The same sort of alternative formulation can be given by writing $f(x) = a_0 + \sum \cdots$ and replacing the first formula with $a_0 = \frac{1}{2p} \int_{-p}^{p} f(x)\, dx$.

Find the Fourier series of f

$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ -2, & 0 \le x < 1 \end{cases}$$

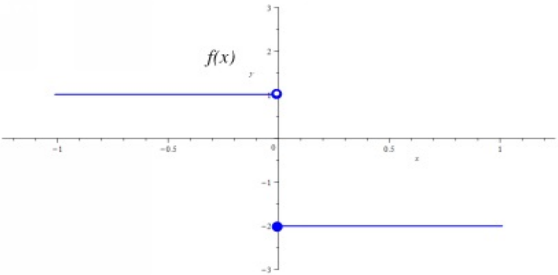


Figure: A plot of the piecewise constant function f. Note that f is defined on the interval $(-1, 1)$.

Example

First, we note that f is defined on the interval $(-1,1)$ so that $p=1$. We apply the formulas for the Fourier coefficients, with $p=1$. The integrals must be computed in pieces since f changes value at $x=0$.

$$\begin{aligned}
a_0 &= \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} dx + \int_{0}^{1}(-2)\,dx = -1\\
a_n &= \frac{1}{1}\int_{-1}^{1} f(x)\cos\left(\frac{n\pi x}{1}\right)\,dx\\
&= \int_{-1}^{0}\cos(n\pi x)\,dx + \int_{0}^{1}(-2)\cos(n\pi x)\,dx = 0\\
b_n &= \frac{1}{1}\int_{-1}^{1} f(x)\sin\left(\frac{n\pi x}{1}\right)\,dx\\
&= \int_{-1}^{0}\sin(n\pi x)\,dx + \int_{0}^{1}(-2)\sin(n\pi x)\,dx = \frac{3((-1)^n-1)}{n\pi}
\end{aligned}$$

We have $a_0=-1$, $a_n=0$ for all $n\geq 1$ and $b_n=\frac{3((-1)^n-1)}{n\pi}$.

Example Continued...

Putting the coefficients into the expansion, we get

$$f(x) = -\frac{1}{2} + \sum_{n=1}^{\infty} \frac{3((-1)^n - 1)}{n\pi} \sin(n\pi x).$$

Important Observation

This example raises some interesting questions. The function f is not continuous on the interval $(-1, 1)$. However, each term in the Fourier series, and any partial sum thereof, is obviously continuous. We can ask about various properties of the series, e.g.,

- *Does the series converge for all x in $(-1, 1)$?*
- *Is the series continuous?*
- *What is the connection between f and its Fourier series at the point of discontinuity of f?*
- *What does the series converge to (if anything) if x is taken outside of the interval $(-1, 1)$?*

This is the convergence issue alluded to earlier.

Convergence in the Mean

Theorem: Convergence of the Series

Let f be defined on $(-p, p)$ and consider the Fourier series representation for f given by (32)–(35). If f is continuous at x_0 in $(-p, p)$, then the series converges to $f(x_0)$ at that point. If f has a jump discontinuity at the point x_0 in $(-p, p)$, then the series **converges in the mean** to the average value

$$\frac{f(x_0-)+f(x_0+)}{2} \stackrel{\text{def}}{=} \frac{1}{2}\left(\lim_{x\to x_0^-} f(x) + \lim_{x\to x_0^+} f(x)\right)$$

at that point.

Period Extension

The series is also defined for x outside of the original domain $(-p, p)$. The extension to all real numbers is $2p$-periodic.

Graphical Example of a Fourier Series

$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ -2, & 0 \le x < 1 \end{cases}, \quad f(x) = -\frac{1}{2} + \sum_{n=1}^{\infty} \frac{3((-1)^n - 1)}{n\pi} \sin(n\pi x).$$

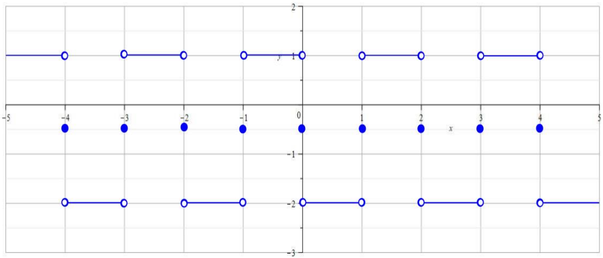


Figure: Here is a plot of the Fourier series of f given in the previous example. We see the basic profile is repeated so that the series is 2-periodic. At each jump, the series converges in the mean to the mid point between the left and right limits. The series converges to $f(x)$ on $(-1, 0)$ and $(0, 1)$. While $f(0) = -2$, the series converges to $\frac{1+(-2)}{2} = -\frac{1}{2}$ at $x = 0$.

Find the Fourier Series for $f(x) = x$, $-1 < x < 1$

Again the value of $p = 1$. Computing the coefficients in the usual way.

$$\begin{aligned}
a_0 &= \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} x\,dx = 0 \\
a_n &= \frac{1}{1}\int_{-1}^{1} f(x)\cos\left(\frac{n\pi x}{1}\right)\,dx \\
&= \int_{-1}^{1} x\cos(n\pi x)\,dx = 0 \\
b_n &= \frac{1}{1}\int_{-1}^{1} f(x)\sin\left(\frac{n\pi x}{1}\right)\,dx \\
&= \int_{-1}^{1} x\sin(n\pi x)\,dx = \frac{2(-1)^{n+1}}{n\pi}
\end{aligned}$$

That is,

$$a_0 = a_n = 0 \quad \text{for all} \quad n \geq 1, \quad \text{and} \quad b_n = \frac{2(-1)^{n+1}}{n\pi}.$$

Example Continued...

