May 30 Math 3260 sec. 51 Summer 2023

A Random Motivational Example

In a certain city, ABC shipping has one receiving (A) and two distribution hubs (B & C). On a given day, 80 packages enter center A and will be distributed to hubs B and C for delivery. Twenty packages will go to a major client from hub C, the rest are to be distributed in quantities $x_1, \dots, x_4$ among the hubs and out for delivery.

Motivating Example

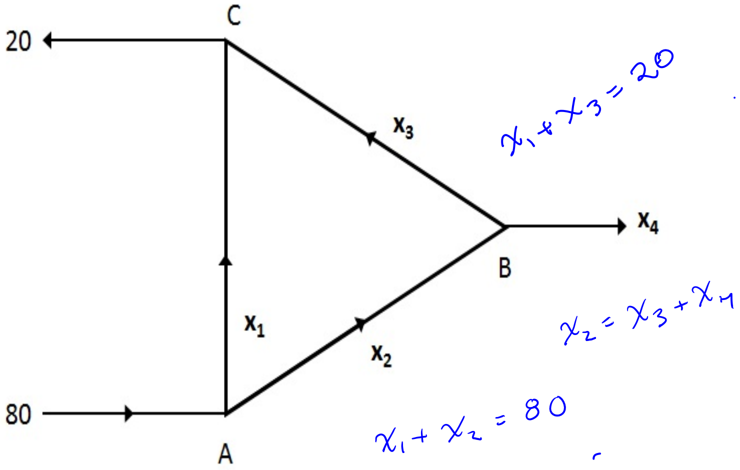


Figure: Distribution Scheme

Equations for Package Quantities

Assuming all of the packages are delivered to customers outside of the shipping company, the quantities $x_1, \ldots, x_4$ have to satisfy the equations

$$\begin{array}{rcrcrcrcr} x_1 & & & + & x_3 & & & = & 20 \\ & & x_2 & - & x_3 & - & x_4 & = & 0 \\ x_1 & + & x_2 & & & & & = & 80 \end{array}$$

Questions

- Is there a set of numbers $x_1, \dots, x_4$ that satisfy all of the equations?
- If there is a set of numbers, is it the only one?
- If we could find numbers $x_1, \dots, x_4$, and then the input 80 changed (say on another day), do we have to do all the work again? Or is there a way to generalize our finding?

(This is just to illustrate the kinds of questions addressed by **Linear Algebra.** We'll leave answering these questions for another day.)

We'll work in a variety of settings...

Linear sys.
$$\begin{array}{rcrcrcrcr} x_1 & & & + & x_3 & & & = & 20 \\ & & x_2 & - & x_3 & - & x_4 & = & 0 \\ x_1 & + & x_2 & & & & & = & 80 \end{array}$$

Matrix eqns.
$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \\ 80 \end{bmatrix}$$

More Matrices
$$\begin{bmatrix} 1 & 0 & 1 & 0 & 20 \\ 0 & 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & 0 & 80 \end{bmatrix}$$

Vector eqns.
$$x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \\ 80 \end{bmatrix}$$

Two main abstractions we'll be interested in are **Linear Transformations** and **Vector Spaces**.

Section 1.1: Systems of Linear Equations

We begin with a linear (*algebraic*) equation in n variables x_1, x_2, ..., x_n for some positive integer n.

A **linear equation** can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

The numbers $a_1, \ldots, a_n$ are called the *coefficients*. These numbers and the right side b are real (or complex) constants that are **known**.

Linear Equation in n Variables

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

Notice the main structure on the left side. The unknowns/variables $(x_1, \ldots, x_n)$ are

- multipled by numbers (a.k.a. coefficients), and
- added together.

Other types of actions (squaring, multiplying variables, taking variable's reciprocal, etc.) aren't allowed if an equation is **linear**.

Examples of Equations that are or are not Linear

$$2x_1 = 4x_2 - 3x_3 + 5 \quad \text{and} \quad 12 - \sqrt{3}(x + y) = 0$$

These equations are linear.

$2x_1 - 4x_2 + 3x_3 = 5$

$\sqrt{3}x + \sqrt{3}y = 12$

Examples of Equations that are or are not Linear

$$x_1 + 3x_3 = \frac{1}{x_2} \quad \text{and} \quad xyz = \sqrt{w}$$

These equations are NOT linear.

No reciprocals , no square roots

A *Linear System* is a collection of linear equations in the same variables

Example 1:

$$\begin{array}{rcrcrcrcr} 2x_1 & + & x_2 & - & 3x_3 & + & x_4 & = & -3 \\ -x_1 & + & 3x_2 & + & 4x_3 & - & 2x_4 & = & 8 \end{array}$$

Example 1 is a linear system that has two equations in four variables.

