Mechanical and Circuit Formulas Math 2306

Sping-Mass-Damping: For an object of mass m attached to a spring with spring constant k subject to linear damping (i.e. proportional to velocity) with damping coefficient b, the displacement x relative to equilibrium satisfies

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = f(t)$$

where f is the external driving force. If there is no external force, then the right side is zero. In the absence of damping, $b = 0$.

Simple Harmonic Motion: In the absence of damping or driving, we have

$$x'' + \omega^2 x = 0 \quad \text{where} \quad \omega = \sqrt{\frac{k}{m}}.$$

If the initial displacement $x(0) = x_0$ and initial velocity is $x'(0) = x_1$, then the motion has

- Period $T = \frac{2\pi}{\omega}$ and frequency $f = 1/T$
- Amplitude $A = \sqrt{x_0^2 + x_1^2/\omega^2}$
- Circular (and resonance frequency) ω

Hooke's Law: If an object of mass m displaces a spring δx units, the force imparted by the spring is

$$W = k\delta x \quad \text{where weight} \quad W = mg$$

Displacement in Equilibrium: If an object displaces a spring δx units *in equilibrium*, then

$$\omega^2 = \frac{k}{m} = \frac{g}{\delta x}.$$

LRC-series Circuit: For inductance L henries, resistance R ohms, capacitance C farads, and applied force E volts, the charge on the capacitor q satisfies

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E, \quad q(0) = q_0, \quad q'(0) = i_0$$

Current i is rate of change of charge, $i = \frac{dq}{dt}$. The **steady state** and **transient** charges are the particular solution (q_p) and the complementary solution (q_c), respectively.

Damping types: Overdamped = two distince real roots, Critically damped = one repeated real root, Underdamped = complex conjugate roots. (For both mechanical and circuits without external applied force.)

Damping Ratio: Relative to the differential equation

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0,$$

the ratio

$$\zeta = \frac{b}{2\sqrt{mk}}$$

is called the **damping ratio**.

The damping type can be determined from this value as follows:

$\zeta < 1$	under damped
$\zeta = 1$	critically damped
$\zeta > 1$	over damped

This criterion is identical to the one I'm using. In particular, note that this is related to the roots of the characteristic equation as follows.

ζ value	b, m, k values	Discriminant	Characteristic Root Case	Damping Type
$\zeta < 1$	$b < 2\sqrt{mk}$	$b^2 - 4mk < 0$	complex roots	under damped
$\zeta = 1$	$b = 2\sqrt{mk}$	$b^2 - 4mk = 0$	1 repeated real root	critically damped
$\zeta > 1$	$b > 2\sqrt{mk}$	$b^2 - 4mk > 0$	2 distinct real roots	over damped