Having determined the coefficients, we have the Fourier series

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi x)$$

Observation:

f is an odd function. It is not surprising then that there are no nonzero constant or cosine terms (which have even symmetry) in the Fourier series for f.

The following plots show f, f plotted along with some partial sums of the series, and f along with a partial sum of its series extended outside of the original domain $(-1, 1)$.

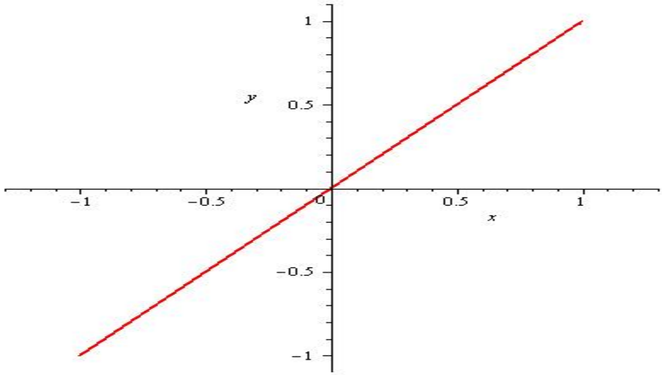


Figure: Plot of $f(x) = x$ for $-1 < x < 1$

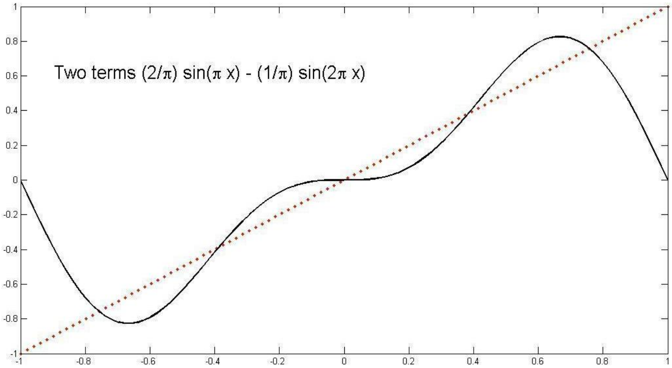


Figure: Plot of $f(x) = x$ for $-1 < x < 1$ with two terms of the Fourier series. This is only the first two terms, so the truncated series is not a good approximation. It does seem to have the general 45° tilt expected as well as the odd symmetry.

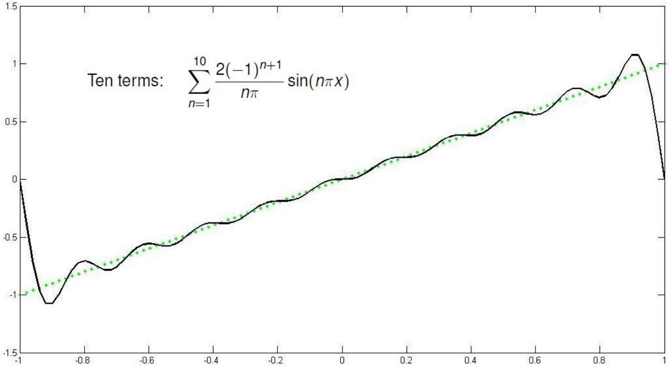


Figure: Plot of $f(x) = x$ for $-1 < x < 1$ with 10 terms of the Fourier series. With ten terms, we see that the sum of sines starts to hug the 45° degree line segment with less wiggling. Note that at the ends, the series takes the value of zero. No amount of added terms will eliminate this effect. This is convergence in the mean.

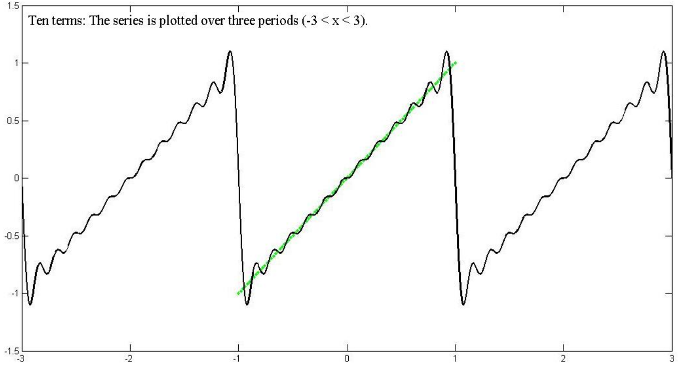
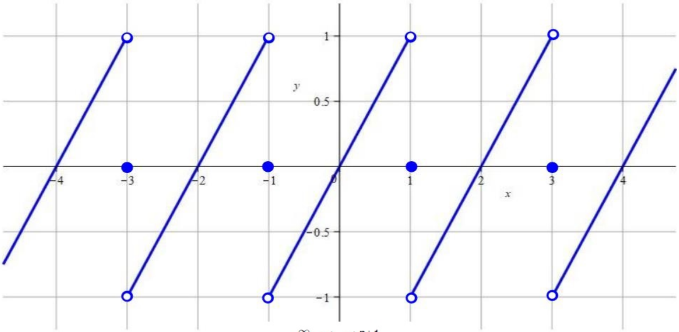


Figure: Plot of $f(x) = x$ for $-1 < x < 1$ with ten terms of the Fourier series plotted on $(-3, 3)$. While f is not defined outside of the interval $(-1, 1)$, we see that the series is 2-periodic. This truncated series is still continuous, but we can see that at each jump, the series will converge to the midpoint $(1 + (-1))/2 = 0$.



$$S_f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi x)$$

Figure: Here, we see a plot of the series—what the whole infinite series converges to. We see the periodicity and convergence in the mean. **Note:** A plot like this is determined by our knowledge of the generating function and Fourier series, not by evaluating the series itself.