Example 2:

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 4 \\ 3x & & & + & 12z & = & 0 \\ 2x & + & 2y & - & 5z & = & -6 \end{array}$$

Example 2 is a linear system that has three equations in three variables.

Some Preliminary Terms

Solutions

A **solution** is a list of numbers $(s_1, s_2, \ldots, s_n)$ that reduce each equation in the system to a true statement upon substitution.

A **solutions set** is the set of all possible solutions of a linear system.

Equivalent Systems

Two systems are called **equivalent** if they have the same solution set.

Note: If the variables are $x_1, x_2, \ldots, x_n$, then a list such as $(s_1, s_2, \ldots, s_n)$ given as an n-tuple is understood to mean that upon substitution we set $x_1 = s_1$, $x_2 = s_2$, and so forth. If the variables are x, y, z, then (s_1, s_2, s_3) would mean $x = s_1$, $y = s_2$, and $z = s_3$.

An Example

$$\begin{array}{rcrcr} 2x_1 & - & x_2 & = & -1 \\ -4x_1 & + & 2x_2 & = & 2 \end{array}$$

(a) Show that $(1, 3)$ is a solution.

We can set $x_1 = 1$ and $x_2 = 3$ and show that both equations reduce to an identity.

$2(1) - (3) \stackrel{?}{=} -1 \qquad 2 - 3 = -1 \quad$ yes!

$-4(1) + 2(3) \stackrel{?}{=} 2 \qquad -4 + 6 = 2. \quad$ yes

Both equations become true.

An Example Continued

$$\begin{array}{rcrcr} 2x_1 & - & x_2 & = & -1 \\ -4x_1 & + & 2x_2 & = & 2 \end{array}$$

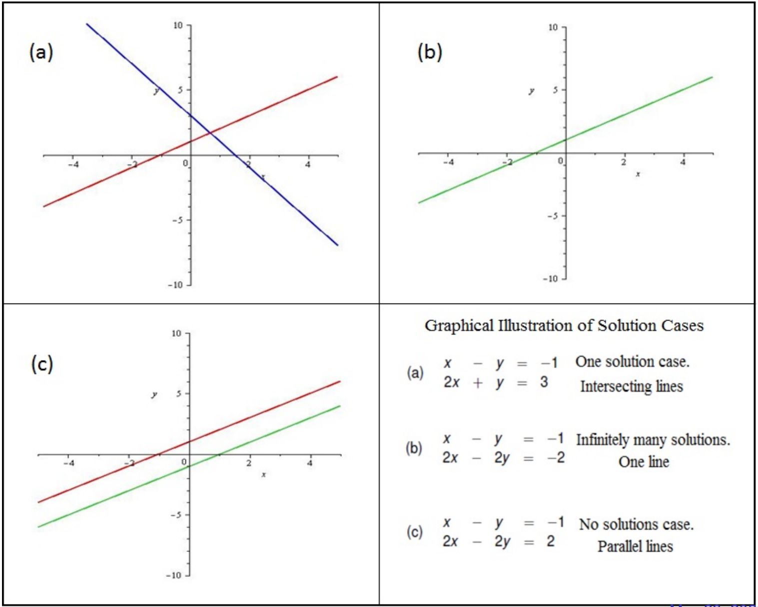
(b) The **solution set** for this system is

$$\left\{ (x_1, x_2) \mid x_1 = -\frac{1}{2} + \frac{1}{2}x_2 \right\}.$$

Notice that setting $x_1 = -\frac{1}{2} + \frac{1}{2}x_2$ in each equation we get the pair of true statements

$$\begin{array}{rcrcrl} 2\left(-\frac{1}{2} + \frac{1}{2}x_2\right) & - & x_2 & = & -1, & \quad \text{and} \\ -4\left(-\frac{1}{2} + \frac{1}{2}x_2\right) & + & 2x_2 & = & 2. & \end{array}$$

The Geometry of 2 Equations with 2 Variables



Theorem

Theorem

A linear system of equations has exactly one of the following:

i No solution, or

ii Exactly one solution, or

iii Infinitely many solutions.

A system is called **inconsistent** if it does not have any solutions (case i), and it's called **consistent** if it has any solution(s) (cases ii & iii).

Note: This theorem speaks to those two big questions:

- Existence: Is there a solution/does a solution exist?
- Uniqueness: Is there a unique solution or multiple solutions?