Section 18: Sine and Cosine Series

In this final section, we will continue to consider Fourier series. There are two topics in this section. First, we'll discuss Fourier series for functions defined on $(-p, p)$ that have either even or odd symmetry.

Then, we'll discuss what are sometimes called *half-range* Fourier series. These are series created for a function defined on the interval $(0, p)$, for some $p > 0$. We can generate a couple of different series for such a function by *pretending* to extend it to the interval $(-p, 0)$ as either an even or an odd function. Since the extension is an artifice, we can generate two such functions, one of sines and one of cosines.

Let's recall the definitions of even and odd symmetry.

Functions with Symmetry

Definition

Suppose f is defined on an interval containing x and $-x$. Then

- If $f(-x) = f(x)$ for all x, then f is said to be **even**.
- If $f(-x) = -f(x)$ for all x, then f is said to be **odd**.

The classic example of a function with symmetry is the monomial $f(x) = x^n$. When n is an even number, the function is even. And when n is odd, the function is odd. This is where the names even and odd come from. But other functions may have such symmetry, and in fact all of the basic trigonometric functions have symmetry. Most notably, $g(x) = \cos(x)$ is even and $h(x) = \sin(x)$ is odd.

Graphical Interpretation

Graphs of symmetric functions have a characteristic appearance. An even function's graph is symmetric about the *y*-axis so that the curve to the right of this axis is a mirror image of the curve to the left. An odd function's graph is symmetric with respect to the origin so that the curve to the right of the *y*-axis appears to have been reflected in both coordinate axes.

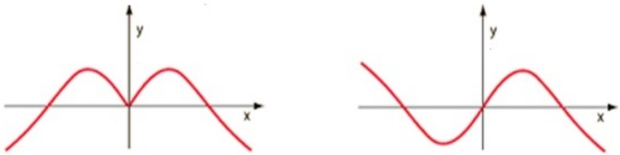


Figure: The plot of an even function on the left as compared to an odd function on the right.

The images seen here suggest the following useful result related to the integral of an even or odd function over an interval that is symmetric about zero.

Integrals on symmetric intervals

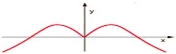
Integrals & Symmetry

If f is an even function on $(-p, p)$, then

$$\int_{-p}^{p} f(x)\,dx = 2\int_{0}^{p} f(x)\,dx.$$

If f is an odd function on $(-p, p)$, then

$$\int_{-p}^{p} f(x)\,dx = 0.$$



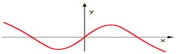


Figure: These integration results are a consequence of the relationship between integration and the sum of signed areas between curves.

Products of Even and Odd functions

Products of Symmetric Functions

If f and g are functions with symmetry on $(-p, p)$, then the product fg has symmetry. And we can characterize the symmetry of the product based on the symmetry of the factors.

- If f is even and g is even, then fg is even.
- If f is even and g is odd, then fg is odd.
- If f is odd and g is odd, then fg is even.

Remark: Note what this tells us. For every integer n,

- if f is **even** on $(-p, p)$, then

$$f(x)\cos\left(\frac{n\pi x}{p}\right) \text{ is } \textbf{even} \quad \text{and} \quad f(x)\sin\left(\frac{n\pi x}{p}\right) \text{ is } \textbf{odd}.$$

- if f is **odd** on $(-p, p)$, then

$$f(x)\sin\left(\frac{n\pi x}{p}\right) \text{ is } \textbf{even} \quad \text{and} \quad f(x)\cos\left(\frac{n\pi x}{p}\right) \text{ is } \textbf{odd}.$$

Fourier Series of an Even Function

Fourier Series: Even Function

If f is even on $(-p, p)$, then the Fourier series of f will not have any sine terms. Moreover

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right), \quad \text{where}$$

$$a_0 = \frac{2}{p}\int_0^p f(x)\,dx, \quad \text{and} \quad a_n = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)\,dx.$$

This is not a new set of formulas for the Fourier series. It is simply the result of the symmetry of f that $b_n = 0$ for every n, and the integrals for the a's can be stated to take advantage of the symmetry.

Fourier Series of an Odd Function

Fourier Series: Odd Function

If f is odd on $(-p,p)$, then the Fourier series of f will have no constant or cosine terms. In fact,

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right), \quad \text{where}$$

$$b_n=\frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)\,dx.$$

Again, this is simply what the general Fourier series reduces to in the case that f is odd. We saw this with the last example shown in the previous section where $f(x)=x$ on $(-1,1)$. Let's consider an example of looking for and taking advantage of symmetry.

Find the Fourier series of f

$$f(x) = \begin{cases} x+\pi, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases}$$

It may not be clear from looking at the way f is defined, but a quick plot of this piecewise linear function shows that it is even.

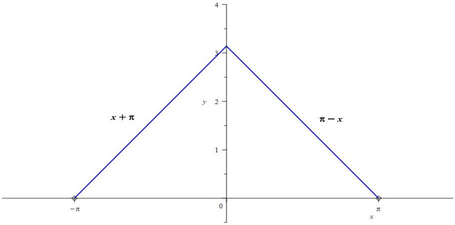


Figure: A plot of f obtained by simply plotting the two lines $y = \pi + x$ and $y = \pi - x$ over the appropriate intervals.

Example Continued...

Since f is even, we know that the Fourier series will not have any sine terms. (If we go through the hassle of computing the b's, they would all end up being zero!) Moreover, even though f is defined differently on $(-\pi, 0)$ and $(0, \pi)$, we can use the symmetry to compute the integrals on only the right half and double it. That is, using the symmetry versions of the coefficient formulas, we find

$$a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx = \frac{2}{\pi}\int_0^\pi (\pi - x)\,dx = \pi$$

and

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^\pi (\pi - x)\cos(nx)\,dx =$$

$$= \frac{2(1-(-1)^n)}{n^2\pi}$$

Example Continued...

Making use of the coefficients we just found,

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n^2\pi}\cos(nx)$$

Remark

If we hadn't noticed the symmetry and computed the series is the usual way, we would end up with precisely this same result. It's worth noticing how much effort was avoided by checking for and using the symmetry. If we had used the full integral formulas for all of the a's and the b's, each computation would involve two integrals. That's a total of six integrals (two for a_0, two for a_n and two for b_n). And four of those require integration by parts! We managed to get there with two integrals, one of which required integration by parts.

Half Range Sine and Half Range Cosine Series

Next, we want to consider the options for generating Fourier series representations for a function defined on an interval $(0, p)$ for some $p > 0$. A most notable application of this comes from solving partial differential equations in which the spatial domain is an object of fixed length. For example, suppose we wish to track the temperature distribution in a thin rod 1 meter long. The temperature may depend on time and vary along the length of the rod so that the temperature $u = u(x, t)$ where t is the independent variable representing time and x is the variable representing position. We construct a coordinate system putting one end of the rod at the origin, and our variable x satisfies $0 \leq x \leq 1$ (assuming appropriate units).

An approach is to *pretend* that our function is defined on $(-p, p)$ in such a way that it is either even or odd. This introduces some choice which allows for other considerations—for example, perhaps we want the functions to be zero at the ends (sines) or we want zero change at the ends (cosines).

Even Extension of a Function

If f is defined on $(0, p)$ and we artificially extend it to be even, we can visual this as reflecting the graph of f in the y-axis.

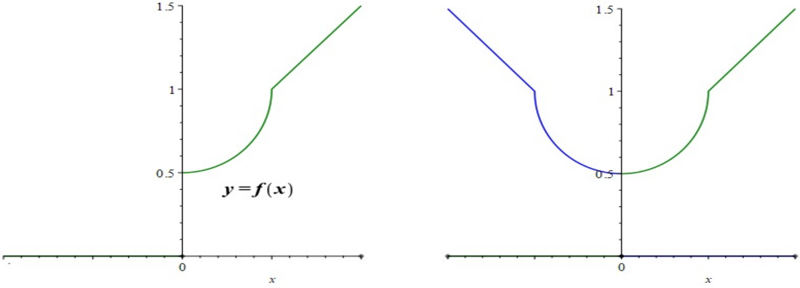


Figure: The graph of some function f shown along with an even extension.

Odd Extension of a Function

We can similarly visualize an odd extension as resulting from reflecting the curve in the x-axis and then the y-axis

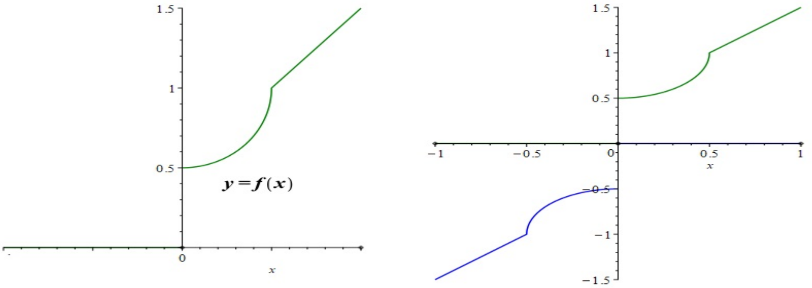


Figure: The graph of some function f shown along with an odd extension.

In either case, the extension is just a convenient idea. We don't actually have to consider a symbolic or graphical representation of the new function on $(-p, 0)$.

Half Range Cosine Series

Cosine Series

Suppose f is piecewise continuous on $(0, p)$ for some $p > 0$. The **half-range sine series** of f is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right)$$

where $\quad a_0 = \frac{2}{p}\int_0^p f(x)\,dx \quad$ and $\quad a_n = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)\,dx.$

Remark:

Notice that the assumption that the extension is even means that we can use the coefficient formulas that take advantage of even symmetry. In other words, we never have to evaluate or work with f for values of $x < 0$.

Half Range Sine Series

SineSeries

Suppose f is piecewise continuous on $(0, p)$ for some $p > 0$. The **half-range sine series** of f is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$$

$$\text{where} \quad b_n = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)\,dx.$$

Remark:

Again, this takes advantage of the assumption of odd symmetry. No value of f on $(-p, 0)$ is ever referenced!

Half Range Series

Note that these two results give us two distinct series for the same function. It's reasonable to wonder how they are related, for example, what does each converge to on $(0, p)$ or on $(-p, p)$.

For f piecewise continuous on $(0, p)$, both the cosine and sine series converge to f in the mean. Not too surprising, on $(-p, p)$ the cosine series will converge (in the mean) to an even extension of f, whereas the sine series will converge to an odd extension of f.

If we consider larger intervals, both the cosine and sine series will converge to a $2p$-periodic extension of the corresponding extension.

Let's look at an example.

Find the Half Range Sine Series of f

$$f(x) = 2 - x, \quad 0 < x < 2$$

Here, the value $p = 2$. Using the formula for the coefficients of the sine series

$$\begin{aligned} b_n &= \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right)\,dx \\ &= \int_0^2 (2-x)\sin\left(\frac{n\pi x}{2}\right)\,dx \\ &= \frac{4}{n\pi} \end{aligned}$$

The series is $\quad f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi}\sin\left(\frac{n\pi x}{2}\right).$

Find the Half Range Cosine Series of f

$$f(x) = 2 - x, \quad 0 < x < 2$$

Using the formulas for the cosine series

$$\begin{aligned}
a_0 &= \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^2 (2-x)\,dx = 2 \\
a_n &= \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)\,dx \\
&= \int_0^2 (2-x)\cos\left(\frac{n\pi x}{2}\right)\,dx \\
&= \frac{4(1-(-1)^n)}{n^2\pi^2}
\end{aligned}$$

Example Continued...