3 Equations in 3 Variables

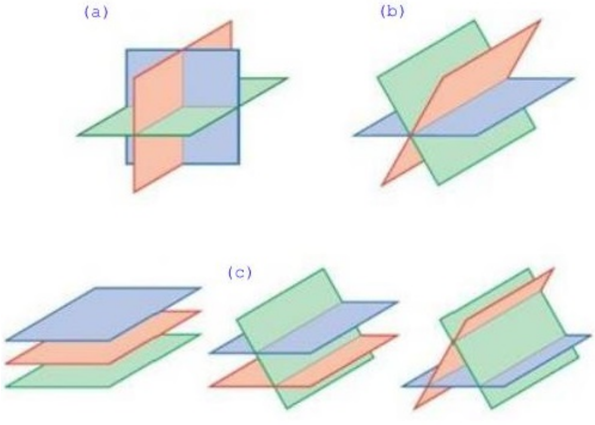


Figure: The graph of $ax + by + cz = d$ is a plane. Three may (a) intersect in a single point, (b) intersect in infinitely many points, or (c) not intersect in various ways.

Matrices

Definition:

A **matrix** is a rectangular array of numbers. It's **size** (a.k.a. dimension/order) is $m \times n$ (read "m by n") where m is the number of rows and n is the number of columns the matrix has.

Examples:

$$\begin{bmatrix} 2 & 0 & -1 & 3 \\ 1 & 1 & 13 & -4 \\ 12 & -3 & 2 & -2 \end{bmatrix}, \qquad \begin{bmatrix} 2 & 0 \\ 4 & 4 \\ 3 & -5 \end{bmatrix}$$

3×4 $\qquad$ 3×2

Linear System & Matrices

Given any linear system of equations, we can associate two matrices with the system. These are the **coefficient** matrix and the **augmented** matrix.

Example:
$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

Before we start to set up these matrices, we write our system in the form shown above. Note that all variables are on the left side, and like variables have the same order in each equation (they are aligned vertically).

Linear System: Coefficient Matrix

The **coefficient** matrix has one row for each equation and one column for each variable. The entries are the coefficients of the variables in our system.

$$\text{Example:}\quad \begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

m = # equations

n = # variables

Here 3x3

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Linear System: Augmented Matrix

The **augmented** matrix has one row for each equation, one column for each variable, and one extra, right most column. The entries in the first columns match the coefficient matrix, and the right most column has the numbers from the right hand side of each equation.

Example:

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

m = # of equations

n = # of variables + 1

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & 6 \end{bmatrix}$$

Here 3x4

Legitimate Operations for Solving a System

There are three operations that we can perform on a system of equations that result in an **equivalent** system. We can use these operations to *elinimate* variables. We can

- **swap** the order of any two equations,
- **scale** an equation by multiplying it by any **nonzero** number, and
- **replace** an equation with the sum[1] of itself and a nonzero multiple of any other equation.

We will use some standard notation for these operations. Let's call the first equation E_1, the second equation E_2 and so forth.

[1] Adding equations means adding like variables.

Swap

To indicate that we are swapping equations E_i and E_j, we'll write

$$E_i \leftrightarrow E_j$$

For example

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array} \quad \boxed{E_1 \leftrightarrow E_3} \quad \begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 6 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & 2x_2 & - & x_3 & = & -4 \end{array}$$

Scale

To indicate that we are scaling equation E_i by the nonzero factor k, we'll write

$$kE_i \to E_i$$

For example

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array} \quad \boxed{-2E_3 \to E_3} \quad \begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ -2x_1 & - & 2x_2 & - & 2x_3 & = & -12 \end{array}$$

Replace

To indicate that we are replacing equation E_j with the sum of itself adn k times equation E_i, we'll write

$$kE_i + E_j \to E_j$$

For example

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array} \quad \boxed{-2E_1 + E_2 \to E_2} \quad \begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ & - & 4x_2 & + & 3x_3 & = & 15 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array}$$

Note

$$\begin{array}{lrcrcrcr} & -2x_1 & - & 4x_2 & + & 2x_3 & = & 8 \\ & 2x_1 & & & + & x_3 & = & 7 \\ \hline \text{(add)} & 0x_1 & - & 4x_2 & + & 3x_3 & = & 15 \end{array}$$

Example

Use some sequence of these three operations to solve the following system by eliminating variable(s). Keep track of the augmented matrix at each step.

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ 2x_1 & & & + & x_3 & = & 7 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array} \qquad \begin{bmatrix} 1 & 2 & -1 & -4 \\ 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & 6 \end{bmatrix}$$

We can use x_1 in the first equation to **eliminate** x_1 from equations E_2 and E_3. Perform $-2E_1 + E_2 \to E_2$ and then $-E_1 + E_3 \to E_3$.