We can write out the half range cosine series

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right).$$

We have two different series representations for this function each of which converge to $f(x)$ on the interval $(0, 2)$. The following plots show graphs of f along with partial sums of each of the series. When we plot over the interval $(-2, 2)$ we see the two different symmetries. Plotting over a larger interval such as $(-6, 6)$ we can see the periodic extensions of the two symmetries.

Odd Extension $f(x) = 2 - x, \quad 0 < x < 2$

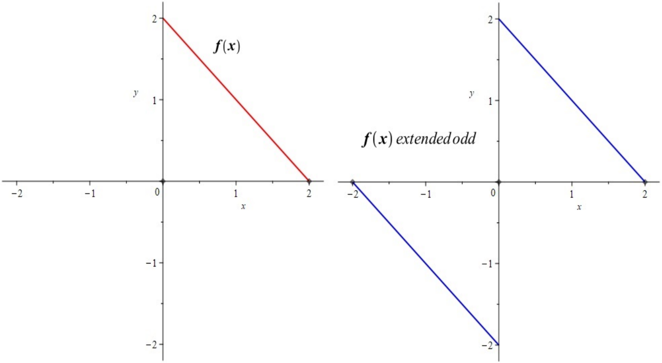


Figure: We have the plot of f on the left along with the odd extension on the right. The half-range sine series is a full Fourier series for the extended version of f.

The next two images show the first ten terms of the sine series plotted on $(-2, 2)$ and on $(-6, 6)$.

Plots of f with Half range series

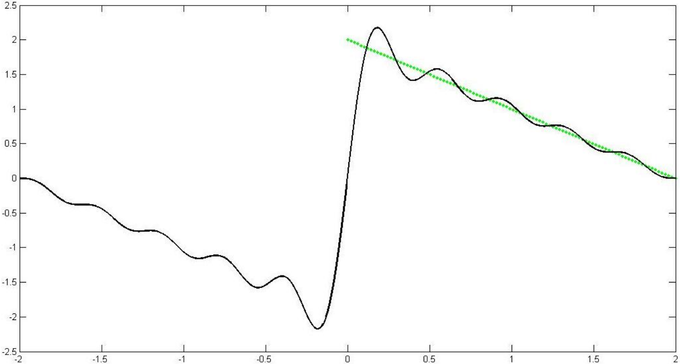


Figure: $f(x) = 2 - x$, $0 < x < 2$ with 10 terms of the sine series. Note the odd symmetry. At $x = 0$, the whole series would converge in the mean to the midpoint of the jump, $\frac{-2+2}{2} = 0$.

Plots of f with Half range series

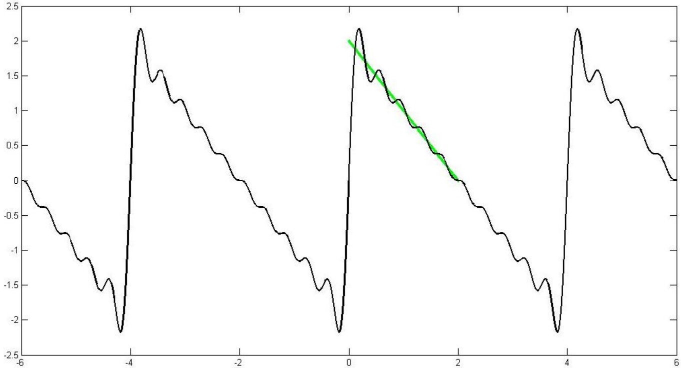


Figure: $f(x) = 2 - x$, $0 < x < 2$ with 10 terms of the sine series, and the series plotted over $(-6, 6)$. Note that the series is 4-periodic.

$$f(x) = 2 - x \text{ for } 0 < x < 2$$

Sine Series Generated from f: $\quad S_{s_f}(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin\left(\frac{n\pi x}{2}\right)$

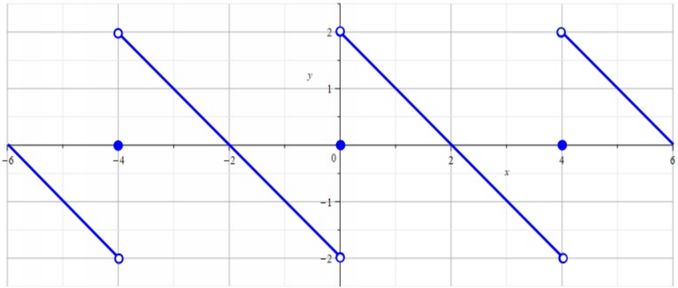


Figure: This plot shows what the half range sine series converges to. The interval shown is $(-6, 6)$. We see the odd extension, the convergence in the mean, and the $2p$-periodicity (with $p = 2$).

Even Extension $f(x) = 2 - x, \quad 0 < x < 2$

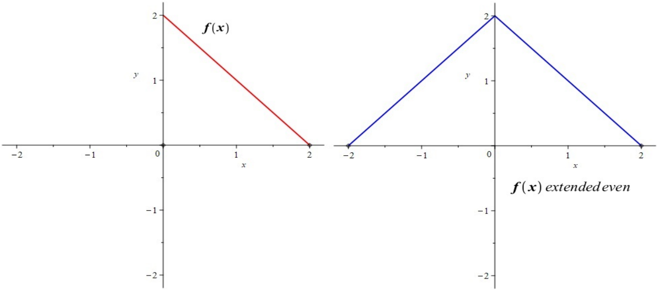


Figure: We have the plot of f on the left along with the even extension on the right. The half-range cosine series is a full Fourier series for the extended version of f.

The next two images show the first five terms of the cosine series plotted on $(-2, 2)$ and on $(-6, 6)$.

Plots of f with Half range series

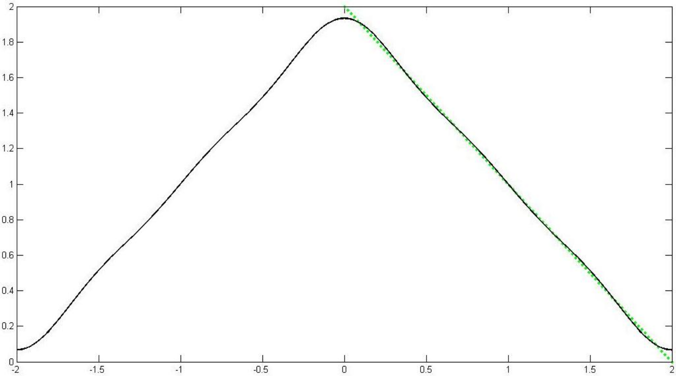


Figure: $f(x) = 2 - x$, $0 < x < 2$ with 5 terms of the cosine series. Note the even symmetry.

Plots of f with Half range series

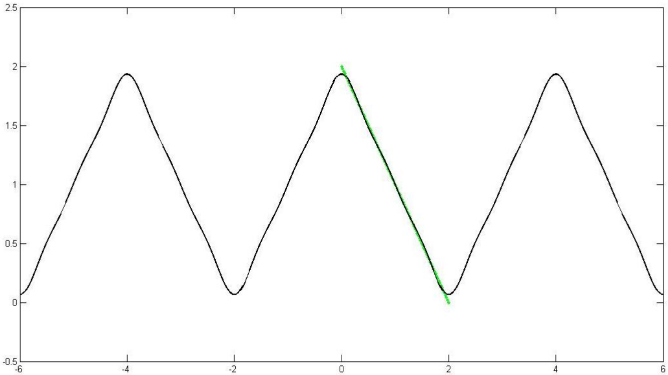


Figure: $f(x) = 2 - x$, $0 < x < 2$ with 5 terms of the cosine series, and the series plotted over $(-6, 6)$. Note that the series is 4-periodic.

$f(x) = 2 - x$ for $0 < x < 2$

Cosine Series Generated from f: $S_{C_f}(x) = 1 + \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right)$

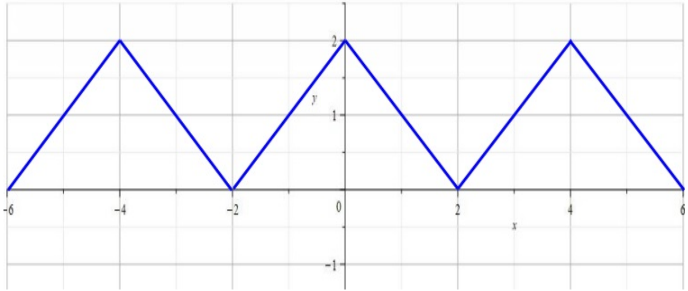


Figure: This plot shows what the half range cosine series converges to. The interval shown is $(-6, 6)$. This even extension ends up continuous, and again is $2p$-periodicity (with $p = 2$).

Notes on Convergence

We have two series for $f(x) = 2 - x$, $0 < x < 2$. These were

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin\left(\frac{n\pi x}{2}\right) \quad \text{and}$$

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right)$$

If we look back at the plots of partial sums, we might notice that taking ten terms of the sine series doesn't seem to give the same level of accuracy as taking only five terms of the cosine series. Looking at the series, this isn't too surprising. For one thing, the coefficients for the cosine series are zero whenever n is even since

$$1 - (-1)^n = 1 - 1 = 0 \quad \text{whenever } n \text{ is even.}$$

So five nonzero terms of this series include the indices, 1, 3, 5, and 7 (including the zeroth term).

Notes on Convergence

Even taking this into account, the cosine series should converge must faster. If we fix some value of x in $(0,2)$, say $x = x_0$ and look at the corresponding numerical series, the n^{th} terms are

$$\text{Sine:}\quad \frac{4\sin\left(\frac{n\pi x_0}{2}\right)}{n\pi},\quad \text{and}\quad \text{Cosine}\quad \frac{4(1-(-1)^n)\cos\left(\frac{n\pi x_0}{2}\right)}{n^2\pi^2}.$$

Granted, the sine and cosine values there will take on various values between -1 and 1. But notice that these look something like

$$\text{Sine:}\quad \frac{C_n}{n},\quad \text{and}\quad \text{Cosine:}\quad \frac{K_n}{n^2}.$$

The values C_n and K_n aren't fixed, but we see that the coefficients in the sine series tend to zero sort of like $\frac{1}{n}$, whereas the coefficients of the cosine series go to zero at the faster rate like $\frac{1}{n^2}$. With this observation, it's to be expected that partial sums of the cosine series will give a closer approximation to f on $(0,2)$. Which series is used in practice tends to depend on other considerations. So we can't say that the sine series is of less interest.

Series Series Solution of a Differential Equation

Let's return to the motivational example used to open the discussion on Fourier series. While the power of Fourier series is most often used in working with partial differential equations, we can see an example of their application to ODEs. We'll recall the problem statement:

An undamped spring mass system has a mass of 2 kg attached to a spring with spring constant 128 N/m. The mass is driven by an external force $f(t) = 2t$ for $-1 < t < 1$ that is 2-periodic so that $f(t+2) = f(t)$ for all $t > 0$. Determine a particular solution x_p for the displacement for $t > 0$.