$-2E_1 + E_2 \to E_2$

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ & & -4x_2 & + & 3x_3 & = & 15 \\ x_1 & + & x_2 & + & x_3 & = & 6 \end{array} \qquad \begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & -4 & 3 & 15 \\ 1 & 1 & 1 & 6 \end{bmatrix}$$

$-E_1 + E_3 \to E_3$

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -4 \\ -4x_2 + 3x_3 &= 15 \\ -x_2 + 2x_3 &= 10 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & -4 & 3 & 15 \\ 0 & -1 & 2 & 10 \end{bmatrix}$$

$E_2 \leftrightarrow E_3$

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -4 \\ -x_2 + 2x_3 &= 10 \\ -4x_2 + 3x_3 &= 15 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & -1 & 2 & 10 \\ 0 & -4 & 3 & 15 \end{bmatrix}$$

$-4E_2 + E_3 \to E_3$

$$x_1 + 2x_2 - x_3 = -4$$
$$-x_2 + 2x_3 = 10$$
$$-5x_3 = -25$$

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & -1 & 2 & 10 \\ 0 & 0 & -5 & -25 \end{bmatrix}$$

$-\frac{1}{5}E_3 \to E_3 \qquad -E_2 \to E_2$

$$x_1 + 2x_2 - x_3 = -4$$
$$x_2 - 2x_3 = -10$$
$$x_3 = 5$$

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & 1 & -2 & -10 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$2E_3 + E_2 \to E_2$

$$x_1 + 2x_2 - x_3 = -4$$
$$x_2 = 0$$
$$x_3 = 5.$$

$$\begin{bmatrix} 1 & 2 & -1 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$E_3 + E_1 \to E_1$

$$\begin{aligned} x_1 + 2x_2 \quad &= 1 \\ x_2 \quad &= 0 \\ x_3 &= 5 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$-2E_2 + E_1 \to E_1$

$$\begin{aligned} x_1 \quad &= 1 \\ x_2 \quad &= 0 \\ x_3 &= 5 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

There is one solution,

$$(1, 0, 5).$$

Elementary Row Operations

Elementary row operations are operations we can perform on the rows of a matrix. There are three of them, and they are analogous to operations on the equations in a system. We use a similar notation using R_i for the i^{th} row.

Elementary Row Operations

i Interchange row i and row j (**swap**), $R_i \leftrightarrow R_j$.

ii Multiply row i by any nonzero constant k (**scale**), $kR_i \to R_i$.

iii Replace row j with the sum of itself and k times row i (**replace**), $kR_i + R_j \to R_j$.

Row Equivalent Matrices

Definition

Two matrices are called **row equivalent** if one can be obtained from the other by performing a sequence of elementary row operations.

Theorem

If the augmented matrices of two linear systems of equations are row equivalent, then the linear systems of equations are equivalent (i.e., they have the same solution set).

A key here is *structure*!

Consider the following augmented matrix. Determine if the associated system is consistent or inconsistent. If it is consistent, determine the solution set.

(a) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$

$x_1 = 3$

$x_2 = 1$

$x_3 = -2$

Yes, it's consistent w/ solution

$(3, 1, -2)$

(b) $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

The third equation reads

as

$0x_1 + 0x_2 + 0x_3 = 3$

$0 = 3$ always false

The system is inconsistent

(c) $\begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x_1 \quad - 2x_3 = -3$

$x_2 + x_3 = 4$

$0 = 0$

The system is consistent w/ ∞ solutions

$x_1 = -3 + 2x_3$

$x_2 = 4 - x_3$

and x_3 can be any real #

We can write the solution set as

$$\{(x_1, x_2, x_3) \mid x_1 = -3 + 2x_3,\ x_2 = 4 - x_3\}.$$

Section 1.2: Row Reduction and Echelon Forms

Definition

A matrix is in **echelon form**, also called *row echelon form (ref)*, if it has the following properties:

i Any row of all zeros are at the bottom.

ii The first nonzero number (called the *leading entry*) in a row is to the right of the first nonzero number in all rows above it.

iii All entries below a leading entry are zeros.[a]

[a] This condition is superfluous but is included for clarity.

an ref

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 7 \end{bmatrix}$$

not an ref

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Reduced Echelon Form

Definition

A matrix is in **reduced echelon form**, also called *reduced row echelon form (rref)* if it is in echelon form and has the additional properties

iv The leading entry of each row is 1 (called a *leading* 1), and

v each leading 1 is the only nonzero entry in its column.

an rref

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

not an rref

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$