Let's recall that the model for this system with mass m and spring constant k is $mx'' + kx = f(t)$, which here gives

$$2x'' + 128x = f(t). \tag{36}$$

Series Solution of a Differential Equation

The forcing function f can be expressed in terms of a sine series. In fact, the series for this function was determined in the previous section, see page (463). It was the last example[67] in that section as it illustrated how odd symmetry affected a Fourier series. The Fourier sine series we found for the forcing function computed on page (463) is

$$f(t) = 2\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t).$$

Let's put (36) in standard form by dividing through by our mass. Then making use of the series representation for f, we have

$$x'' + 64x = \frac{1}{2}f(t) \quad \Longrightarrow \quad x'' + 64x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t).$$

[67] Here, we'll include an extra factor of 2 and call the independent variable t as opposed to x, but we can otherwise copy the series obtained on page (463).

Series Solution of a Differential Equation

We're looking for a particular solution, x_p. To this end, let's assume that x_p can be represented by a sine series. Set

$$x_p = \sum_{n=1}^{\infty} B_n \sin(n\pi t),$$

where we hope to determine the coefficients B_n.

Remark:

Before we get into the computational details, lets notice what we're doing here. The right side of the ODE is a sum, $\frac{2}{\pi}\sin(\pi t) - \frac{1}{\pi}\sin(2\pi t) + \frac{2}{3\pi}\sin(3\pi t) - \cdots$. So we're looking for our particular solution as a sum; that's the **principle of superposition**.

Moreover, we've assumed a form for x_p with unknown coefficients, $B_1\sin(\pi t) + B_2\sin(2\pi t) + B_3\sin(3\pi t) + \cdots$. This is the **method of undetermined coefficients**[a].

We are taking something of a leap here in assuming that these principles and methods apply when working with infinite series, but this isn't a completely new process.

[a] We could assume $x_p = \sum_{n=1}^{\infty} (A_n\cos(n\pi t) + B_n\sin(n\pi t))$, in keeping with the general expectation that cosines are included with sines. We would find that $A_n = 0$ for each n.

Series Solution of a Differential Equation

Now, we'll substitute our assumed form for x_p into the differential equation in the usual way. This requires that we take the first two derivatives of x_p. In doing so, we will assume that our series for x_p converges in such a way that it is valid to take the derivative term by term. That is, we'll assume that

$$x_p' = \frac{d}{dt}\left(\sum_{n=1}^{\infty} B_n \sin(n\pi t)\right) = \sum_{n=1}^{\infty} \frac{d}{dt}\left(B_n \sin(n\pi t)\right) = \sum_{n=1}^{\infty} n\pi B_n \cos(n\pi t).$$

Taking the second derivative in the same fashion, we get

$$x_p'' = \frac{d}{dt}\left(\sum_{n=1}^{\infty} n\pi B_n \cos(n\pi t)\right) = \sum_{n=1}^{\infty} -n^2\pi^2 B_n \sin(n\pi t).$$

Series Solution of a Differential Equation

Substituting into the left side of the differential equation gives

$$\begin{aligned} x_p'' + 64x_p &= \sum_{n=1}^{\infty} -n^2\pi^2 B_n \sin(n\pi t) + 64\sum_{n=1}^{\infty} B_n \sin(n\pi t) \\ &= \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t) \end{aligned}$$

We'll collect the two series on the left side of the ODE into a single summation and factor the common $B_n \sin(n\pi t)$

$$\sum_{n=1}^{\infty} -n^2\pi^2 B_n \sin(n\pi t) + \sum_{n=1}^{\infty} 64B_n \sin(n\pi t) = \sum_{n=1}^{\infty} (64 - n^2\pi^2)B_n \sin(n\pi t).$$

The equation becomes

$$\sum_{n=1}^{\infty} (64 - n^2\pi^2)B_n \sin(n\pi t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t).$$

Series Solution of a Differential Equation

Finally, we match like terms. For each n, the coefficient of $\sin(n\pi t)$ on the left must match the corresponding coefficient on the right producing a sequence of equations

$$(64 - n^2\pi^2)B_n = \frac{2(-1)^{n+1}}{n\pi}, \quad \text{for each } n \geq 1.$$

We note here that the coefficient $64 - n^2\pi^2 \neq 0$ for every integer n, which means that we can divide through and solve for our series coefficients for x_p.

$$B_n = \frac{2(-1)^{n+1}}{n\pi(64 - n^2\pi^2)}.$$

Series Solution of a Differential Equation

Having found the coefficients, the particular solution can now be expressed

$$x_p = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi(64 - n^2\pi^2)} \sin(n\pi t).$$

This completes the problem.

Remark:

What if the specifics of the problem had resulted in a value of n, say n_k, for which B_{n_k} could not be solved (i.e. if $\omega^2 - n_k^2\pi^2 = 0$)? This would indicated a pure resonance term. The above approach would still yield the remaining B values. The resonance term would have to be considered separately. We could assume, using the principle of superposition, that

$$x_p = A_{n_k} t\cos(n_k\pi t) + B_{n_k} t\sin(n_k\pi t) + \sum_{\substack{n=1 \\ n\neq n_k}}^{\infty} B_n \sin(n\pi t).$$

This completes our introduction to Fourier series.

Table of Laplace Transforms

$f(t) = \mathscr{L}^{-1}\{F(s)\}$	$F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$
1	$\frac{1}{s} \quad s > 0$
$t^n \quad n = 1, 2, \dots$	$\frac{n!}{s^{n+1}} \quad s > 0$
$t^r \quad r > -1$	$\frac{\Gamma(r+1)}{s^{r+1}} \quad s > 0$
e^{at}	$\frac{1}{s-a} \quad s > a$
$\sin(kt) \quad k \neq 0$	$\frac{k}{s^2+k^2} \quad s > 0$
$\cos(kt)$	$\frac{s}{s^2+k^2} \quad s > 0$
$e^{at} f(t)$	$F(s-a)$
$\mathscr{U}(t-a) \quad a > 0$	$\frac{e^{-as}}{s} \quad s > 0$
$\mathscr{U}(t-a) f(t-a) \quad a > 0$	$e^{-as} F(s)$
$\mathscr{U}(t-a) g(t) \quad a > 0$	$e^{-as} \mathscr{L}\{g(t+a)\}$
$\delta(t-a) \quad a \geq 0$	e^{-as}
$f'(t)$	$sF(s) - f(0)$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$
$tf(t)$	$-\frac{d}{ds} F(s)$
$t^n f(t) \quad n = 1, 2, \dots$	$(-1)^n \frac{d^n}{ds^n} F(s)$

$$\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathscr{L}\{f(t)\} + \beta \mathscr{L}\{g(t)\}$$

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha \mathscr{L}^{-1}\{F(s)\} + \beta \mathscr{L}^{-1}\{G(s)\}$$

Appendix: Cramer's Rule

Cramer's rule is an approach to solving a linear, algebraic system of equations under the conditions that

a. the number of equations matches the number of unknowns (i.e. the system is square), and

b. the system is uniquely solvable (i.e. there is exactly one solution).

While Cramer's rule can be used with any size system, we'll restrict ourselves to the 2×2 case. We obtain the solution in terms of ratios of determinants. First, let's see how the method plays out in general, and then we illustrate with an example.

Note:

Cramer's rule will produce the same solution as any other approach. Its advantage is in its computational simplicity (which gets lost the larger the system is).

Appendix: Cramer's Rule

We begin with a linear system of two equations in two unknowns.

$$\begin{array}{ccccc} ax & + & by & = & e \\ cx & + & dy & = & f \end{array}$$

The unknowns are x and y, and the parameters a, b, c, d, e, and f are constants[a].

We're going to form 3 matrices. The first is the coefficient matrix for the system. I'll call that A. That is,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

---

[a]We can allow any of a, $b, \ldots, f$ to be unknown parameters as long as they don't depend on x or y. Hence here we consider them *constant*.

Appendix: Cramer's Rule

$$\begin{array}{ccccc} ax & + & by & = & e \\ cx & + & dy & = & f \end{array}$$

Next, we form two more matrices that I'll call A_x and A_y. These matrices are obtained by replacing one column of A with the values from the right side of the system. For A_x we replace the first column (the one with x's coefficients), and for A_y we replace the second column (the one with y's coefficients). We have

$$A_x = \begin{bmatrix} e & b \\ f & d \end{bmatrix}, \quad \text{and} \quad A_y = \begin{bmatrix} a & e \\ c & f \end{bmatrix}.$$

Appendix: Cramer's Rule

$$\begin{array}{ccccc} ax & + & by & = & e \\ cx & + & dy & = & f \end{array}$$

Now we have the three matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A_x = \begin{bmatrix} e & b \\ f & d \end{bmatrix}, \quad \text{and} \quad A_y = \begin{bmatrix} a & e \\ c & f \end{bmatrix}.$$

The condition that the system is uniquely solvable[a] guarantees that $\det(A) \neq 0$. Here's the punch line: The solution to the system is

$$x = \frac{\det(A_x)}{\det(A)} \quad \text{and} \quad y = \frac{\det(A_y)}{\det(A)}.$$

[a]This is a well known result that can be found in any elementary discussion of Linear Algebra.

Appendix: Cramer's Rule

Let's look at a simple example.

Solve the system of equations
$$\begin{array}{rcrcr} 2x & - & 3y & = & -4 \\ 3x & + & 7y & = & 2 \end{array}.$$

Let's form the three matrices and verify that the determinant of the coefficient matrix isn't zero.

$$A = \begin{bmatrix} 2 & -3 \\ 3 & 7 \end{bmatrix}, \quad A_x = \begin{bmatrix} -4 & -3 \\ 2 & 7 \end{bmatrix}, \quad \text{and} \quad A_y = \begin{bmatrix} 2 & -4 \\ 3 & 2 \end{bmatrix}.$$

$$\det(A) = 2(7) - (-3)(3) = 23.$$

Appendix: Cramer's Rule

Okay, we can proceed by finding the determinants of the two matrices A_x and A_y. We get

$$\det(A_x) = -4(7) - 2(-3) = -22 \quad \text{and} \quad \det(A_y) = 2(2) - 3(-4) = 16.$$

Together with $\det(A) = 23$, the solution to the system is

$$x = -\frac{22}{23} \quad \text{and} \quad y = \frac{16}{23}.$$

It's worth taking a moment to substitute those values back into the system to verify that it does indeed solve it. It's not hard to imagine, looking at the solution, that solving it with substitution or elimination is probably more tedious.

Appendix: Cramer's Rule

This process can be extended in the obvious way to larger systems of equations provided they are square and uniquely solvable. You form the coefficient matrix. Then for each variable, form another matrix by replacing that variable's coefficient column with the values on the right side of the system. Each variable's solution value will be the ratio of the corresponding determinants.

For larger systems (perhaps bigger than 3×3) one must weigh the computational intensity of computing determinants with that of other options such as elimination or substitution. The approach also breaks down if the coefficient matrix has zero determinant. The system may have solutions (or not). Some other approach is needed to determine whether such a system has solutions and to characterize solutions in the case that they exist.